Topological Fluid Dynamics: Theory and Applications

Dynamics of scaled norms of vorticity for the three-dimensional Navier-Stokes and Euler equations

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Abstract

A series of numerical experiments is suggested for the three-dimensional Navier-Stokes and Euler equations on a periodic domain based on a set of L^{2m}-norms of vorticity Ω_m for $m \geq 1$. These are scaled to form the dimensionless sequence $D_m = (\varpi_0^{-1}\Omega_m)^{\alpha_m}$ where ϖ_0 is a constant frequency and $\alpha_m = 2m/(4m-3)$. A numerically testable Navier-Stokes regularity criterion comes from comparing the relative magnitudes of D_m and D_{m+1} while another is furnished by imposing a critical lower bound on $\int_0^t D_m \, d\tau$. The behaviour of the D_m is also important in the Euler case in suggesting a method by which possible singular behaviour might also be tested.

1. Introduction

The challenges that face those concerned with the numerical integration of the three-dimensional incompressible Euler and Navier-Stokes equations for a velocity field $\boldsymbol{u}(x, t)$ on a 3D periodic domain $V = [0, L]_{per}^3$

$$\frac{D\boldsymbol{u}}{Dt} = \nu\Delta\boldsymbol{u} - \nabla p \qquad \operatorname{div}\boldsymbol{u} = 0 \qquad (1)$$

($\nu = 0$ for the Euler equations) are also reflected in the challenges faced by analysts in their attempts to understand the regularity of these equations. The best known and most effective result in which analysis has guided numerics is the Beale-Kato-Majda (BKM) theorem [1, 2] , which says that solutions of the three-dimensional incompressible Euler equations are controlled from above by $\int_0^t \|\boldsymbol{\omega}\|_\infty \, d\tau$. If this is finite then no blow-up can occur at time t. Moreover, any numerical singularity in the vorticity field of the type $\|\boldsymbol{\omega}\|_\infty \sim (t_0 - t)^{-p}$ must have $p \geq 1$ for the singularity not to be a numerical artefact. The BKM criterion has become a standard feature in Euler computations : see the papers in the special volume [3].

This present paper is concerned with regularity criteria that form a consistent framework for the Euler and Navier-Stokes equations and which are testable numerically. In both cases it is often not clear when a numerically observed

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spike in the vorticity or strain fields remains finite or is a manifestation of a singularity. It is well known that monitoring the global enstrophy, or H_1-norm $\|\boldsymbol{\omega}\|_2$, pointwise in time determines Navier-Stokes regularity [4, 5], while the monitoring of $\|\boldsymbol{\omega}\|_\infty$ likewise determines the fate of Euler solutions. However, the range of L^p-norms between these may be useful. The basic objects are a set of frequencies based on L^{2m}-norms of the vorticity field $\boldsymbol{\omega} = \mathrm{curl}\,\boldsymbol{u}$

$$\Omega_m(t) = \left(L^{-3}\int_{\mathcal{V}}|\boldsymbol{\omega}|^{2m}dV\right)^{1/2m} \qquad 1 \leq m \leq \infty\,. \tag{2}$$

Hölder's inequality insists that $\Omega_m \leq \Omega_{m+1}$. The Navier-Stokes and Euler equations are invariant under the re-scaling $x' = \epsilon x$; $t' = \epsilon^2 t$; $\boldsymbol{u} = \epsilon\boldsymbol{u}'$; $p = \epsilon^2 p'$. If the domain length L is also re-scaled as $L' = \varepsilon L$ then Ω_m re-scales as $\Omega_m = \varepsilon^2\Omega_m'$ as expected. If, however, L is not re-scaled but kept fixed then Ω_m re-scales as

$$\Omega_m^{\alpha_m} = \epsilon\Omega_m'^{\,\alpha_m}\,, \tag{3}$$

where

$$\alpha_m = \frac{2m}{4m-3}\,. \tag{4}$$

It turns out that this strange scaling is particularly important and provides a motivation for the definition of the set of dimensionless quantities

$$D_m(t) = \left(\varpi_0^{-1}\Omega_m\right)^{\alpha_m}\,, \qquad 1 \leq m \leq \infty \tag{5}$$

where $\alpha_1 = 2$ and $\alpha_\infty = 1/2$. For the Navier-Stokes equations the frequency ϖ_0 is easily defined as $\varpi_0 = \nu L^{-2}$. The case of the Euler equations is more difficult as there is no obvious material constant to replace ν in the definition of ϖ_0. A circulation Γ has the same dimensions as that of ν but it must be taken around some chosen initial data: for instance, in [6], initial data was taken to be a pair of anti-parallel vortex tubes, in which case Γ could be chosen as the circulation around one of these. For a discussion of the variety of conclusions that can be drawn from numerical experiments see [6, 7, 8, 9, 10, 11].

