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An approach to Hopf algebras via Frobenius coordinates II

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Abstract

We make a study of finitely generated, projective Hopf algebras over commutative rings from the point of view of P-Frobenius algebras. Defining modular function as the composite of counit with Nakayama automorphism, we establish the Radford formula (Amer. J. Math. 98 (1976) 333–335) [34] by means of comparing and transforming a Frobenius system. We then study when such Hopf algebras are separable and strongly separable, their Hopf subalgebras as Frobenius extensions of a third kind, their Drinfeld double and a change to a ground ring with trivial Picard group.

MSC: Primary; 16W30; Secondary; 16L60

1. Introduction

Perhaps the first beginnings of relating Frobenius algebras to Hopf algebras was the example by Berkson [4]. He proved that the restricted universal enveloping algebra of a finite dimensional restricted Lie algebra is a Frobenius algebra. Together with the well-known Frobenius algebra examples of finite group algebras, this raised the question if a finitely generated, projective Hopf k-algebra H is Frobenius. This was established by Larson and Sweedler [22] for k a principal ideal domain and their results were generalized by Pareigis [31] for k a commutative ring with trivial Picard group. Later, Hopf H-Galois extensions [19,6] and Hopf subalgebras [29,35] have been shown to be

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Frobenius extensions of the first and the second kinds for k a commutative ring (with a proviso that a Hopf subalgebra be a k-direct summand or pure k-submodule in H).

Although quantum groups, being deformations of the universal enveloping algebras or the algebra of polynomial functions on Lie groups, have been studied as Hopf algebras over fields, we would expect that any study of the deformations of affine group schemes would naturally involve Hopf algebras over commutative rings [39].

We began a study of a Hopf algebra H over commutative ring k from the point of view of Frobenius algebras and extensions in [13] starting with previous results in [22,31,2]. In [13] we studied a certain class of Hopf algebras called FH-algebras under the condition that H is a Frobenius algebra. A purely Frobenius approach to proving the Radford formula for the fourth power of the antipode $S: H \to H$ was taken there. In this paper we use this approach to the Radford formula for a general H. The idea of this proof in [13] and in the present paper is the following conceptually. First, from a complete set of Frobenius data called a Frobenius (coordinate) system for a Hopf algebra, we obtain another Frobenius system by applying the antipodal anti-automorphism. Second, we obtain two Nakayama automorphisms with formulas involving $S^{\pm 2}$ acted on from the right and left, respectively, by the left modular function for H. Third, the principle that any two Frobenius systems are unique up to an invertible element, called the (Radon–Nikodym) derivative, leads after a computation to the modular function for H^*, $b \in H$ as derivative. Finally, since the two Nakayama automorphisms are related by an inner automorphism determined by the derivative, we arrive at a conceptually simplified proof for the Radford formula for S^4. In principle, this technique might produce nice formulas or new proofs wherever one deals with examples of Frobenius algebras or extensions.

In this paper we will see that a good working principle is that a general Hopf algebra H is very close to being an FH-algebra [32]. As noted above, our main example of this principle is to make a Frobenius proof of Radford's formula work for a general finite projective Hopf algebra H. The first part of our paper is organized around this task as follows. In the Section 2, we present preliminary material on a general theory of P-Frobenius algebras [25,26,33] with Frobenius homomorphism, dual bases and Nakayama automorphisms, which we also call *a Frobenius system* for H. To this we add the conceptually useful *comparison theorem* and *transformation theorem* for P-Frobenius algebras. In Section 3, we continue a review of preliminaries with the basic integral theory for a finite projective Hopf algebra H the conclusion of which is that H is a P-Frobenius algebra with Frobenius homomorphism ψ very similar to a left integral and dual bases determined by a left norm N. In Section 4, we face the problem that for H the usual definition of modular function does not work: the usual definition depends on the norm element being a free generator of the space of integrals, but the space of integrals in H^* is not freely generated by a left norm. We instead define a modular function as the Nakayama automorphism composed with the counit [13], and prove that this plays a successful role. In Section 5, we find a formula for the Nakayama automorphism of H, similar to the formulas in [29,9], which eventually leads to the proof of Radford's formula in this general case. Then we transform the P-Frobenius system for H by the antipode S and prove that the derivative is proportional to the distinguished group-like $b \in H$. We finally apply the comparison theorem and obtain a complete but conceptual proof of Radford's formula for S^4 in this general case.

The rest of the paper is organized as follows. In Section 6, we show that a finite projective Hopf algebra H is separable precisely when the counit of its norm is invertible in some generalized sense for modules. We show that if H is separable and involutive, then it is strongly separable in Kanzaki's sense; conversely, as a corollary of Etingof and Gelaki [8], if H is separable and coseparable, it is involutive (given that 2 is not a zero-divisor in k). In Section 7, we show that a Hopf subalgebra pair forms a Frobenius extension of a *third kind*, which is an exotic generalization of Frobenius extensions of the second kind [28] and the P-Frobenius algebras of Section 2. This kind of Frobenius extension depends not only on a relative Nakayama automorphism but also on the two Picard group elements of k represented by the space of integrals of K^* and H^*. The relative homological algebra of Frobenius extensions [16,30] of the first kind and Frobenius extensions of the second and third kinds differs only in that the functors of co-induction and induction are naturally equivalent for the first kind and differ by a Morita auto-equivalence of the module category of the subalgebra for the second and third kinds. In Section 8, we return to the idea that a finite projective Hopf subalgebra H is close to being an FH-algebra by proving that H is a Hopf subalgebra of an FH-algebra in two ways. First, we prove that the Drinfel'd double $D(H)$ is an FH-algebra. Second, we find a ring extension $k \subset K$ such that $\mathrm{Pic}(K) = 0$: therefore the FH-algebra $H \otimes_k K$ is a flat extension of H.

2. Preliminaries: P-Frobenius algebras

In this section, we sketch the theory of P-Frobenius algebras which generalizes ordinary Frobenius algebras and will be needed in the later sections (except Proposition 2.2). The material in this section is folkloric and straightforward applications of for example [25,26,33,12]. We include short proofs since these have not appeared in published form. The material after and including Theorem 2.7 is, however, somewhat new.

Let k be a commutative ring throughout this paper. A tensor $\otimes$ without subscript means $\otimes_k$ as will a homomorphism group $\mathrm{Hom} = \mathrm{Hom}_k$. The k-dual of a k-module V is denoted by V^*. If A is a k-algebra, its V-dual $\mathrm{Hom}(A, V)$ has a standard A-bimodule structure given by $(bfc)(a) := f(cab)$ for every $f \in \mathrm{Hom}(A, V), a, b, c \in A$.

Let P be an invertible k-module throughout, i.e. P is finite projective of constant rank 1 [37]. The functor represented by $P \otimes -$ is a Morita auto-equivalence of the category of k-modules, denoted by $\mathscr{M}_k$, and P represents an isomorphism class in the Picard group $\mathrm{Pic}(k)$ of k [1,37]. Let Q be its inverse as an element of $\mathrm{Pic}(k)$, so $Q \cong P^*$, and both $P \otimes Q \cong k$ and $Q \otimes P \cong k$ are given by canonical isomorphisms ϕ_1 and ϕ_2, respectively, which we choose so that associativity holds

$$(qp)q' = q(pq') \tag{1}$$

for every $p \in P$ and $q, q' \in Q$, and a corresponding associativity equation on $P \otimes Q \otimes P$ [1], where the values of these isomorphisms are denoted simply by $p \otimes q \mapsto pq$ and

$q \otimes p \mapsto qp$. Since $\phi_2 \circ \varsigma \circ \phi_1^{-1}$ is an automorphism of k, where $\varsigma : P \otimes Q \to Q \otimes P$ is the ordinary twist map, we have $\chi, \gamma \in k$ such that $\chi\gamma = 1_k$ and

$$pq = \gamma qp,$$
$$qp = \chi pq \qquad (2)$$

for every $p \in P$, $q \in Q$. Since P is canonically identified with P^{**}, we will set $\gamma = \chi = 1$.

Definition 2.1. A k-algebra A is said to be a P-Frobenius algebra if

- A is finite projective as a k-module;
- $A_A \cong \mathrm{Hom}_k(A, P)_A$.

