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Rényi divergence and L_p-affine surface area for convex bodies

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Abstract

We show that the fundamental objects of the L_p-Brunn–Minkowski theory, namely the L_p-affine surface areas for a convex body, are closely related to information theory: they are exponentials of Rényi divergences of the cone measures of a convex body and its polar.

We give geometric interpretations for all Rényi divergences D_α, not just for the previously treated special case of relative entropy which is the case $\alpha = 1$. Now, no symmetry assumptions are needed and, if at all, only very weak regularity assumptions are required.

Previously, the relative entropies appeared only after performing second order expansions of certain expressions. Now already first order expansions make them appear. Thus, in the new approach we detect "faster" details about the boundary of a convex body.

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1. Introduction

There exists a fascinating connection between convex geometric analysis and information theory. An example is the close parallel between geometric inequalities for convex bodies and

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inequalities for probability densities. For instance, the Brunn–Minkowski inequality and the entropy power inequality follow both in a very similar way from the sharp Young inequality (see. e.g., [2]).

In several recent papers, Lutwak et al. [24,26,27,29] established further connections between convexity and information theory. For example, they showed in [26] that the Cramer–Rao inequality corresponds to an inclusion of the Legendre ellipsoid and the polar L_2-projection body. The latter is a basic notion from the L_p-Brunn–Minkowski theory. This L_p-Brunn–Minkowski theory has its origins in the 1960s when Firey introduced his L_p-addition of convex bodies. It evolved rapidly over the past years and due to a number of highly influential works (see, e.g., [4,6–8,10,9,12,13,16–20,22,23,21,25,28,30–33,36–42,44–46,50]), is now a central part of modern convex geometry. In fact, this theory redirected much of the research about convex bodies from the Euclidean aspects to the study of the affine geometry of these bodies, and some questions that had been considered Euclidean in nature turned out to be affine problems. For example, the famous Busemann–Petty Problem (finally laid to rest in [3,5,48,49]), was shown to be an affine problem with the introduction of intersection bodies by Lutwak in [21].

Two fundamental notions within the L_p-Brunn–Minkowski theory are L_p-affine surface areas, introduced by Lutwak in the ground breaking paper [23] and L_p-centroid bodies introduced by Lutwak and Zhang in [30]. See Section 3 for the definition of those quantities.

Based on these quantities, Paouris and Werner [34] established yet another relation between affine convex geometry and information theory. They proved that the exponential of the relative entropy of the cone measure of a symmetric convex body and its polar equals a limit of normalized L_p-affine surface areas. Moreover, also in [34], Paouris and Werner gave geometric interpretations of the relative entropy of the cone measures of a *sufficiently smooth, symmetric* convex body and its polar.

In this paper we show that the very core of the L_p-Brunn–Minkowski theory, namely the L_p-affine surface areas themselves are concepts of information theory: they are exponentials of Rényi divergences of the cone measures of a convex body and its polar. This identification allows to translate known properties from one theory to the other.

Even more is gained. Geometric interpretations for *all* Rényi divergences D_α of cone measures of a convex body and its polar are given for *all* α, not just for the special case of relative entropy which corresponds to the case $\alpha = 1$. We refer to Sections 2 and 3 for the definition of D_α. No symmetry assumptions on K are needed. Nor do these new geometric interpretations require the strong smoothness assumptions of [34].

In the context of the L_p-centroid bodies, the relative entropies appeared only after performing *second order expansions* of certain expressions. The remarkable fact now is that in our approach here, already *first order expansions* make them appear. Thus, these bodies detect “faster” details of the boundary of a convex body than the L_p-centroid bodies.

The paper is organized as follows. In Section 2 we introduce Rényi divergences for convex bodies and describe some of their properties. We also introduce L_p-affine surface areas and mixed p-affine surface areas.