No proof exists, as yet, of the existence and uniqueness of solutions of either the 3D Navier-Stokes or Euler equations for arbitrarily long times. A time-honoured approach has been to look for minimal assumptions that achieve this result [4, 5]. In fact, much of what is known about solutions of both the 3D Navier-Stokes and Euler equations is encapsulated in the sequence of time integrals

$$\int_0^t D_m^2\,d\tau \tag{6}$$

based on the continuum lying between

$$D_1 = \varpi_0^{-2}L^{-3}\int_{\mathcal{V}}|\boldsymbol{\omega}|^2\,dV\,\ldots \qquad \rightarrow \qquad \ldots\,D_\infty = \left(\varpi_0^{-1}\|\boldsymbol{\omega}\|_\infty\right)^{1/2}\,. \tag{7}$$

A well-known time-integral regularity condition for the Navier-Stokes equatons is that the first in the sequence in (6) should be finite [4, 5]: that is $\int_0^t D_1^2\,d\tau < \infty$. In contrast, the boundedness of the last in the sequence in (6) at $m = \infty$ is exactly the Beale-Kato-Majda criterion $\int_0^t \|\boldsymbol{\omega}\|_\infty\,d\tau < \infty$ for the regularity of solutions of the Euler equations [1, 2, 3].

For three-dimensional Navier-Stokes turbulence, it has to be admitted that arbitrarily imposed regularity assumptions, such as $\int_0^t D_m^2\,d\tau < \infty$, while mathematically interesting, have little foundation in physics: see the discussion of this point in [12]. However, what is known, without any assumptions, is that weak solutions (in the sense of Leray [13]) obey the time integral [14]

$$\int_0^t D_m\,d\tau \leq c\left(t\,Re^3 + \eta_1\right)\,, \tag{8}$$

where η_1 is a constant depending upon $D_m(0)$. This result plays two roles. In §3 it is shown that it leads to a definition of a continuum of inverse length scales $L\lambda_m^{-1}$ the upper bound on the first of which is the well-known Kolmogorov

scale proportional to $Re^{3/4}$. The more general upper bound is discussed in §3 and is given by $L\lambda_m^{-1} \le c\,Re^{3/2\alpha_m}$. Thus the λ_m for $m > 1$ correspond to deeper length scales associated with the higher L^{2m}-norms of vorticity implicit within the D_m.

In addition, the magnitude of the bounded time integral in (8) is also significant. Let us consider whether the saturation, or near saturation, of this time integral plays any role in the regularity question. In the forced case it has been shown in [12] that if a critical *lower* bound is imposed on this time integral in terms of the Grashof number Gr then this leads to exponential collapse in the $D_m(t)$. In fact boundedness from above of any one of the D_m also implies the boundedness of D_1 which immediately leads to the existence and uniqueness of solutions. While it can be argued that the imposition of this lower bound is physically artificial the result is nevertheless intriguing because it suggests that if the value of the integral (8) is sufficiently *large* then solutions are under control, which is surely counter-intuitive. Once it dips *below* this critical value then regularity could potentially break down. The importance of this mechanism lies in the role it may play in understanding the phenomenon of Navier-Stokes intermittency . This is discussed in §5 where the critical lower bound is expressed in terms of the more physical Reynolds number Re

$$\left(t\,Re^{3\delta_m} + \eta_2\right) \le \int_0^t D_m\,d\tau\,, \qquad 0 \le \delta_m \le 1\,. \tag{9}$$

The range of δ_m is estimated and it is shown that $\delta_m \searrow \tfrac{1}{2}$ for large m, thus allowing considerable slack between the critical lower bound in (9) and the upper bound in (8).

The D_m are comparatively easy quantities to calculate from a numerical scheme and is thus it is worth exploring whether regularity criteria can be gleaned from the relative magnitudes of the D_m or their time integrals. This is the task of this paper. In §2 the Euler equations are discussed in these terms where two versions of a numerical experiment are suggested for testing singular or non-singular behaviour.