If $P \cong P'$, then a P-Frobenius algebra is also P'-Frobenius. In particular, if $P \cong k$, then a P-Frobenius algebra is an ordinary Frobenius algebra. Thus there are no non-trivial P-Frobenius algebras over ground rings with trivial Picard group. The following converse statement is *false*: if a P-Frobenius algebra is also P'-Frobenius, then $P \cong P'$. This may be somewhat surprising if one recalls that the corresponding statement is true for β-Frobenius extensions [28]. A counterexample is based on the Steinitz isomorphism theorem $A \oplus B \cong R \oplus AB$ for nonzero ideals A, B in a Dedekind domain R [23]:

Proposition 2.2. *Suppose R is a Dedekind domain and I is a non-principal ideal in R such that $I \cong I^{-1}$. Let $A := M_2(R)$. Then*

$${}_A\mathrm{Hom}_R(A, I) \cong {}_A A. \qquad (3)$$

Proof. Let F denote the field of fraction of R, and e_{ij} the matrix units in A. We first note that $\mathrm{Hom}_R(A, I) \cong M_2(I)$, since

$$f \mapsto \begin{pmatrix} f(e_{11}) & f(e_{12}) \\ f(e_{21}) & f(e_{22}) \end{pmatrix}$$

is a left A-isomorphism if we define the left A-module structure on $M_2(I)$ by $X \cdot B := BX^t$ for every $B \in M_2(I), X \in A$.

By the Steinitz isomorphism theorem, $I \oplus I \cong R \oplus R$ as R-modules determined by a matrix $C \in M_2(F)$ as $(x\ y) \mapsto (x\ y)C^t$. Then the mapping $X \mapsto (CX)^t$ for every $X \in M_2(I)$ determines an R-isomorphism $\Psi : M_2(I) \to A$. But for every $Y \in A$ we have

$$\Psi(Y \cdot X) = (CXY^t)^t = YX^tC^t = Y\Psi(X)$$

whence Ψ is a left A-module isomorphism as desired. $\square$

A is of course a well-known example of a Frobenius algebra over R. That it is also an I-Frobenius algebra where $I \ncong R$ follows directly from Theorem 2.4 below. R is for example realized by the ring of integers of an algebraic number field with two element ideal class group.

Recall that an algebra A is QF (quasi-Frobenius) in the sense of Müller [27], if A is finite projective as a k-module, and A_A is isomorphic to a direct summand of the direct sum of n copies of A^*_A, for $n \geqslant 1$. It follows straightaway from Definition 2.1 that:

Proposition 2.3. *A P-Frobenius algebra A is a QF algebra.*

Proof. If $P \oplus N \cong k^n$, then

$$A_A \oplus \mathrm{Hom}_k(A,N) \cong nA^*. \qquad \square$$

Recall that a QF ring A is artinian and injective as a right or left module over itself [20]. If k is an artinian commutative ring, it has trivial Picard group, so A in the proposition is a QF ring if k is a QF ring [27,16].

We shall see below that P-Frobenius algebras are much closer to being Frobenius algebras than QF algebras.

Theorem 2.4. *The following conditions on a k-algebra A are equivalent*:

(1) *A is a P-Frobenius algebra*;

(2) A_k *is finite projective and* ${}_AA \cong {}_A\mathrm{Hom}_k(A,P)$;

(3) *there are* $\phi \in \mathrm{Hom}_k(A,P)$, $x_1,\dots,x_n, y_1,\dots,y_n \in A$ *and* $q_1,\dots,q_n \in Q$ *such that*

$$\sum_i \phi(ax_i)q_i y_i = a \tag{4}$$

for every $a \in A$, *or*

$$\sum_i x_i q_i \phi(y_i a) = a \tag{5}$$

for every $a \in A$. (ϕ *is referred to as a Frobenius homomorphism and* $\{x_i\}, \{q_i\}$, $\{y_i\}$ *as dual bases for* ϕ.)

Proof. (1) $\Rightarrow$ (2) We compute using the Hom–tensor relation:

$$\begin{aligned} {}_A\mathrm{Hom}_k(A,P) &\cong \mathrm{Hom}_k(\mathrm{Hom}_k(A,P)_A, P) \\ &\cong {}_A\mathrm{Hom}_k(A^* \otimes P, P) \\ &\cong {}_A\mathrm{Hom}_k(A^*, k) \cong {}_AA, \end{aligned}$$

since P is an invertible module.

(2) $\Rightarrow$ (3) Given $\Psi : {}_AA \xrightarrow{\cong} {}_A\mathrm{Hom}_k(A,P)$ and $\phi := \Psi(1_A)$, then $\Psi(a) = a\phi$ for every $a \in A$. Then ${}_AA \otimes Q \cong {}_AA^*$ via $a \otimes q \mapsto a\phi q$. If $\{y_i \in A\}$, $\{f_i \in A^*\}$ is a finite projective base for A_k, one finds $x_{ij} \in A$, $q_{ij} \in Q$ such that $\sum_j x_{ij}\phi q_{ij} = f_i$. Setting $y_{ij} := y_i$ for each i and j, we have for every $a \in A$,

$$\begin{aligned} a &= \sum_i f_i(a) y_i \\ &= \sum_{i,j} (x_{ij}\phi)(a) q_{ij} y_{ij} = \sum_{i,j} \phi(a x_{ij}) q_{ij} y_{ij}. \end{aligned}$$

We merely reindex to get Eq. (4). Eq. (5) follows from a computation showing $\Psi(\sum_i x_i q_i \phi(y_i a))(x) = \Psi(a)(x)$ for $x, a \in A$, which is similar to [9, 1.3].

(3) $\Rightarrow$ (1) Suppose $\sum_{i=1}^n x_i q_i(\phi y_i) = \mathrm{id}_A$. Then A is finite projective. Define $\Psi : A_A \to \mathrm{Hom}_k(A, P)_A$ by $\Psi(a) := \phi a$ for every $a \in A$. Then Ψ is epi since for every $f \in \mathrm{Hom}_k(A, P)$ we have $\Psi(\sum_i f(x_i) q_i y_i)(a) = f(a)$ for every $a \in A$. Since $\Psi : A \to \mathrm{Hom}_k(A, P) \cong A^* \otimes P$ is an epimorphism between finite projective modules of the same local rank, (i.e. $\mathscr{P}$-rank for every prime ideal $\mathscr{P}$ in k), Ψ is bijective [37,31].

A similar argument shows that we may establish Condition 2 from Eq. (4). □

Throughout this section, we continue our use of the notation ϕ and x_i, q_i, y_i for the Frobenius homomorphism and dual base of a P-Frobenius algebra A. A QF ring has a Nakayama permutation on the set of simples modules induced by taking the socle of the corresponding projective indecomposable modules [20]. Frobenius algebras moreover have Nakayama *automorphisms* [16]. We next see that P-Frobenius algebras also have Nakayama automorphisms.

Corollary 2.5. *In a P-Frobenius algebra A there is an algebra automorphism $v: A \to A$ given by*

$$a\phi = \phi v(a) \tag{6}$$

for every $a \in A$. (Call v the Nakayama automorphism.)

Proof. In the proof of the last theorem we established $3 \Rightarrow 1$ by showing $a \mapsto \phi a$, for every $a \in A$, is an isomorphism. As we noted, we may equally well establish $3 \Rightarrow 2$ in this proof by showing that $a \mapsto a\phi$ is an isomorphism ${}_A A \cong {}_A \mathrm{Hom}_k(A, P)$. Since $a\phi \in \mathrm{Hom}_k(A, P)$ for each $a \in A$, it follows that there is a unique $a' \in A$ such that $a\phi = \phi a'$. One defines $v(a) = a'$ and easily checks that v is an automorphism. □

In this respect a P-Frobenius algebra is almost Frobenius: of course, v measures the deviation of ϕ from satisfying the trace condition $\phi(ab) = \phi(ba)$ for every $a, b \in A$. If v is inner, A will possess such a trace-like Frobenius homomorphism and is called a *symmetric P-Frobenius algebra*. We fix the data (ϕ, x_i, q_i, y_i, v) for the rest of this section and refer to this as the *Frobenius system* of A in this paper.

Proposition 2.6. *Given a P-Frobenius algebra A, the dual base tensor $\sum_i x_i \otimes q_i \otimes y_i$ satisfies $\forall a \in A$:*

1. $\sum_i a x_i \otimes q_i \otimes y_i = \sum_i x_i \otimes q_i \otimes y_i a$, *and*
2. $\sum_i x_i a \otimes q_i \otimes y_i = \sum_i x_i \otimes q_i \otimes v(a) y_i$.

Proof. We give only the proof of the second equation, the first being similar. By Eqs. (5), (1), (6) and (4), we compute:

$$\begin{aligned}\sum_i x_i a \otimes q_i \otimes y_i &= \sum_{i,j} x_j q_j \phi(y_j x_i a) \otimes q_i \otimes y_i \\ &= \sum_{i,j} x_j \otimes q_j \otimes \phi(y_j x_i a) q_i y_i\end{aligned}$$

$$= \sum_{i,j} x_j \otimes q_j \otimes \phi(v(a) y_j x_i) q_i y_i$$

$$= \sum_{j} x_j \otimes q_j \otimes v(a) y_j \qquad \square$$

We next prove that P-Frobenius systems for A are unique up to an invertible element in A, which we call the *comparison theorem*.