The main observations are Theorems 2.4 and 2.5 which show that L_p-affine surface areas and mixed p-affine surface areas are exponentials of Rényi divergences. These identifications allow to translate known properties from one theory to the other—this is done in the rest of Section 2 and in Section 3. Also, in Section 3, we give geometric interpretations for Rényi divergences D_α of cone measures of convex bodies for all α, including new ones for the relative entropy not requiring the (previously necessary) strong smoothness and symmetry assumptions on the body.

Further notation.

Throughout the paper, we will assume that the centroid of a convex body K in $\mathbb{R}^n$ is at the origin. We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm. $B_2^n(x,r)$ is the ball centered at x with radius r. We write $B_2^n = B_2^n(0,1)$ for the Euclidean unit ball centered at 0 and S^{n-1} for the unit sphere. The volume is denoted by $|\cdot|$ or, if we want to emphasize the dimension, by $\mathrm{vol}_d(A)$ for a d-dimensional set A. $K^\circ = \{y \in \mathbb{R}^n : \langle x, y\rangle \le 1 \text{ for all } x \in K\}$ is the polar body of K.

For a point $x \in \partial K$, the boundary of K, $N_K(x)$ is the outer unit normal in x to K and $\kappa_K(x)$ is the (generalized) Gauss curvature in x. We write $K \in C^2_+$, if K has C^2 boundary ∂K with everywhere strictly positive Gaussian curvature κ_K. μ_K is the usual surface area measure on ∂K. σ is the usual surface area measure on S^{n-1}.

Let K be a convex body in $\mathbb{R}^n$ and let $u \in S^{n-1}$. Then $h_K(u)$ is the support function of K in direction $u \in S^{n-1}$, and $f_K(u)$ is the curvature function, i.e. the reciprocal of the Gaussian curvature $\kappa_K(x)$ at the point $x \in \partial K$ that has u as an outer normal.

2. Rényi divergences for convex bodies

Let (X,μ) be a measure space and let $dP = p\,d\mu$ and $dQ = q\,d\mu$ be probability measures on X that are absolutely continuous with respect to the measure μ. Then the Rényi divergence of order α, introduced by Rényi [35] for $\alpha > 0$ and $\alpha \neq 1$, is defined as

$$D_\alpha(P \parallel Q) = \frac{1}{\alpha-1}\log\int_X p^\alpha q^{1-\alpha}\,d\mu. \tag{1}$$

It is the convention to put $p^\alpha q^{1-\alpha} = 0$, if $p = 0$ or $q = 0$, even if $\alpha < 0$ and $\alpha > 1$. The integrals

$$\int_X p^\alpha q^{1-\alpha}\,d\mu \tag{2}$$

are also called *Hellinger integrals*. See e.g. [15] for those integrals and additional information.

Usually, in the literature, $\alpha \ge 0$. However, we will also consider $\alpha < 0$, provided the expressions exist. Also, usually in the literature, the measures are probability measures. Therefore we normalize the measures.

Special cases.

(i) The case $\alpha = 1$ is also called the *Kullback–Leibler divergence* or *relative entropy* from P to Q (see [1]). It is obtained as the limit as $\alpha \uparrow 1$ in (1) and one gets

$$D_{KL}(P\|Q) = D_1(P\|Q) = \lim_{\alpha\uparrow 1} D_\alpha(P\|Q) = \int_X p\log\frac{p}{q}\,d\mu. \tag{3}$$

(The limit $\alpha \to 1$ may not exist but limit $\alpha \uparrow 1$ exists [14].)

(ii) The case $\alpha = 0$ gives for $q \neq 0$ (with the convention that $0^0 = 1$) that

$$D_0(P\|Q) = 0, \tag{4}$$

as $dQ = q\,d\mu$ is a probability measure on X. If $q = 0$, then $D_0(P\|Q) = -\infty$. Note however, that the case $q = 0$ is an exceptional case as it does not give rise to a probability measure.