2. The incompressible $3D$ Euler equations

Whether the three-dimensional Euler equations develop a singularity in a finite time still remains an open problem but a variety of super-weak solutions have recently been shown to exist [15, 16, 17, 18, 19, 20]. Let Γ be the circulation around some chosen initial data such that $\varpi_0 = \Gamma L^{-2}$, as discussed in §1. This defines ϖ_0 within the definition of D_m.

Proposition 1 *Provided solutions of the three-dimensional Euler equations exist, for $1 \le m < \infty$ the D_m formally satisfy the following differential inequality*

$$\dot{D}_m \le c_m\,\varpi_0 \left(\frac{D_{m+1}}{D_m}\right)^{\xi_m} D_m^3\,, \qquad \xi_m = \tfrac{1}{2}(4m+1)\,. \tag{10}$$

Proof : The time derivative of the Ω_m obeys

$$
\begin{aligned}
2mL^3\Omega_m^{2m-1}\dot{\Omega}_m &= \frac{d}{dt}\int |\boldsymbol{\omega}|^{2m}\,dV \\
&\le 2m\int_0^t \boldsymbol{\omega}^{2m}|\nabla\boldsymbol{u}|\,dV \\
&\le 2m\left(\int |\boldsymbol{\omega}|^{2m}\,dV\right)^{\frac{1}{2}}\left(\int |\boldsymbol{\omega}|^{2(m+1)}\,dV\right)^{\frac{m}{2(m+1)}}\left(\int |\nabla u|^{2(m+1)}\,dV\right)^{\frac{1}{2(m+1)}} \\
&\le 2mL^3 c_{1,m}\,\Omega_{m+1}^{m+1}\,\Omega_m^m
\end{aligned}
\tag{11}
$$

where we have used $\|\nabla u\|_p \le c_p\|\omega\|_p$, for $1 \le p < \infty$. Thus it transpires that

$$\dot{\Omega}_m \le c_{1,m}\left(\frac{\Omega_{m+1}}{\Omega_m}\right)^{m+1}\Omega_m^2\,, \qquad 1 \le m < \infty\,. \tag{12}$$

Note that the case $m = \infty$ is excluded : it was shown in [1] that a logarithmic $H_3 = \int_{\mathcal{V}} |\nabla^3\boldsymbol{u}|^2\,dV$ factor is needed such that $\|\nabla u\|_\infty \le c\,\|\omega\|_\infty\,(1 + \ln H_3)$.

We wish to convert inequality (12) to one in terms of D_m

$$\left(\frac{\Omega_{m+1}}{\Omega_m}\right)^{m+1}\Omega_m \;\; = \;\; \varpi_0\left(\frac{D_{m+1}}{D_m}\right)^{\frac{m+1}{\alpha_{m+1}}} D_m^{\left(\frac{m+1}{\alpha_{m+1}}-\frac{m+1}{\alpha_m}\right)+\frac{1}{\alpha_m}} = \varpi_0\left(\frac{D_{m+1}}{D_m}\right)^{\frac{1}{2}(4m+1)} D_m^2$$

having used the fact that

$$\left(\frac{1}{\alpha_{m+1}}-\frac{1}{\alpha_m}\right)\beta_m = 2\,, \qquad\qquad \beta_m = \tfrac{4}{3}m(m+1)\,. \tag{13}$$

Substitution into (12) completes the proof. ∎

There are now at least two interesting routes for the integration of (10).

1. Firstly if a finite-time singularity is suspected then divide (10) by $D_m^{3-\varepsilon}$ with $0 \le \varepsilon < 2$ to obtain

$$[D_m(t)]^{2-\varepsilon} \le \frac{1}{[D_m(t_0)]^{-(2-\varepsilon)} - F_{1,\varepsilon}(t)} \tag{14}$$

   where

$$F_{1,\varepsilon}(t) = c_m(2-\varepsilon)\varpi_0 \int_{t_0}^{t}\left(\frac{D_{m+1}}{D_m}\right)^{\xi_m} D_m^{\varepsilon}\,d\tau\,, \qquad\qquad \xi_m = \tfrac{1}{2}(4m+1)\,. \tag{15}$$