Theorem 2.7 ("Comparison Theorem"). *Suppose* (ϕ, x_i, q_i, y_i) *and* $(\phi', x'_j, q'_j, y'_j)$ *are two P-Frobenius systems for a P-Frobenius algebra A. Then there is* $d \in A^\circ$ *such that*

$$\phi' = \phi d \tag{7}$$

and

$$\sum_j x'_j \otimes q'_j \otimes y'_j = \sum_i x_i \otimes q_i \otimes d^{-1} y_i. \tag{8}$$

If v, v' *are the Nakayama automorphisms of* ϕ *and* ϕ', *then* $\forall a \in A$,

$$v'(a) = d^{-1} v(a) d. \tag{9}$$

Proof. Since ϕ and ϕ' freely generate $\mathrm{Hom}_k(A, P)$ as right A-modules, Eq. (7) is clear with d an invertible in A.

To verify Eq. (8), we note that

$$\sum_i x_i q_i \phi d(d^{-1} y_i a) = a \tag{10}$$

for every $a \in A$. There is an isomorphism

$$A \otimes Q \otimes A \cong \mathrm{End}_k(A)$$

given by $a \otimes q \otimes b \mapsto aq\phi' b$, for every $a, b \in A, q \in Q$, since $A \otimes A^* \cong \mathrm{End}_k(A)$ and $Q \otimes A \cong A^*$. Eq. (8) follows from the injectivity of this mapping and Eq. (10).

We note that for every $x, a \in A$

$$\phi'(xa) = \phi'(v'(a)x) \Leftrightarrow \phi(dxa) = \phi(v(a)dx) = \phi(dv'(a)x) \tag{11}$$

the last equation implying that for all $a \in A$,

$$v(a)d = dv'(a),$$

which is equivalent to Eq. (9). $\square$

We also need to know the effect of an algebra anti-automorphism on a Frobenius system, as given in the following *transformation theorem.*

Theorem 2.8 ("Transformation Theorem"). *Let A be a P-Frobenius algebra with Frobenius system (ϕ, x_i, q_i, y_i, v). If α is a k-algebra anti-automorphism of A, then*

$$(\alpha\phi,\ \bar{\alpha}(y_i),\ q_i,\ \bar{\alpha}(x_i),\ \bar{\alpha} \circ \bar{v} \circ \alpha) \tag{12}$$

is another Frobenius system for A, where $\bar{\alpha}$ and $\bar{v}$ denote the inverses of α and v, and $\alpha\phi := \phi \circ \alpha$.

Proof. We compute using the identity $\alpha(ab) = \alpha(b)\alpha(a)$ for all $a, b \in A$:

$$a = \sum_i x_i q_i \phi(y_i a) = \sum_i (\alpha\phi)(\bar{\alpha}(a)\bar{\alpha}(y_i))q_i x_i$$

and by applying $\bar{\alpha}$ to both sides we obtain

$$\bar{\alpha}(a) = \sum_i (\alpha\phi)(\bar{\alpha}(a)\bar{\alpha}(y_i))q_i\bar{\alpha}(x_i).$$

It follows from Theorem 2.4 that $\alpha\phi$ is a Frobenius homomorphism with dual bases $\{\bar{\alpha}(y_i)\}$, $\{q_i\}$, $\{\bar{\alpha}(x_i)\}$.

We compute the Nakayama automorphism η for $\alpha\phi$ in terms of α and v: for all $a, b \in A$,

$$\phi(\alpha(a)\alpha(b)) = (\alpha\phi)(ba) = (\alpha\phi)(\eta(a)b) = \phi(\alpha(b)\alpha\eta(a)) = \phi((v\alpha\eta)(a)\alpha(b))$$

by applying Eq. (6) twice. Since ϕ freely generates A^*, it follows that $v \circ \alpha \circ \eta = \alpha$, whence

$$\eta = \bar{\alpha} \circ \bar{v} \circ \alpha. \qquad \square \tag{13}$$

We will need the following lemma in our last section.

Lemma 2.9. *If A is a P-Frobenius algebra and B is a Q-Frobenius algebra, then the tensor product algebra $A \otimes B$ is a $P \otimes Q$-Frobenius algebra.*

Proof. First, $C := A \otimes B$ is finite projective as a k-module. Secondly,

$${}_C C \cong {}_A A \otimes {}_B B \cong {}_A\mathrm{Hom}(A, P) \otimes {}_B\mathrm{Hom}(B, Q) \cong {}_C\mathrm{Hom}(C, P \otimes Q),$$

since A, B, P and Q are finite projective k-modules. $\square$

3. Preliminaries II: Hopf Algebras as *P*-Frobenius Algebras

Let H be a Hopf algebra over a commutative ring k, which is finite (i.e., finitely generated) projective as a k-module, throughout this paper unless otherwise stated. In this section, we review the Hopf module structure on the dual Hopf algebra H^* [22,31] and the P-Frobenius structure on H [33]. For the convenience of the reader we offer proofs for the propositions that have not been published.

For the Hopf algebra H we denote its comultiplication by $\Delta : H \to H \otimes H$, its counit by ε, and its antipode by S. The values of Δ are denoted by $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$.

If M is a right comodule over H the values of its coaction on an element $m \in M$ is denoted by $\sum m_{(0)} \otimes m_{(1)}$. The dual of H is itself a Hopf algebra H^* where its multiplication is the convolution product (dual to Δ), comultiplication is the dual of multiplication on H, the counit is $1 \in H \cong H^{**}$ ($x \mapsto$ evaluation at x). We also denote its antipode by S where the context is clear. The notation $g \rightharpoonup a := \sum a_{(1)} g(a_{(2)})$ and $a \leftharpoonup g := \sum g(a_{(1)}) a_{(2)}$ denotes the usual left and right module actions of the convolution algebra H^* on $H \cong H^{**}$. Note that the $\rightharpoonup$ and $\leftharpoonup$ actions of H on H^* are exactly the usual ones defined in the previous section.

Proposition 3.1. *If H is a finite projective Hopf algebra, then H^* is right Hopf module.*

Proof (Sketch, Larson and Sweedler [22] and Pareigis [31]).
The natural left H^*-module structure on the dual algebra H^* induces a comodule structure mapping $\chi : H^* \to H^* \otimes H$, determined by

$$gh = \sum h_{(0)} g(h_{(1)}) \tag{14}$$

for every $g, h \in H^*$. The right H-module structure on H^* is given by $(h^* \cdot h)(x) = h^*(xS(h))$ for every $x, h \in H$ and $h^* \in H^*$. A rather long computation shows this compatible with the H^*-comodule structure in the sense of Hopf modules. $\square$

Proposition 3.2. *A right Hopf module M over a finite projective Hopf algebra H is isomorphic to the trivial Hopf module, $M \cong P(M) \otimes H$, where*

$$P(M) = \{m \in M \mid \chi(m) = m \otimes 1_H\}$$

is a k-direct summand of M and $\chi : M \to M \otimes H$ denotes the right H-comodule structure mapping.

Proof (Sketch, Pareigis [31]).
One shows that the map $M \to M$ given by $m \mapsto \sum S(m_{(0)}) m_{(1)}$ is a k-linear projection onto $P(M)$. Then the mapping $\beta : M \to P(M) \otimes H$ given by $\beta(m) = \sum m_{(0)} S(m_{(1)}) \otimes m_{(2)}$ has inverse given by the Hopf module map $\alpha : P(M) \otimes H \to M$ defined by $\alpha(m \otimes h) = mh$.

Corollary 3.3. *The k-module $P(H^*)$ associated to a Hopf algebra H by Propositions* 3.1 *and* 3.2 *is an invertible k-direct summand in H^*.*

Proof. Since $P(H^*) \otimes H \cong H^*$ and H, H^* have the same local ranks, it follows that the finite projective k-module $P(H^*)$ has constant rank 1. Then $P(H^*) \otimes P(H^*)^* \cong k$ and $P(H^*)$ is invertible [37]. $\square$

We note that $P(H^*)$ is the space of left integrals $\int_{H^*}^{\ell}$ in H^*:

$$P(H^*) = \{f \in H^* \mid gf = g(1) f\} \tag{15}$$

which follows from Eq. (14) since $\sum f_{(0)} \otimes f_{(1)} = f \otimes 1$.

Proposition 3.4. *The antipode S of a finite projective Hopf algebra H is bijective.*

Proof (Sketch, Pareigis [31]).
Assuming that $S(x) = 0$, one then notes that multiplication from the right by x on $P(H^*) \otimes H$ is zero by the existence of the (H-module) isomorphism $\alpha : P(H^*) \otimes H \to H^*$ in Proposition 3.2. If k is field $P(H^*) \cong k$ and it is clear that x is then zero. The general case follows from a localization argument. Surjectivity for S is apparent if k is a field, and the general case follows again from a localization argument. □

Denote the composition-inverse of S by $\bar{S}$.