(iii) The case $\alpha = \frac{1}{2}$ gives

$$D_{\frac{1}{2}}(P\|Q) = D_{\frac{1}{2}}(Q\|P) = -2\log\int_X p^{\frac{1}{2}}q^{\frac{1}{2}}\,d\mu. \tag{5}$$

The expression $\int_X p^{\frac{1}{2}}q^{\frac{1}{2}}\,d\mu$ is also called the *Bhattacharyya coefficient* or *Bhattacharyya distance* of p and q.
(iv) The cases $\alpha = \infty$ and $\alpha = -\infty$.

$$D_\infty(P\|Q) = \log\left(\sup_x \operatorname{ess}\frac{p(x)}{q(x)}\right), \tag{6}$$

and

$$D_{-\infty}(P\|Q) = -\left(\sup_x \operatorname{ess}\frac{q(x)}{p(x)}\right) = -D_\infty(Q\|P). \tag{7}$$

Note that for all $-\infty \le \alpha \le \infty,\ \alpha \ne 1$,

$$D_\alpha(Q\|P) = \frac{\alpha}{1-\alpha}D_{1-\alpha}(P\|Q). \tag{8}$$

As $\alpha \uparrow 1$, the limit on the left and the limit on the right of (8) exist and are both equal to $D_1(Q\|P) = \int_X q\log\frac{q}{p}\,d\mu$. Thus (8) holds for all $-\infty \le \alpha \le \infty$.
We will now consider Rényi divergence for convex bodies K in $\mathbb{R}^n$. Let

$$p_K(x) = \frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^n\, n|K^\circ|}, \qquad q_K(x) = \frac{\langle x, N_K(x)\rangle}{n\,|K|}. \tag{9}$$

Then

$$P_K = p_K\,\mu_K \quad \text{and} \quad Q_K = q_K\,\mu_K \tag{10}$$

are probability measures on ∂K that are absolutely continuous with respect to μ_K.

Recall that the normalized cone measure cm_K on ∂K is defined as follows: for every measurable set $A \subseteq \partial K$

$$cm_K(A) = \frac{1}{|K|}\left|\left\{ta:\ a\in A,\ t\in[0,1]\right\}\right|. \tag{11}$$

The next proposition is well known. See e.g. [34] for a proof. It shows that the measures P_K and Q_K defined in (10) are the cone measures of K and K°. $N_K : \partial K \to S^{n-1}$, $x \to N_K(x)$ is the Gauss map.

Proposition 2.1. *Let K be a convex body in $\mathbb{R}^n$. Let P_K and Q_K be the probability measures on ∂K defined by (10). Then*

$$Q_K = cm_K,$$

or, equivalently, for every measurable subset A in ∂K $Q_K(A) = cm_K(A)$.

If K is in addition in C^2_+, then

$$P_K = N_K^{-1}N_{K^\circ}cm_{K^\circ}$$

or, equivalently, for every measurable subset A in ∂K

$$P_K(A) = c m_{K^\circ}\Big(N_{K^\circ}^{-1}\big(N_K(A)\big)\Big).$$

For $\alpha = 1$, the relative entropy of a convex body K in $\mathbb{R}^n$ was considered in [34], namely

$$\begin{aligned} D_1(P_K\|Q_K) &= D_{KL}(P_K\|Q_K) \\ &= \int_{\partial K} \frac{\kappa_K(x)}{n|K^\circ|\langle x, N_K(x)\rangle^n} \log\left(\frac{|K|\kappa_K(x)}{|K^\circ|\langle x, N_K(x)\rangle^{n+1}}\right) d\mu_K(x) \\ D_1(Q_K\|P_K) &= D_{KL}(Q_K\|P_K) \\ &= \int_{\partial K} \frac{\langle x, N_K(x)\rangle}{n|K|} \log\left(\frac{|K^\circ|\langle x, N_K(x)\rangle^{n+1}}{|K|\kappa_K(x)}\right) d\mu_K(x), \end{aligned}$$

provided the expressions exist.