   For instance, for $\varepsilon = 0$ we have

$$D_m^2(t) \le \frac{1}{[D_m(t_0)]^{-2} - F_{1,0}(t)} \tag{16}$$

   where

$$F_{1,0}(t) = 2c_m\varpi_0 \int_{t_0}^{t}\left(\frac{D_{m+1}}{D_m}\right)^{\xi_m}\,d\tau\,. \tag{17}$$

   A singularity in the upper bound of inequality (16) is not necessarily significant. What is more significant is whether the solution tracks this singular upper bound. This suggests the following numerical test :

   (a) Is $F_{1,0}$ linear in t?
   (b) If so, then test whether
$$D_m^2\left(T_{c,m}-t\right) \to C_m \qquad\qquad \text{with} \qquad\qquad T_{c,m} \to T_c \tag{18}$$
   uniformly in m. If such behaviour occurs it suggests, but does not prove, that the D_m may be blowing up close to the upper bound.

2. If exponential or super-exponential growth is suspected then divide (10) only by D_m (the case $\varepsilon = 2$) and integrate

$$D_m \le D_m(t_0)\exp\varpi_0 \int_{t_0}^{t}\left(\frac{D_{m+1}}{D_m}\right)^{\xi_m} D_m^2\,d\tau\,. \tag{19}$$

   The rate of growth of the integral with respect to t is of interest. Does it remain finite for as long as the integrated solution remains reliable?

The companion paper is this volume by Kerr addresses some of these questions [21].

3. Weak solutions of Navier-Stokes and a range of scales

Weak solutions are natural for the global enstrophy $\|\boldsymbol{\omega}\|_2^2$ because of the properties of projection operators. The original argument used by Leray [13] gives us the textbook result from his energy inequality [4, 5]. In terms of D_1 this is

$$\langle D_1 \rangle_T \leq c\, Re^3 + O\left(T^{-1}\right) \tag{20}$$

where the time average up time T given by $\langle \cdot \rangle_T$ is defined by

$$\langle F(\cdot) \rangle_T = \frac{1}{T} \limsup_{F_0} \int_0^T F(\tau)\, d\tau \,. \tag{21}$$

To obtain similar results for $\|\boldsymbol{\omega}\|_{2m}$ for $m > 1$ looks difficult not only because the properties of projection operators do not naturally extend to the higher spaces but also because $\|\boldsymbol{\omega}\|_{2m}$ does not appear naturally in an energy inequality. However, these problems have been circumvented in [14], the main result from which will be stated below and its very short proof repeated for the benefit of the reader :

Theorem 1 *For $1 \leq m \leq \infty$, weak solutions obey*

$$\langle D_m \rangle_T \leq c\, Re^3 + O\left(T^{-1}\right) \,, \tag{22}$$

where c is a uniform constant.

Proof : The proof is based on a result of Foias, Guillopé & Temam (1981) [22] (their Theorem 3.1) for weak solutions. Doering and Foias [23] used the square of the averaged velocity $U_0^2 = L^{-3} \left\langle \|\boldsymbol{u}\|_2^2 \right\rangle_T$ to define the Reynolds number $Re = U_0 L \nu^{-1}$ which enables us to convert estimates in Gr to estimates in Re. Thus the result of Foias, Guillopé & Temam [22] in terms of Re becomes

$$\left\langle H_N^{\frac{1}{2N-1}} \right\rangle_T \leq c_N L^{-1} \nu^{\frac{2}{2N-1}} Re^3 + O\left(T^{-1}\right) \,, \tag{23}$$

where

$$H_N = \int_{\mathcal{V}} \left|\nabla^N \boldsymbol{u}\right|^2 dV = \int_{\mathcal{V}} \left|\nabla^{N-1} \boldsymbol{\omega}\right|^2 dV \,, \tag{24}$$

and where $H_1 = \int_{\mathcal{V}} |\nabla \boldsymbol{u}|^2 dV = \int_{\mathcal{V}} |\boldsymbol{\omega}|^2 dV$. Then an interpolation between $\|\boldsymbol{\omega}\|_{2m}$ and $\|\boldsymbol{\omega}\|_2$ is written as

$$\|\boldsymbol{\omega}\|_{2m} \leq c_{N,m} \|\nabla^{N-1}\boldsymbol{\omega}\|_2^a \|\boldsymbol{\omega}\|_2^{1-a} \,, \qquad a = \frac{3(m-1)}{2m(N-1)} \,, \tag{25}$$

for $N \geq 3$. $\|\boldsymbol{\omega}\|_{2m}$ is raised to the power A_m, which is to be determined.