Proposition 3.5 (Pareigis [33]). *If H is a finite projective Hopf algebra and*

$$P := P(H^*)^*,$$

then H is a P-Frobenius algebra.

Proof. We set $\Phi : P(H^*) \otimes H \xrightarrow{\cong} H^*$, $f \otimes x \mapsto f \cdot x$, where we note that the right H-module structure is related to the standard left H-module structure on H^* via a twist by S: for every $g \in H^*, x, y \in$

$$(g \cdot x)(y) = g(yS(x)) = (S(x)g)(y).$$

Let $Q := P(H^*)$, which is canonically isomorphic to the dual of P, and satisfies $P \otimes Q \cong k$ by Corollary 3.3.
Define $\Psi' : H \to \mathrm{Hom}_k(H, P)$ as the composite of the right H-module isomorphisms

$$H \longrightarrow P \otimes Q \otimes H \xrightarrow{1 \otimes \Phi} P \otimes H^* \longrightarrow \mathrm{Hom}_k(H, P).$$

It is easy to check that

$$\Psi'(x)(y)(q) := \Phi(q \otimes x)(y) = q(yS(x)) \tag{16}$$

for all $x, y \in H$ and $q \in Q$.
Now let $\Psi := \Psi' \circ \bar{S}$. Ψ is a Frobenius isomorphism ${}_H H \cong {}_H \mathrm{Hom}_k(H, P)$, since $\bar{S}$ is an anti-automorphism of H and

$$\Psi(xy) = \Psi'(\bar{S}(y)\bar{S}(x)) = \Psi(y) \cdot \bar{S}(x) = x\Psi(y). \qquad \square$$

Gabriel has an example of a finite projective Hopf algebra which is *not* a Frobenius algebra [31].

Corollary 3.6. *The Frobenius homomorphism $\psi : H \to P$ defined by the theorem satisfies for every $a \in H$*

$$\sum a_{(1)} \otimes \psi(a_{(2)}) = 1 \otimes \psi(a). \tag{17}$$

Proof. We note that the Frobenius homomorphism $\psi := \Psi(1) = \Psi'(1)$ satisfies by Eq. (16), for every $q \in P(H^*), a \in H$

$$\psi(a)(q) = q(a)$$

and

$$q(a)1_H = \sum a_{(1)}q(a_{(2)})$$

since $q \in \int_{H^*}^{\ell}$.

Since $P = Q^*$ and H is finite projective over k, we canonically identify $H \otimes P \cong \mathrm{Hom}_k(Q, H)$, and compute $\forall q \in Q, a \in H$

$$\left(\sum a_{(1)} \otimes \psi(a_{(2)})\right)(q) = \sum a_{(1)}\psi(a_{(2)})(q)$$

$$= \sum a_{(1)}q(a_{(2)}) = 1_H q(a) = (1 \otimes \psi(a))(q)$$

whence Eq. (17). □

If $\int_{H^*}^{\ell} \cong k$, we see from the theorem and the corollary that H is an ordinary Frobenius algebra with Frobenius homomorphism a left integral in H^*: this is called an *FH-algebra* [32,13]. Conversely, we have the following result.

Proposition 3.7. *If H is a Frobenius algebra and Hopf algebra, then H is an FH-algebra.*

Proof. We use the fact that the k-submodule of integrals of an augmented Frobenius algebra is free of rank 1 (cf. Lemma 4.4, [31, Theorem 3] or [13, Proposition 3.1]). Then $\int_{H}^{\ell} \cong k$. It follows from Proposition 3.5 that the dual Hopf algebra H^* is a Frobenius algebra. Whence $\int_{H^*}^{\ell} \cong k$ and H is an FH-algebra. □

Next, we obtain as in [33] a left norm for the Frobenius homomorphism $\psi : H \to P$ and study its properties. Since $x \mapsto x\psi$ is an isomorphism ${}_H H \to {}_H \mathrm{Hom}(H, P)$ and $\mathrm{Hom}(H, P) \otimes Q \cong H^*$ affords a canonical identification, it follows that there are elements $N_i \in H, q_i \in Q$ such that the counit of H,

$$\varepsilon \overset{\cong}{\mapsto} \sum_i N_i\psi \otimes q_i. \tag{18}$$

Call $N := \sum_i N_i \otimes q_i$ in $H \otimes Q$ the *left norm* of ψ, and note that $\sum_i \psi(aN_i)q_i = \varepsilon(a)$ for every $a \in H$. In the natural left H-module ${}_H H \otimes Q$ we have

$$aN = \varepsilon(a)N, \tag{19}$$

since both aN and $\varepsilon(a)N$ map to $\varepsilon(a)\varepsilon$ under the composite isomorphism, $H \otimes Q \xrightarrow{\cong} \mathrm{Hom}_k(H, P) \otimes Q \xrightarrow{\cong} H^*$ given by $a \otimes q \mapsto a\psi q$.

For all $p \in P$, we note that

$$\sum_i N_i q_i(p) \in \int_H^{\ell}, \tag{20}$$

since this follows by applying Eq. (19) to p.

Proposition 3.8 (Pareigis [33]). *If H is a Hopf algebra with Frobenius homomorphism ψ given above and left norm $\sum_i N_i \otimes q_i$, then the dual bases for ψ is given by*

$$\{N_{i(2)}\},\ \{q_i\},\ \{\bar{S}(N_{i(1)})\}. \tag{21}$$

Proof. We compute as in [33, Lemma 3.16], using Eq. (17) at first and Eq. (19) next (for every $a \in A$):

$$\begin{aligned}\sum \psi(aN_{i(2)})q_i\bar{S}(N_{i(1)}) &= \sum a_{(1)}N_{i(2)}(\psi(a_{(2)}N_{i(3)})q_i)\bar{S}(N_{i(1)})\\ &= \sum a_{(1)}\psi(a_{(2)}N_i)q_i\\ &= \sum a_{(1)}\varepsilon(a_{(2)})\psi(N_i)q_i = a\varepsilon(1) = a.\end{aligned}$$

It follows from Theorem 2.4 that $\{N_{i(2)}\}$, $\{q_i\}$, $\{\bar{S}(N_{i(1)})\}$ are dual bases for ψ. □

4. Pinning down the modular functions

In this section we give a definition of modular function in Eq. (24) based on [13], and find two formulas, Eqs. (25) and (27) which will be used later. The rest of this section is somewhat technical and might be browsed on a first reading.

It follows from applying S to the equation in the last proof, and setting $a = 1$, that

$$\sum_i (\psi\, q_i) \rightharpoonup N_i = 1, \tag{22}$$

where $\psi q_i \in H^*$ is the mapping $a \mapsto \psi(a)q_i$ for each i and $a \in H$. Of course $1 \in H^{**} \cong H$ is the counit of H^*. It follows from Eqs. (18) and (22) that the antipode on the dual Hopf algebra H^* is given by

$$S(g) = \sum N_i(g(\psi q_i)_{(2)})(\psi q_i)_{(1)} \tag{23}$$

since one computes that $\sum g_{(1)}S(g_{(2)}) = g(1)\varepsilon$ for every $g \in H^*$.

Theorem 4.1. *If H is a Hopf algebra and P-Frobenius algebra, then H^* is a Hopf algebra and P^*-Frobenius algebra with Frobenius homomorphism induced by the left norm $N \in H \otimes Q$.*

Proof. Let $Q = P^*$. We continue with the $q_i \in Q$, $N_i \in H$ defined above, and let $p_i \in P$ be such that $\sum_i q_i p_i = 1_k$. Clearly, H^* is a Hopf algebra and finite projective over k. Let ψ^* denote the mapping induced by $N \in H \otimes Q$ canonically identified with $\mathrm{Hom}(H^*, Q)$; i.e., $\psi^*(g) = \sum_i g(N_i)q_i$ for all $g \in H^*$. We will show that the right H^*-module mapping $\mathscr{F}: H^* \to \mathrm{Hom}(H^*, Q)$ given by $f \mapsto \psi^* f$ is a Frobenius *isomorphism*. In detail, we note that

$$(\psi^* g)(f) = \sum_i g(N_{i(1)})f(N_{i(2)})q_i.$$

We use the inverse Frobenius isomorphism $\Psi^{-1}:\ \mathrm{Hom}(H,P) \to H$ given by $g \mapsto \sum_i N_{i(2)} q_i g(\bar{S}(N_{i(1)}))$. Ψ^{-1} induces $H^* \xrightarrow{\cong} \mathrm{Hom}(H^*, Q)$ given by

$$g \mapsto \left(f \mapsto \sum_i f(N_{i(2)}) q_i g(\bar{S}(N_{i(1)})\right)$$

($g, f \in H^*$) via canonical maps $H \otimes Q \cong \mathrm{Hom}(H^*, Q)$ given by $a \otimes q \mapsto (g \mapsto g(a)q)$, $H^* \cong H^* \otimes P \otimes Q$ given by $f \mapsto \sum_i f \otimes p_i \otimes q_i$, and $H^* \otimes P \cong \mathrm{Hom}(H, P)$ given by $g \otimes p \mapsto (x \mapsto g(x)p)$.