We now define the Rényi divergence of K of order α for all other α, $-\infty \le \alpha \le \infty,\ \alpha \neq 1$.

Definition 2.2. Let K be a convex body in $\mathbb{R}^n$ and let $-\infty < \alpha < \infty,\ \alpha \neq 1$. Then the Rényi divergences of order α of K are

$$D_\alpha(Q_K\|P_K) = \frac{1}{\alpha-1}\log\left(\frac{\int_{\partial K}\frac{\kappa_K^{1-\alpha}d\mu_K}{\langle x, N_K(x)\rangle^{n-\alpha(n+1)}}}{n|K|^\alpha|K^\circ|^{1-\alpha}}\right) \tag{12}$$

$$D_\alpha(P_K\|Q_K) = \frac{1}{\alpha-1}\log\left(\frac{\int_{\partial K}\frac{\kappa_K^{\alpha}d\mu_K}{\langle x, N_K(x)\rangle^{\alpha(n+1)-1}}}{n|K|^{1-\alpha}|K^\circ|^{\alpha}}\right) \tag{13}$$

$$D_\infty(Q_K\|P_K) = \log\left(\sup_{x\in\partial K} \text{ess } \frac{|K^\circ|\ \langle x, N_K(x)\rangle^{n+1}}{|K|\ \kappa_K(x)}\right) \tag{14}$$

$$D_\infty(P_K\|Q_K) = \log\left(\sup_{x\in\partial K} ess\ \frac{|K|\ \kappa_K(x)}{|K^\circ|\ \langle x, N_K(x)\rangle^{n+1}}\right) \tag{15}$$

and

$$D_{-\infty}(Q_K\|P_K) = -D_\infty(P_K\|Q_K), \qquad D_{-\infty}(P_K\|Q_K) = -D_\infty(Q_K\|P_K), \tag{16}$$

provided the expressions exist.

Remarks.

(i) By (8) for all $-\infty \le \alpha \le \infty,\ \alpha \neq 1$,

$$D_\alpha(Q_K\|P_K) = \frac{\alpha}{1-\alpha} D_{1-\alpha}(P_K\|Q_K).$$

This identity also holds for $\alpha \uparrow 1$. Therefore, it is enough to consider only one of the two, $D_\alpha(Q_K \parallel P_K)$ or $D_\alpha(P_K\|Q_K)$.

(ii) If we put $N_K(x) = u \in S^{n-1}$, then $\langle x, N_K(x)\rangle = h_K(u)$. If K is in C_+^2, then, with the inverse N_K^{-1} of the Gauss map, $d\mu_K(N_K^{-1}(u)) = f_K(u)d\sigma(u)$. Hence, in that case, we can express the

Rényi divergences also as

$$D_\alpha(Q_K \| P_K) = \frac{1}{\alpha-1} \log \left(\frac{\int_{S^{n-1}} \frac{f_K(u)^\alpha \, d\sigma(u)}{h_K(u)^{n-\alpha(n+1)}}}{n|K|^\alpha |K^\circ|^{1-\alpha}} \right) \tag{17}$$

$$D_\alpha(P_K \| Q_K) = \frac{1}{\alpha-1} \log \left(\frac{\int_{S^{n-1}} \frac{f_K(u)^{1-\alpha} \, d\sigma(u)}{h_K(u)^{\alpha(n+1)-1}}}{n|K|^{1-\alpha} |K^\circ|^\alpha} \right), \tag{18}$$

accordingly for $D_{KL}(Q_K \| P_K)$ and $D_{KL}(P_K \parallel Q_K)$.