$$\begin{aligned}
\left\langle \|\boldsymbol{\omega}\|_{2m}^{A_m} \right\rangle_T &\leq c_{N,m}^{A_m} \left\langle \|\nabla^{N-1}\boldsymbol{\omega}\|_2^{aA_m} \|\boldsymbol{\omega}\|_2^{(1-a)A_m} \right\rangle_T \\
&= c_{N,m}^{A_m} \left\langle \left(H_N^{\frac{1}{2N-1}}\right)^{\frac{1}{2}aA_m(2N-1)} H_1^{\frac{1}{2}(1-a)A_m} \right\rangle_T \\
&\leq c_{N,m}^{A_m} \left\langle H_N^{\frac{1}{2N-1}} \right\rangle_T^{\frac{1}{2}aA_m(2N-1)} \left\langle H_1^{\frac{(1-a)A_m}{2-aA_m(2N-1)}} \right\rangle_T^{1-\frac{1}{2}aA_m(2N-1)}
\end{aligned} \tag{26}$$

An explicit upper bound in terms of Re is available only if the exponent of H_1 within the average is unity ; that is

$$\frac{(1-a)A_m}{2-aA_m(2N-1)} = 1 \qquad \Rightarrow \qquad A_m = \frac{2m}{4m-3} = \alpha_m \tag{27}$$

as desired. Using the estimate in (23), and (20) for $\langle H_1 \rangle$, the result follows. $c_{N,m}$ can be minimized by choosing $N = 3$. $c_{3,m}$ does not blow up even when $m = \infty$; thus we take the largest value of $c_{3,m}^{\alpha_m}$ and call this c.   ∎

Following the statement of Theorem 1 and motivated by the definition of the Kolmogorov length for $m = 1$, a continuum of length scales λ_m can be defined thus:

$$\langle D_m \rangle_T := \left(L\lambda_m^{-1} \right)^{2\alpha_m} , \tag{28}$$

in which case (28) becomes

$$L\lambda_m^{-1} \leq c \, Re^{3/2\alpha_m} . \tag{29}$$

When $m = 1$, $\alpha_1 = 2$, and thus $L\lambda_1^{-1} \leq c \, Re^{3/4}$, which is consistent with Kolmogorov's statistical theory [22, 23]. However, the bounds on λ_m^{-1} become increasingly large with increasing m reflecting how the L^{2m}-norms can detect finer scale motions.

4. A regularity criterion based on the relative sizes of D_m and D_{m+1}

Consider two m-dependent constants $c_{1,m}$ and $c_{2,m}$ and two frequencies $\varpi_{1,m}$ and $\varpi_{2,m}$ defined by

$$\varpi_{1,m} = \varpi_0 \alpha_m c_{1,m}^{-1} \qquad\qquad \varpi_{2,m} = \varpi_0 \alpha_m c_{2,m} . \tag{30}$$

In [12], using a standard contradiction strategy on a finite interval of existence and uniqueness $[0, T^*)$, it was shown that for the decaying Navier-Stokes equations the D_m obey the following theorem in which the dot represents differentiation with respect to time:

Theorem 2 *For $1 \leq m < \infty$ on $[0, T^*)$ the $D_m(t)$ satisfy the set of inequalities*

$$\dot{D}_m \leq D_m^3 \left\{ -\varpi_{1,m} \left(\frac{D_{m+1}}{D_m} \right)^{\rho_m} + \varpi_{2,m} \right\} , \tag{31}$$

where $\rho_m = \frac{2}{3}m(4m + 1)$. In the forced case there is an additive term $\varpi_{3,m} Re^2 D_m$.

The obvious conclusion is that solutions come under control pointwise in t provided

$$D_{m+1}(t) \geq c_{\rho_m} D_m(t) \tag{32}$$

where

$$c_{\rho_m} = [c_{1,m} c_{2,m}]^{1/\rho_m} \tag{33}$$

This is a numerically testable criterion: if (32) holds then the D_m must decay in time. It was shown in [12] that this can be weakened to a time integral result:

Theorem 3 *For any value of $1 \leq m < \infty$, if the integral condition is satisfied*

$$\int_0^t \ln \left(\frac{1 + Z_m}{c_{4,m}} \right) d\tau \geq 0 , \qquad\qquad Z_m = D_{m+1}/D_m \tag{34}$$

with $c_{4,m} = \left[2^{\rho_m - 1} \left(1 + c_{1,m} c_{2,m} \right) \right]^{\rho_m^{-1}}$, then $D_m(t) \leq D_m(0)$ on the interval $[0, t]$.