Now the two displayed equations differ only by an application of the bijective map $\bar{S}$. I.e., the commutative triangle below shows that the Frobenius map $\mathscr{F}$ is an isomorphism. □

$$\begin{tikzcd} H^* \arrow[rr, "\cong"] \arrow[dr, "\cong", "\bar{S}"'] & & \mathrm{Hom}(H^*, Q) \\ & H^* \arrow[ur, "\mathscr{F}"] & \end{tikzcd}$$

Remark 4.2. Note that the last theorem shows that H^* is Q-Frobenius where $Q \cong P(H^*)$. At the same time, the theory in the previous section shows that the Hopf algebra H^* is $P(H)^*$-Frobenius. Caenepeel informs us that in fact $P(H)^*$ is canonically isomorphic to $P(H^*)$; we see this by showing ψ^* is a left integral in H^*, in the sense of Eq. (17), by the following computation. We use a pairing $\langle\, ,\, \rangle$ of $H^* \otimes Q$ and H with values in Q. For every $a \in H$,

$$\begin{aligned} \langle g_{(1)} \otimes \psi^*(g_{(2)}), a \rangle &= \sum_i g_{(1)}(a) g_{(2)}(N_i) q_i \\ &= \sum_i g(aN_i) q_i \\ &= \langle \varepsilon \otimes \psi^*(g), a \rangle \end{aligned}$$

by Eq. (19). Moreover, Eq. (22) shows that ψ^* is a left norm in $\mathrm{Hom}(H^*, Q)$. Then H^* with this norm has dual base $\{(\psi q_j)_{(2)}\}$, $\{p_j\}$, $\{\bar{S}(\psi q_j)_{(1)})\}$.

We next define a *left modular function* for a Hopf algebra H. We continue the notation established in the previous section.

Definition 4.3. Define the *left modular function*, or *left distinguished group-like element*, $m: H \to k$ *by*

$$m := \varepsilon \circ v, \tag{24}$$

where v is the Nakayama automorphism of H relative to ψ (cf. Corollary 2.5).

First note that m does not depend on the choice of Nakayama automorphism, since $\varepsilon(dv(a)d^{-1}) = \varepsilon(v(a))$ for every $a \in A$. Next, note that m is an algebra homomorphism (an augmentation in fact), and therefore a group-like element in the dual Hopf algebra H^*. With respect to the natural right H-module $H_H \otimes_k Q$, we note that for all $a \in H$,

$$Na = Nm(a), \tag{25}$$

since Na is mapped into $\sum_i N_i a\psi \otimes q_i = \sum_i N_i \psi\, v(a) \otimes q_i$, then into $\varepsilon(v(a))\varepsilon = m(a)\varepsilon$, under the canonical isomorphism $H \otimes Q \cong H^*$.

Let A be an algebra with augmentation ε, ${}_A M_A$ an A-bimodule and define the k-module of left integrals in M as $\int_M^\ell := \{x \in M | ax = \varepsilon(a)x\}$. For a Hopf algebra and P-Frobenius algebra H we consider the natural H-bimodule ${}_H H_H \otimes Q$ in the lemma below.

Lemma 4.4. *Given Hopf algebra H and Frobenius homomorphism ψ, $\int_{H\otimes Q}^\ell$ is a sub-bimodule freely generated by the left norm $N = \sum_i N_i \otimes q_i$ and a k-direct summand of $H \otimes Q$.*

Proof. N is left integral by Eq. (19). We recall the isomorphism $H \otimes Q \xrightarrow{\cong} \mathrm{Hom}(H, P) \otimes Q \xrightarrow{\cong} H^*$ given by $a \otimes q \mapsto (a\psi)q$. Given $T = \sum_i T_i \otimes q_i' \in \int_{H\otimes Q}^\ell$, denote $\phi(T) := \sum_i \psi(T_i) q_i' \in k$, and note that, for all $x \in H$,

$$\sum_i \psi(xT_i)q_i' = \varepsilon(x)\phi(T) = \sum_i \psi(xN_i)q_i\phi(T),$$

whence

$$T = \phi(T)N. \tag{26}$$

Thus, N generates $\int_{H\otimes Q}^\ell$ and the mapping of $H \otimes Q \to \int_{H\otimes Q}^\ell$ given by $x \otimes q \mapsto \psi(x)qN$ is a k-linear projection.

If $\lambda \in k$ such that $\lambda N = 0$, then

$$0 = \sum_i \psi(N_i)q_i\lambda = \varepsilon(1)\lambda = \lambda,$$

so N freely generates $\int_{H\otimes Q}^\ell$. $\square$

We similarly define right integrals in a bimodule over an augmented algebra, and prove a right-handed version of the lemma. It follows from Lemma 4.4 that $T := \sum_i \bar{S}(N_i) \otimes q_i$ is a right integral that freely generates $\int_{H\otimes Q}^r$, since $\varepsilon \circ \bar{S} = \varepsilon$ and $\bar{S}$ is an anti-automorphism of H. By Proposition 3.8, we compute

$$\begin{aligned} T &= \sum_{i,j,(N_i)} \psi(\bar{S}(N_j)N_{i(2)})q_i\bar{S}(N_{i(1)}) \otimes q_j \\ &= \sum \psi(\bar{S}(N_j))q_i\bar{S}(N_{i(1)})\varepsilon(N_{i(2)}) \otimes q_j \\ &= T\left(\sum_j q_j\psi(\bar{S}(N_j))\right), \end{aligned}$$

whence

$$\sum_j q_j \psi(\bar{S}(N_j))) = 1_k. \tag{27}$$

It follows that T is a right norm in the sense that $\sum_i q_i \psi \bar{S}(N_i) = \varepsilon$.

Lemma 4.5. *ψ and $\psi \circ \bar{S}$ are left and right norms in the natural H-bimodule $H^* \otimes P \cong \mathrm{Hom}_k(H, P)$.*

Proof. Theorem 4.1 states that $N \in H \otimes Q$ is a Frobenius homomorphism for the dual Hopf algebra H^*. The concepts of left and right norm relative to N make sense in the H^*-bimodule $H^* \otimes Q$. But Eq. (22) implies that $\psi \in \mathrm{Hom}(H, P) \cong H^* \otimes P$ is a left norm for N. Similarly, $\sum_i S(N_i) \otimes q_i$ is a Frobenius homomorphism $H^* \to Q$ by applying the anti-automorphism S as in Theorem 2.8, and $\bar{S}\psi$ is a right norm. □

One easily checks that $\sum_i S(N_i) \otimes q_i$ is a right norm in $H \otimes Q$ for $\psi \circ \bar{S}$. Since H^* is a Q-Frobenius algebra, it has a Nakayama automorphism v^*, which we make formal use of below.

Definition 4.6. Let $b \in H$, where H is canonically identified with H^{**}, be the left modular function defined by

$$b = \eta \circ v^*, \tag{28}$$

where η is the counit of H^* defined by $\eta(f) = f(1)$ for every $f \in H^*$.

It follows from Eq. (25) and Lemma 4.5 that for every $f \in H^*$,

$$\psi f = \psi f(b), \tag{29}$$

where $\psi \in H^* \otimes P$ has the natural H^*-bimodule structure.

5. An application to Radford's formula

We now compute a formula for the Nakayama automorphism of $\psi : H \to P$ in terms of the square of the antipode and m.

Theorem 5.1. *The Nakayama automorphism v for $\psi : H \to P$ is given by*

$$v(a) = \bar{S}^2(m \rightharpoonup a) = m \rightharpoonup \bar{S}^2(a). \tag{30}$$

Proof. The rightmost equation follows from noting that m is a group-like element in H^*, whence $m \circ S = m^{-1}$ and $m \circ S^2 = m$: i.e., S^2 and $\bar{S}^2$ fix m.