Let $K_1, \dots, K_n$ be convex bodies in $\mathbb{R}^n$. Let $u \in S^{n-1}$. For $1 \le i \le n$, define

$$p_{K_i}(u) = \frac{1}{n^{\frac{1}{n}} |K_i^\circ|^{\frac{1}{n}} h_{K_i}(u)}, \qquad q_{K_i}(u) = \frac{f_{K_i}(u)^{\frac{1}{n}} h_{K_i}(u)^{\frac{1}{n}}}{n^{\frac{1}{n}} \, |K_i|^{\frac{1}{n}}} \tag{19}$$

and measures on S^{n-1} by

$$P_{K_i} = p_{K_i} \, \sigma \quad \text{and} \quad Q_{K_i} = q_{K_i} \, \sigma. \tag{20}$$

Then we define the Rényi divergences of order α for convex bodies $K_1, \dots, K_n$ by the following.

Definition 2.3. Let $K_1, \dots, K_n$ be convex bodies in $\mathbb{R}^n$. Then for $-\infty < \alpha < \infty, \ \alpha \neq 1$

$$D_\alpha(Q_{K_1} \times \cdots \times Q_{K_n} \| P_{K_1} \times \cdots \times P_{K_n}) = \frac{\log \left(\int_{S^{n-1}} \prod_{i=1}^n \frac{f_{K_i}^{\frac{\alpha}{n}} h_{K_i}^{\frac{\alpha}{n}-(1-\alpha)}}{n^{\frac{1}{n}} \ |K_i|^{\frac{\alpha}{n}} |K_i^\circ|^{\frac{1-\alpha}{n}}} d\sigma \right)}{\alpha - 1}$$

$$D_\alpha(P_{K_1} \times \cdots \times P_{K_n} \| Q_{K_1} \times \cdots \times Q_{K_n}) = \frac{\log \left(\int_{S^{n-1}} \prod_{i=1}^n \frac{f_{K_i}^{\frac{1-\alpha}{n}} h_{K_i}^{\frac{1-\alpha}{n}-\alpha}}{n^{\frac{1}{n}} \ |K_i|^{\frac{1-\alpha}{n}} \ |K_i^\circ|^{\frac{\alpha}{n}}} d\sigma \right)}{\alpha - 1}$$

provided the expressions exist.

For $\alpha = 1$ the definitions were given in [34]:

$$D_1(Q_{K_1} \times \cdots \times Q_{K_n} \| P_{K_1} \times \cdots \times P_{K_n}) \\ = \int_{S^{n-1}} \prod_{i=1}^n \frac{f_{K_i}^{\frac{1}{n}} h_{K_i}^{\frac{1}{n}}}{n^{\frac{1}{n}} \ |K_i|^{\frac{1}{n}}} \log \left(\prod_{i=1}^n \frac{|K_i^\circ|^{\frac{1}{n}} \ f_{K_i}^{\frac{1}{n}} h_{K_i}^{1+\frac{1}{n}}}{|K_i|^{\frac{1}{n}}} \right) d\sigma$$

$$D_1(P_{K_1} \times \cdots \times P_{K_n} \| Q_{K_1} \times \cdots \times Q_{K_n}) \\ = \int_{S^{n-1}} \prod_{i=1}^n \frac{h_{K_i}^{-1}}{n^{\frac{1}{n}} \ |K_i^\circ|^{\frac{1}{n}}} \log \left(\prod_{i=1}^n \frac{|K_i|^{\frac{1}{n}}}{|K_i^\circ|^{\frac{1}{n}} f_{K_i}^{\frac{1}{n}} h_{K_i}^{1+\frac{1}{n}}} \right) d\sigma,$$

provided the expressions exist.