Given the nature of $c_{4,m} \searrow 2$ it is clear that there must be enough regions of the time axis where $D_{m+1} > (c_{4,m} - 1)D_m$ to make the integral positive. The D_m are easily computable from Navier-Stokes data. Therefore, an interesting numerical experiment would be to test:

1. Whether the D_m are ordered as time evolves such that $D_m \geq D_{m+1}$ or $D_m \leq D_{m+1}$?
2. Do the D_m cross over from one regime to the other?
3. How significant are the initial conditions and the Reynolds number in this behaviour?

5. Body-forced Navier-Stokes equations

5.1. A critical lower bound on $\int_0^t D_m \, d\tau$ in terms of Re

In [12, 26] the body-forced Navier-Stokes equations were considered in terms of the Grashof number Gr. It is more useful to to consider this in terms of the Reynolds number Re. The inclusion of the forcing in (31) modifies[1] this but requires the introduction of a third frequency $\varpi_{3,m} = \varpi_0 \alpha_m c_{3,m}$

$$\dot{D}_m \leq D_m^3 \left\{ -\varpi_{1,m} \left(\frac{D_{m+1}}{D_m} \right)^{\rho_m} + \varpi_{2,m} \right\} + \varpi_{3,m} Re^2 D_m \,, \tag{35}$$

where $\rho_m = \frac{2}{3} m(4m+1)$ and $\gamma_m = \frac{1}{2} \alpha_{m+1} \left(m^2 - 1 \right)^{-1}$. Let Δ_m be defined by ($2 \leq \Delta_m \leq 6$)

$$\Delta_m = 3 \left\{ \delta_m (2 + \rho_m \gamma_m) - \rho_m \gamma_m \right\} \tag{36}$$

The following result shows that if a critical lower bound is set on $\int_0^t D_m \, d\tau$ then D_m will decay exponentially. Note that the case $m = 1$ is excluded :

Theorem 4 *If there exists a value of m lying in the range $1 < m < \infty$, with initial data $[D_m(0)]^2 < C_m Re^{\Delta_m}$, for which the integral lies on or above the critical value*

$$c_m \left(t \, Re^{3\delta_m} + \eta_2 \right) \leq \int_0^t D_m \, d\tau \tag{37}$$

and δ_m and η_2 lie in the ranges

$$\frac{2/3 + \rho_m \gamma_m}{2 + \rho_m \gamma_m} < \delta_m < 1 \,, \qquad and \qquad \eta_2 \geq \eta_1 Re^{3(\delta_m - 1)} \,, \tag{38}$$

then $D_m(t)$ decays exponentially on $[0, t]$.

Remark: $\delta_m \searrow 1/2$ for large m so enough slack lies between the upper and lower bounds on $\int_0^t D_m \, d\tau$.
Proof : To proceed, divide by D_m^3 to write (35) as

$$\frac{1}{2} \frac{d}{dt} \left(D_m^{-2} \right) \geq X_m \left(D_m^{-2} \right) - \varpi_{2,m} \qquad X_m = \varpi_{1,m} \left(\frac{D_{m+1}}{D_m} \right)^{\rho_m} D_m^2 - \varpi_{3,m} Re^2 \,. \tag{39}$$

A lower bound for $\int_0^t X_m d\tau$ can be estimated thus :

$$\int_0^t D_{m+1} \, d\tau = \int_0^t \left[\left(\frac{D_{m+1}}{D_m} \right)^{\rho_m} D_m^2 \right]^{\frac{1}{\rho_m}} D_m^{\frac{\rho_m - 2}{\rho_m}} \, d\tau$$

$$\leq \left(\int_0^t \left(\frac{D_{m+1}}{D_m} \right)^{\rho_m} D_m^2 \, d\tau \right)^{\frac{1}{\rho_m}} \left(\int_0^t D_m \, d\tau \right)^{\frac{\rho_m - 2}{\rho_m}} t^{1/\rho_m} \tag{40}$$

and so

$$\int_0^t X_m d\tau \geq \varpi_{1,m} t^{-1} \frac{\left(\int_0^t D_{m+1} \, d\tau \right)^{\rho_m}}{\left(\int_0^t D_m \, d\tau \right)^{\rho_m - 2}} - \varpi_{3,m} t \, Re^2 \,. \tag{41}$$

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[1] For the forced case the definition of Ω_m requires an additive ϖ_0 term to act as a lower bound [12, 26].