The leftmost equation is computed below and follows [33, Satz 3.17] until (31): for every $a \in H$,

$$\begin{aligned} S^2(v(a)) &= S^2\left(\sum \psi(N_{i(2)}a)q_i\bar{S}(N_{i(1)})\right) \\ &= \sum S(N_{i(1)})\psi(N_{i(2)}a)q_i \\ &= \sum S(N_{i(1)})N_{i(2)}a_{(1)}\psi(N_{i(2)}a_{(2)})q_i \\ &= \sum a_{(1)}\psi(N_i a_{(2)})q_i \\ &= \sum a_{(1)}m(a_{(2)})\psi(N_i)q_i \\ &= m \rightharpoonup a \end{aligned} \tag{31}$$

by Eqs. (6), (17), (25) and (18), respectively. □

Since H has Frobenius system $(\psi, N_{i(2)}, q_i, \bar{S}(N_{i(1)}), v)$, it follows from Theorem 2.8 that we obtain another Frobenius system by applying the algebra (and coalgebra) anti-automorphism $\bar{S}$:

Proposition 5.2. *A Hopf algebra H with left norm N has Frobenius system*

$$(\bar{S}\psi,\ N_{i(1)},\ q_i,\ S(N_{i(2)}),\ \alpha), \tag{32}$$

where $\bar{S}\psi$ satisfies a "right integral-like equation",

$$(\bar{S}\psi)(x) \otimes 1_H = \sum (\bar{S}\psi)(x_{(1)}) \otimes x_{(2)} \tag{33}$$

and the Nakayama automorphism,

$$\alpha(x) = S^2(x) \leftharpoonup m \tag{34}$$

for every $x \in H$.

Proof. The dual bases (32) follows directly from Theorem 2.8 and Proposition 3.8. Eq. (33) follows from Eq. (17) since $\bar{S}$ is a coalgebra anti-automorphism.

To compute the Nakayama automorphism we first need to find the inverse of Eq. (13): for all $a \in H$,

$$\bar{v}(a) = S^2(m^{-1} \rightharpoonup a) = m^{-1} \rightharpoonup S^2(a). \tag{35}$$

Next we apply Eq. (13) where $\bar{S}$ is the anti-automorphism:

$$\begin{aligned} \alpha(x) &= (S \circ \bar{v} \circ \bar{S})(x) \\ &= S(m^{-1} \rightharpoonup S(x)) \\ &= S\left(\sum S(x_{(2)})m^{-1}(S(x_{(1)}))\right) \\ &= S^2(x) \leftharpoonup m, \end{aligned}$$

since $m \circ S = m^{-1}$ and S^2 is an algebra and coalgebra *automorphism*. □

By the comparison theorem, we know that the two Frobenius homomorphisms ψ and $\bar{S}\psi$ are related by an invertible element d called the derivative: $\bar{S}\psi = \psi\, d$. The next proposition shows that d is proportional to the left distinguished group-like element b of H^*.

Proposition 5.3. *If ψ is a Frobenius homomorphism for the Hopf algebra H, then*

$$\psi \circ \bar{S} = \psi b. \tag{36}$$

Proof. We first show that ψb is a right integral in the H^*-bimodule $H^* \otimes P$. Recall that $H^* \otimes P$ is canonically identified with $\mathrm{Hom}_k(H,P)$ Let $f \in H^*$, then

$$(\psi b)f = [\psi(fb^{-1})]b = [\psi((fb^{-1})(b))]b = (\psi b)f(1)$$

since $\Delta(b) = b \otimes b$.

Since $\psi \circ \bar{S}$ is a right norm it follows that there is $\lambda \in k$ such that $\psi \circ \bar{S} = \lambda(\psi b)$. But comparing Eq. (27) to the application below of Eq. (19):

$$\sum_i q_i(\psi b)(N_i) = \varepsilon(b)\varepsilon(1) = 1,$$

shows that $\lambda = 1$ (cf. Eq. (2)). □

Theorem 5.4. *If H is a finite projective Hopf algebra with left distinguished group-like elements $b \in H$ and $m \in H^*$, then for every $a \in H$,*

$$S^4(a) = b^{-1}(m \rightharpoonup a \leftharpoonup m^{-1})b. \tag{37}$$

Proof. On the one hand, the Nakayama automorphism $\alpha : H \to H$ for the Frobenius homomorphism $\bar{S}\psi$ is by Proposition 5.2 given by

$$\alpha(a) = S^2(a) \leftharpoonup m = S^2(a \leftharpoonup m)$$

for every $a \in H$. On the other hand, the Nakayama automorphism v of H for the Frobenius homomorphism $\psi \in \mathrm{Hom}(H,P)$ is by Theorem 5.1

$$v(a) = \bar{S}^2(m \rightharpoonup a) = m \rightharpoonup \bar{S}^2(a),$$

for every $a \in H$. By Proposition 5.3, $\psi \circ \bar{S} = \psi b$, so by the comparison theorem

$$\alpha(a) = b^{-1}v(a)b$$

for every $a \in H$.

Substituting the first two equations in the third yields,

$$S^2(a) = b^{-1}\bar{S}^2(m \rightharpoonup a)b \leftharpoonup m^{-1}$$

which is equivalent to Eq. (37) since S^2 fixes b and m, and for every group-like $a \in H$, we have $m \rightharpoonup (axa^{-1}) = a(m \rightharpoonup x)a^{-1}$. □

Remark 5.5. In [13] it was shown that a group-like element g in a finite projective Hopf algebra over a Noetherian ring k has finite order dividing the least common multiple N of the local ranks of H. Since m and b are group-like elements in H^*

and H, respectively, it follows from the general Radford formula and Eq. (30) that the antipode S and the Nakayama automorphism $v: H \to H$ have finite order dividing $4N$ and $2N$, respectively.

Waterhouse sketches a different method of how to extend the Radford formula to a finite projective Hopf algebra and show that S has finite order [38]. Schneider has established Radford's formula by different methods for $k=$ field [36]. Radford's formula is generalized to double Frobenius algebras over fields by Koppinen [18].

6. When Hopf algebras are separable

In this section, we give a criterion in terms of the left norm N for when a finite projective Hopf algebra H is separable. We first need a proposition closely related to some results on when Frobenius algebras/extensions/bimodules are separable [11,5,12]. Let k be a commutative ground ring.

Proposition 6.1. *Suppose A is a P-Frobenius algebra with system (ψ, x_i, q_i, y_i). Then A is k-separable if and only if there is $d \in P \otimes A$ such that*

$$\sum_i x_i q_i d y_i = 1_A.$$

Proof. The forward implication is proven by first letting $\sum_j a_j \otimes b_j$ be the separability element for A. Next set $d := \sum_j \psi(a_j) \otimes b_j \in P \otimes A$. Then

$$\sum_i x_i q_i d y_i = \sum_{i,j} x_i q_i \psi(a_j) b_j y_i = \sum_j \sum_i x_i q_i \psi(y_i a_j) b_j = \sum_j a_j b_j = 1_A.$$

The reverse implication is proven by noting that $e := \sum_i x_i \otimes q_i d y_i$ is a separability element for A. By hypothesis, $\mu(e) = 1$ where $\mu: A \otimes A \to A$ is the multiplication mapping. e is in the center $(A \otimes A)^A$ of the natural A-bimodule $A \otimes A$ as a consequence of Proposition 2.6. □

Next, let P be an invertible k-module with inverse Q. We shall say that $q \in Q$ is *Morita-invertible* if there is $p \in P := Q^*$ such that $qp = 1_k$. We note that if $q \in Q$ is Morita-invertible, then Q and P are free of rank 1, since $q' \mapsto q'p$ is epi $Q \to k$, whence an isomorphism. More generally, we say that $\sum_i q_i \otimes a_i \in Q \otimes A$ is Morita-invertible where A is a k-algebra if there is $\sum_j p_j \otimes b_j \in P \otimes A$ such that $\sum_{i,j} q_i p_j a_i b_j = 1_A$. The next theorem generalizes results in [29,2].

Theorem 6.2. *Suppose H is a finite projective Hopf algebra with P-Frobenius homomorphism ψ satisfying Eq.* (17) *and left norm $N = \sum_i N_i \otimes q_i$. Then H is k-separable if and only if $\sum_i \varepsilon(N_i) q_i$ is Morita-invertible.*

Proof. We make use of the dual bases $\{N_{i(2)}\}, \{q_i\}, \{\bar{S}(N_{i(1)})\}$ given by Proposition 3.8. If H is k-separable, then by the proposition above there is $d := \sum_j p_j \otimes a_j \in P \otimes H$

such that

$$\sum_{i,(N_i)} N_{i(2)} q_i d \bar{S}(N_{i(1)}) = 1_H.$$

Applying ε we obtain

$$\sum \varepsilon(\varepsilon(N_{i(1)}) N_{i(2)}) q_i p_j \varepsilon(a_j) = \sum_i \varepsilon(N_i) q_i \sum_j p_j \varepsilon(a_j) = 1_k,$$

whence $\sum_i \varepsilon(N_i) q_i$ is Morita-invertible.