Recall that $\rho_m = \frac{2}{3}m(4m+1)$ and $\gamma_m = \frac{1}{2}\alpha_{m+1}\left(m^2-1\right)^{-1}$. It is not difficult to prove that $\Omega_m^{m^2} \leq \Omega_{m+1}^{m^2-1}\Omega_1$ for $m > 1$, from which, after manipulation into the D_m becomes

$$D_m \leq D_{m+1}^{\alpha_m/2\gamma_m m^2} D_1^{\alpha_m/2m^2} \qquad\qquad \frac{\alpha_m}{2m^2}\left(1+\gamma_m^{-1}\right) = 1 \tag{42}$$

and therefore a Hölder inequality gives

$$\frac{\int_0^t D_{m+1}\,d\tau}{\int_0^t D_m\,d\tau} \geq \left(\frac{\int_0^t D_m\,d\tau}{\int_0^t D_1\,d\tau}\right)^{\gamma_m}. \tag{43}$$

$$\int_0^t X_m d\tau \;\geq\; \varpi_{1,m}t^{-1}\frac{\left(\int_0^t D_m\,d\tau\right)^{\rho_m\gamma_m+2}}{\left(\int_0^t D_1\,d\tau\right)^{\rho_m\gamma_m}} - \varpi_{3,m}t\,Re^2. \tag{44}$$

Inequality (39) integrates to

$$[D_m(t)]^2 \leq \frac{\exp\left\{-2\int_0^t X_m\,d\tau\right\}}{[D_m(0)]^{-2} - 2\varpi_{2,m}\int_0^t \exp\left\{-2\int_0^\tau X_m\,d\tau'\right\}d\tau}. \tag{45}$$

(41) can be re-written as

$$
\begin{aligned}
\int_0^t X_m d\tau &\;\geq\; \varpi_{1,m}t^{-1}\frac{\left(\int_0^t D_m\,d\tau\right)^{\rho_m\gamma_m+2}}{\left(\int_0^t D_1\,d\tau\right)^{\rho_m\gamma_m}} - \varpi_{3,m}t\,Re^2 \\
&\;\geq\; c_m t\left(\varpi_{1,m}Re^{\Delta_m} - \varpi_{3,m}Re^2\right)
\end{aligned} \tag{46}
$$

having used the assumed lower bound in the theorem and the upper bound of $\int_0^t D_1\,d\tau$. Moreover, to have the dissipation greater than forcing requires $\Delta_m > 2$ so δ_m must lie in the range as in (38) because $2 < \Delta_m \leq 6$. For large Re the negative Re^2-term in (45) is dropped so the integral in the denominator of (45) is estimated as

$$\int_0^t \exp\left(-2\int_0^\tau X_m d\tau'\right)d\tau \leq [2\tilde{c}_m\varpi_{1,m}]^{-1}Re^{-\Delta_m}\left(1 - \exp\left[-2\varpi_{1,m}\tilde{c}_m t\,Re^{\Delta_m}\right]\right), \tag{47}$$

and so the denominator of (45) satisfies

$$\text{Denominator} \geq [D_m(0)]^{-2} - c_{2,m}c_{1,m}\left(2\tilde{c}_m\right)^{-1}Re^{-\Delta_m}\left(1 - \exp\left[-2\varpi_{1,m}\tilde{c}_m t\,Re^{\Delta_m}\right]\right). \tag{48}$$

This can never go negative if $[D_m(0)]^{-2} > c_{1,m}c_{2,m}\left(2\tilde{c}_m\right)^{-1}Re^{-\Delta_m}$, which means $D_m(0) < C_m Re^{\frac{1}{2}\Delta_m}$. ∎
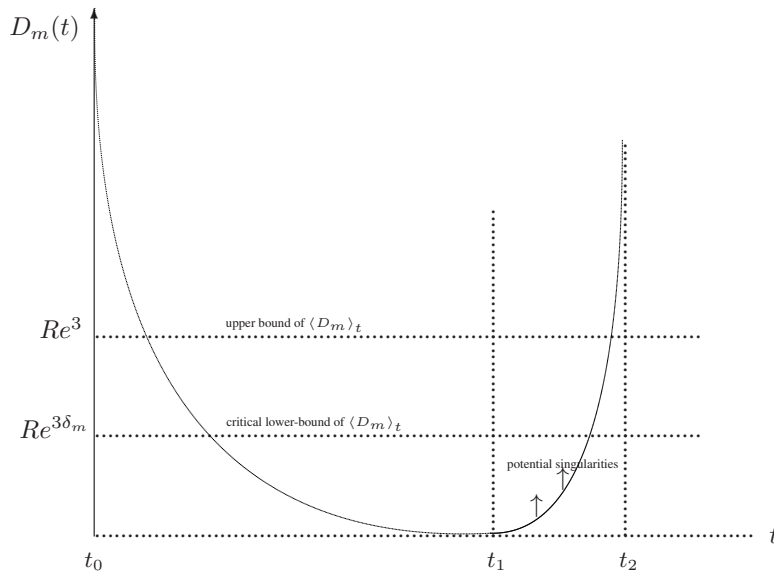
.

Figure 1 : A cartoon of $D_m(t)$ versus t illustrating the phases of intermittency. The range of δ_m lies in Theorem 4. The vertical arrows depict the region where there is the potential for needle-like singular behaviour & thus a break-down of regularity.

5.2. A relaxation oscillator mechanism for intermittency

Experimentally, signals go through cycles of growth and collapse [27, 28, 29, 30] so it is not realistic to expect that the critical lower bound imposed in Theorem 4 should hold for all time. Using the average notation $\langle \cdot \rangle_t$, inequality (45) shows that if $\langle D_m \rangle_t$ lies above critical then $D_m(t)$ collapses exponentially. In Figure 1 the horizontal line at $Re^{3\delta_m}$ is drawn as the critical lower bound on $\langle D_m \rangle_t$.

Above this critical range, $D_m(t)$ will decay exponentially fast. However, because integrals take account of history, there will be a delay before $\langle D_m \rangle_t$ decreases below the value above which a zero in the denominator of (45) can be prevented (at t_1): at this point all constraints are removed and $D_m(t)$ is free to grow rapidly again in $t_1 \leq t \leq t_2$. If the integral drops below critical then it is in this interval that the occurrence of singular events (depicted by vertical arrows) must still formally be considered – if one occurs the solutions fails. Provided a solution still exists, growth in D_m will be such that, after another delay, it will force $\langle D_m \rangle_t$ above critical and the system, with a re-set of initial conditions at t_2, is free to move through another cycle. Thus it behaves like a relaxation oscillator. The vertical arrows in Figure 1 label the region, below critical, where

$$\int_0^t D_m \, d\tau < c_m \left(t \, Re^{3\delta_m} + \eta_2 \right) . \tag{49}$$

It is in this regime where where potentially singular point-wise growth of $D_m(t)$ could occur which contributes little to the growth of the integral $\int_0^t D_m \, d\tau$ and which does not drive it past critical. No control mechanism for this type of growth is known and so the regularity problem remains open.

6. Conclusion

The variables D_m, as defined in (5), have proved useful in expressing the Navier-Stokes and Euler regularity problems in a natural manner. Their use also poses some interesting questions. For instance, while the Ω_m must be ordered because of Hölder's inequality, this is not the case with the D_m because the α_m *decrease* with m. Theorem 2 suggests that the regime $D_{m+1}/D_m \geq c_m$, where c_m is a constant only just above unity, guarantees the decay of D_m and hence control over Navier-Stokes solutions. In terms of numerical experiments, it would be interesting to see, from a variety of initial conditions, which of the two regimes

$$D_{m+1} \geq D_m \qquad \text{or} \qquad D_{m+1} \leq D_m \tag{50}$$

are predominant and whether there is a cross-over from one to the other. If so, does this depend heavily on the initial conditions, such as the contrasting random or anti-parallel vortex initial conditions? Does it also depend on the size of Re? Likewise, do solutions of the Euler equations, when in their intermediate and late growth phases, track a singular upper bound as in (18)?

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