Conversely, if $q := \sum_i \varepsilon(N_i) q_i$ is Morita-invertible with inverse $p \in P$ such that $qp = 1_k$, then we let $d := p \otimes 1_H$. Note that

$$\sum N_{i(2)} q_i d \bar{S}(N_{i(1)}) = \sum_i \varepsilon(N_i) q_i p 1_H = 1_H,$$

whence H is k-separable by Proposition 6.1. □

It follows directly from this theorem that a k-separable projective Hopf algebra is an FH-algebra, since it is P-Frobenius with $P \cong k$. Since a k-separable H has a Morita invertible element, it is P-Frobenius with $P \cong k$; whence the corollary below.

Corollary 6.3. *A separable Hopf algebra H is an FH-algebra.*

As a result, a separable Hopf algebra H is unimodular [13]: i.e. $m = \epsilon$. The following is a corollary and generalization of the main theorem in Etingof and Gelaki [8].

Theorem 6.4. *Suppose* 2 *is not a zero-divisor in k, and Hopf k-algebra H is separable and coseparable. Then $S^2 = \mathrm{id}_H$.*

Proof. First we note that H is unimodular and counimodular. Then it follows from Theorem 5.4 (or cf. [2, Corollary 3.9]) that $S^4 = \mathrm{id}$. Localizing with respect to the set $T = \{2^n, n = 0, 1, \ldots\}$ we may assume that 2 is invertible in k. Then $H = H_+ \oplus H_-$ where $H_\pm = \{h \in H : S^2(h) = \pm h\}$, respectively. We have to prove that $H_- = 0$. It suffices to prove that $(H_-)_m = 0$ for any maximal ideal m in H. Since H_m/mH_m is separable and coseparable over the field k/m, we deduce from the main theorem in [8] that $(H_-)_m \subset mH_m$ and therefore $(H_-)_m \subset m(H_-)_m$. The required result follows from the Nakayama Lemma because H_- is a direct summand in H. □

Next we study when separable Hopf algebras are strongly separable. Recall that an algebra A is *strongly separable* [15,10] if there is $e := \sum_j z_j \otimes w_j \in A \otimes A$ such that $\mu(e) = \sum_j z_j w_j = 1_A$ and for every $a \in A$, we have $\sum_j z_j a \otimes w_j = \sum_j z_j \otimes a w_j$. We will call such an $e \in A \otimes A$ a *Kanzaki separability element*: one may prove that its transpose $\sum_i w_j \otimes z_j$ is an ordinary separability idempotent [10] (cf. [14, Theorem 3.4]). For example, if k is an algebraically closed field of characteristic p, then A is strongly separable if it is semisimple and none of its simple modules have dimension over k divisible by p. We first need a proposition which generalizes part of [14, Proposition 4.1].

Proposition 6.5. *Suppose A is a P-Frobenius algebra with system (ψ, x_i, q_i, y_i) such that*

$$u := \sum_i q_i \otimes y_i x_i \tag{38}$$

is Morita-invertible. Then A is strongly separable.

Proof. Suppose $\sum_j p_j \otimes a_j \in P \otimes A$ satisfies $\sum_{i,j} q_i p_j y_i x_i a_j = 1_A$. From this and Proposition 2.6, we easily see that $e := \sum_{i,j} y_i \otimes x_i q_i p_j a_j$ is a Kanzaki separability element. □

Setting $u^{-1} := \sum_j p_j \otimes a_j$, we can apply Proposition 2.6 to obtain a formula for the Nakayama automorphism:

$$v(a) = uau^{-1}, \tag{39}$$

where we make use of the usual Morita mapping $Q \otimes P \to k$.

Recall that a Hopf algebra H is *involutive* if $S^2 = \mathrm{id}_H$. The next theorem contains a result of Larson [21] as a special case.

Theorem 6.6. *Suppose H is a finite projective, separable, involutive Hopf algebra. Then H is strongly separable.*

Proof. If $(\psi, N_{i(2)}, q_i, \bar{S}(N_{i(1)}))$ is the P-Frobenius system for H given by Proposition 3.8, we note here that $\bar{S} = S$, so that the u-element of Proposition 6.5,

$$u := \sum_i q_i \otimes \sum_{(N_i)} S(N_{i(1)}) N_{i(2)} = \sum_i q_i \varepsilon(N_i) \otimes 1_H$$

is Morita-invertible by Theorem 6.2. □

7. Hopf subalgebras

Throughout this section, k is a commutative ring and we consider a finite projective Hopf algebra H with Hopf subalgebra K which is also finite projective as a k-module. We will show that the functors of induction and co-induction from the category $\mathcal{M}_K$ of K-modules to $\mathcal{M}_H$ are naturally isomorphic up to a Morita auto-equivalence of $\mathcal{M}_K$ determined by a relative Nakayama automorphism and a relative Picard group element. This section generalizes results in [29,35,13].

Let R be an arbitrary ring, $\beta : R \to R$ a ring automorphism, and M_R a module over R. The β-twisted module M_β is defined by $m \cdot r := m\beta(r)$, clearly another R-module. If β is an inner automorphism, is easy to check that $M_R \cong M_\beta$ and $M \otimes_R R_\beta \cong M_\beta$. Then the bimodule ${}_R R_\beta$ induces a Morita auto-equivalence of $\mathcal{M}_R$ via tensoring.

Lemma 7.1. *If A is a P-Frobenius k-algebra with Frobenius homomorphism ϕ and corresponding Nakayama automorphism v, then we have the following bimodule* *isomorphisms*:

$$ {}_A A_A \cong {}_A\mathrm{Hom}(A,P)_\nu \cong {}_{\nu^{-1}}\mathrm{Hom}(A,P). \tag{40}$$

Proof. Since $a\phi = \phi\nu(a)$ in A^* for every $a \in A$, it follows that the Frobenius isomorphisms $a \mapsto a\phi$ and $a \mapsto \phi a$ induce the first and second isomorphisms above (between A and $\mathrm{Hom}(A,P)$). □

As a straightforward extension of Definition 2.1, we define P-Frobenius extension A/S, where P is an invertible S-bimodule (and $-\otimes_S P$ defines a Morita auto-equivalence of $\mathcal{M}_S$ [1]).

Definition 7.2. Suppose S is a subring of ring A and P is an invertible S-bimodule. We say A is a *P-Frobenius extension of S*, *or A/S is a Frobenius extension of the third kind*, if

1. A_S is a finite projective module;
2. ${}_S A_A \cong {}_S\mathrm{Hom}_S(A_S, P_S)_A$

A P-Frobenius extension has a symmetric definition, a Frobenius system like in Section 2, a Nakayama automorphism defined on the centralizer subalgebra $C_S(A)$ of A [33], and a comparison theorem, which we will not need here. As a straightforward consequence of a theorem by Morita [25,26], we state without proof (cf. [9]):

Proposition 7.3. *A is P-Frobenius extension of S if and only if there is a natural isomorphism of right A-modules*,

$$ M \otimes_S A \cong \mathrm{Hom}_S(A_S, M \otimes_S P_S) \tag{41}$$

for every module $M \in \mathcal{M}_S$.

This equivalent condition for a P-Frobenius extension states in other words that the functors of induction and co-induction from $\mathcal{M}_S$ into $\mathcal{M}_A$ form a commutative triangle with the Morita auto-equivalence of $\mathcal{M}_S$ induced by $-\otimes_S P$.

Suppose a Frobenius algebra pair forms a projective ring extension such that the Nakayama automorphism of the overalgebra preserves the subalgebra. We now obtain a theorem that states that such a pair forms a certain P-Frobenius extension.

Theorem 7.4. *Suppose A is a P-Frobenius algebra*, *B is a P'-Frobenius algebra*, *and B is subalgebra of A such that A_B is a finite projective module*, *and a Nakayama automorphism ν_A of A sends B into B*: $\nu_A(B) = B$. *Let ν_B denote a Nakayama automorphism of B. Then A is a W-Frobenius extension of B*, *where*

$$ W = {}_\beta B \otimes Q' \otimes P, \tag{42}$$

$Q' = P'^*$ *and β is the relative Nakayama automorphism given by*

$$ \beta = \nu_B \circ \nu_A^{-1}. \tag{43}$$

Proof. Since A_B is assumed finite projective, we need only show that ${}_BA_A \cong {}_B\mathrm{Hom}_B(A_B, W_B)$. We compute using the hom-tensor adjointness relation and two applications of Lemma 7.1:

$$\begin{aligned} {}_BA_A &\cong {}_{v_A^{-1}}\mathrm{Hom}(A,P)_A \\ &\cong \mathrm{Hom}_k(A \otimes_B B_{v_A^{-1}}, k)_A \otimes P \\ &\cong {}_{v_A^{-1}}\mathrm{Hom}_B(A_B, B_B^*)_A \otimes P \\ &\cong {}_{v_A^{-1}}\mathrm{Hom}_B(A_B, {}_{v_B}B_B \otimes Q')_A \otimes P \\ &\cong {}_B\mathrm{Hom}_B(A_B, {}_{v_B \circ v_A^{-1}}B_B \otimes Q' \otimes P)_A. \qquad \square \end{aligned}$$

Let $K \subseteq H$ be a pair of finite projective Hopf k-algebras where K is a Hopf subalgebra of H (i.e., K is a pure k-submodule of H, $\Delta(K) \subseteq K \otimes K$ and $S(K) = K$) in the next corollary. Let $P(K)^*$, $P(H)^*$ be the k-module of integrals $\int_K^\ell$, $\int_H^\ell$, respectively, v_H, v_K be the respective Nakayama automorphisms and m_H, m_K be the respective left modular functions.

Corollary 7.5. *If $K \subseteq H$ is a finite projective Hopf subalgebra pair, then H/K is a P-Frobenius extension where*

$$P = {}_\beta K \otimes P(K)^* \otimes P(H) \tag{44}$$

and

$$\beta = v_K \circ v_H^{-1}. \tag{45}$$

Proof. The natural module H_K is finite projective as a corollary of the Nichols–Zoeller Freeness theorem [13, Proposition 5.3]. Furthermore, the Nakayama automorphism $v_H^{\pm 1}(a) = m_H^{\pm 1} \rightharpoonup S^{\mp 2}(a)$ for every $a \in H$ by Eq. (30), whence $v_H(K) = K$. Thus the hypotheses of Theorem 7.4 are satisfied. $\square$

It follows from the formulas for v_H and v_K in Eq. (30) that for every $x \in K$,

$$\begin{aligned} \beta(x) &= m_K \rightharpoonup \bar{S}^2(m_H^{-1} \rightharpoonup S^2(x)) \\ &= (m_K * m_H^{-1}) \rightharpoonup x \end{aligned} \tag{46}$$

(cf. [9]).

Remark 7.6. Kasch makes a study in [16] of the relative homological algebra of Frobenius extensions. One can extend this study to a Frobenius extension A/S of the third kind by taking into account some Morita theory. For example, one may show by these means that under the (rather common) additional assumption that S is S-bimodule isomorphic to a direct summand in A, the flat dimension of any S-module is equal to both the flat dimension of its induced A-module and of its co-induced A-module.

In [32] the study in [16] is extended to a cohomology theory for FH-algebras, showing that these have a complete cohomology with cup product, a generalized Tate duality under a certain cocommutativity condition, and a generalized Hochschild–Serre spectral sequence.

8. Embedding H into an FH-algebra

In this section we show that a finite projective Hopf algebra H is a Hopf subalgebra of an FH-algebra in two ways. We first show that H is a Hopf subalgebra of $D(H)$. We let k be a commutative ring. The *quantum double* $D(H)$ of a finite dimensional Hopf algebra, due to Drinfel'd [7], is readily extended to a finite projective Hopf algebra H over k: at the level of coalgebras it is given by

$$D(H) := H^{*\,\mathrm{cop}} \otimes_k H,$$

where $H^{*\,\mathrm{cop}}$ is the co-opposite of H^*, the co-product being Δ^{op}. The multiplication on $D(H)$ is described in two equivalent ways as follows [24, Lemma 10.3.11]. In terms of the notation gx replacing $g \otimes x$ for every $g \in H^*, x \in H$, both H and $H^{*\,\mathrm{cop}}$ are subalgebras of $D(H)$, and for each $g \in H^*$ and $x \in H$,

$$xg := \sum (x_{(1)} \rightharpoonup gS^{-1} \leftharpoonup x_{(3)})x_{(2)} = \sum g_{(2)}(S^{-1}g_{(1)} \rightharpoonup x \leftharpoonup g_{(3)}). \tag{47}$$

The algebra $D(H)$ is a Hopf algebra with antipode $S'(gx) := SxS^{-1}g$, the proof of this proceeding as in [17].

Theorem 8.1. *If H is a finite projective Hopf algebra, then $D(H)$ is an FH-algebra.*

Proof. It is enough to show that $\int^{\ell}_{D(H)^*} \cong k$. As an algebra, $D(H)^* \cong H^{\mathrm{op}} \otimes H^*$, the tensor product algebra of H^* and the opposite algebra of H. Now H is P-Frobenius algebra if and only if H^{op} is, since they have the same Frobenius system with a change of order in the dual base. By Theorem 4.1, H^* is a P^*-Frobenius algebra. It follows from Lemma 2.9 that $D(H)^*$ is a Frobenius algebra, since $P \otimes P^* \cong k$. Now the k-space of integrals of an augmented Frobenius algebra is free of rank one, which proves our theorem. □

Next, we show that H has a ring extension to an FH-algebra $H \otimes_k K$. This will follow right away from the construction of a ring extension $k \subset K$ where K has trivial Picard group. We continue with k as a commutative ring, and let M be the set of all maximal ideals in k. Choose a finite subset $M_\alpha \subset M$.

Let $m_{\alpha_1}, \ldots, m_{\alpha_n}$ be all the elements of M_α, i.e. maximal ideals in k. Then the set

$$K_\alpha = k_{m_{\alpha_1}} \oplus \cdots \oplus k_{m_{\alpha_n}}$$

is a semilocal ring and has trivial Picard group: $\mathrm{Pic}(K_\alpha) = 0$. For any pair $M_\alpha \subset M_\beta$, we have the canonical projection $\pi_{\alpha\beta} : K_\beta \to K_\alpha$ and we may consider the inverse limit ring

$$K := \varprojlim (K_\alpha, \pi_{\alpha\beta}) \tag{48}$$

since finite subsets of M form a partially ordered directed set. Furthermore, for any finite subset M_α of M we have the canonical homomorphism $f_\alpha: k \to K_\alpha$, which is the direct sum of the corresponding localization homomorphisms. The following diagram is clearly commutative:

$$\begin{tikzcd} & k \arrow[dl, "f_\beta"'] \arrow[dr, "f_\alpha"] & \\ K_\beta \arrow[rr, "\pi_{\alpha\beta}"] & & K_\alpha \end{tikzcd}$$

From universality we obtain a homomorphism $f: k \to K$.

Lemma 8.2. *f is a monomorphism.*

Proof. Let f_m be the localization homomorphism $f_m: k \to k_m$. Then it follows easily that $\ker f = \cap_{m \in M} \ker f_m = 0$. □

Now let $\pi_\alpha: K \to K_\alpha$ be the canonical epi. Since the diagram

$$\begin{tikzcd} & K \arrow[dl, "\pi_\beta"'] \arrow[dr, "\pi_\alpha"] & \\ K_\beta \arrow[rr, "\pi_{\alpha\beta}"] & & K_\alpha \end{tikzcd}$$

is commutative, the following diagram is commutative as well:

$$\begin{tikzcd} & Pic(K) \arrow[dl, "Pic(\pi_\beta)"'] \arrow[dr, "Pic(\pi_\alpha)"] & \\ Pic(K_\beta) \arrow[rr, "Pic(\pi_{\alpha\beta})"] & & Pic(K_\alpha) \end{tikzcd}$$

Again from universality we obtain a homomorphism

$$\Phi: \ \mathrm{Pic}(K) \to \varprojlim (\mathrm{Pic}(K_\alpha), \mathrm{Pic}(\pi_{\alpha\beta}))$$

Theorem 8.3. *Φ is injective.*

Proof. We need the following result proved in [3]:

Theorem 8.4. *Suppose I is some directed system and for each ordered $\alpha, \beta \in I$, A_α is a commutative ring and there is an epimorphism $\psi_{\alpha\beta}$ such that the restriction to the group of units $\psi_{\alpha\beta}\colon U(A_\beta) \to U(A_\alpha)$ is a surjection. If*

$$A = \varprojlim (A_\alpha, \psi_{\alpha\beta}),$$

then the induced map

$$\mathrm{Pic}(A) \to \varprojlim (\mathrm{Pic}(A_\alpha), \mathrm{Pic}(\psi_{\alpha\beta}))$$

is injective.

The hypotheses of this proposition are fulfilled by the mappings $\pi_{\alpha\beta}: K_\beta \to K_\alpha$, whence Φ is injective. $\square$

The next corollary follows from recalling that $\mathrm{Pic}(K_\alpha) = 0$.

Corollary 8.5. *Given a commutative ring k and K defined in Eq.* (48), *$k \subset K$ is a ring extension with* $\mathrm{Pic}(K) = 0$.

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