Remark. For $-\infty < \alpha < \infty, \ \alpha \neq 1$,

$$D_\alpha(P_{K_1} \times \cdots \times P_{K_n} \| Q_{K_1} \times \cdots \times Q_{K_n}) \\ = \frac{\alpha}{1-\alpha} D_{1-\alpha}(Q_{K_1} \times \cdots \times Q_{K_n} \| P_{K_1} \times \cdots \times P_{K_n}), \tag{21}$$

and, again, for $\alpha \uparrow 1$, the limits on both sides exist and coincide. Therefore it is enough to consider either $D_\alpha(P_{K_1} \times \cdots \times P_{K_n} \| Q_{K_1} \times \cdots \times Q_{K_n})$ or $D_\alpha(Q_{K_1} \times \cdots \times Q_{K_n} \| P_{K_1} \times \cdots \times P_{K_n})$.

We first present some examples and look at special cases below. In particular, $D_{\pm\infty}(Q_{K_1} \times \cdots \times Q_{K_n} \| P_{K_1} \times \cdots \times P_{K_n})$ will be considered below.

Examples.
(i) If $K = \rho B_2^n$, then $D_\alpha(Q_K \parallel P_K) = D_\alpha(P_K \| Q_K) = 0$ for all $-\infty \le \alpha \le \infty$.
(ii) If K is a polytope, then $\kappa_K = 0$ a.e. on ∂K. Thus, for $\alpha = 1$, $D_1(Q_K \| P_K) = \infty$. For $-\infty < \alpha < 1$, $\int_{\partial K} \frac{\kappa_K^{1-\alpha} d\mu_K}{\langle x, N_K(x)\rangle^{n-\alpha(n+1)}} = 0$ and for $\alpha > 1$, $\int_{\partial K} \frac{\kappa_K^{1-\alpha} d\mu_K}{\langle x, N_K(x)\rangle^{n-\alpha(n+1)}} = \infty$. Hence $D_\alpha(Q_K \| P_K) = \infty$ for all $-\infty < \alpha < \infty$, and K a polytope.

Similarly, $D_1(P_K \| Q_K) = 0$ (with the convention that $0 \cdot \infty = 0$).

$D_\alpha(P_K \| Q_K) = -\infty$, for $1 < \alpha < \infty$ and $-\infty < \alpha < 0$ and K a polytope and $D_\alpha(P_K \| Q_K) = \infty$, for $0 < \alpha < 1$ and K a polytope.

This also shows that D_α need not be continuous at $\alpha = 1$.

For $\alpha = 0$ and $\alpha = \pm\infty$, see below.
(iii) For $1 < r < \infty$, let $K = B_r^n = \{x \in \mathbb{R}^n : \sum_{i=1}^n |x_i|^r \le 1\}$ be the unit ball of l_r^n. We will compute $D_\alpha(Q_K \| P_K)$ and $D_\alpha(P_K \| Q_K)$ for all $-\infty < \alpha < \infty$, $\alpha \ne 1$. The case $\alpha = 1$ was considered in [34]. The cases $\alpha = 0$ and $\alpha = \pm\infty$ are treated below.

If $1 < r < 2$ and $\alpha \ge \frac{1}{2-r}$, then $D_\alpha(P_{B_r^n} \| Q_{B_r^n}) = \infty$. If $1 < r < 2$ and $\alpha \le -\frac{r-1}{2-r}$, then $D_\alpha(Q_{B_r^n} \parallel P_{B_r^n}) = -\infty$. If $2 < r < \infty$ and $\alpha \le \frac{-1}{r-2}$, then $D_\alpha(P_{B_r^n} \| Q_{B_r^n}) = -\infty$. If $2 < r < \infty$ and $\alpha \ge \frac{r-1}{r-2}$, then $D_\alpha(Q_{B_r^n} \| P_{B_r^n}) = \infty$. In all other cases we have

$$D_\alpha(P_{B_r^n} \| Q_{B_r^n}) = \frac{1}{\alpha - 1} \log \left[\left(\frac{\Gamma\left(\frac{n}{r}\right)}{\left(\Gamma\left(\frac{1}{r}\right)\right)^n} \right)^{1-\alpha} \left(\frac{\Gamma\left(n\left(1-\frac{1}{r}\right)\right)}{\left(\Gamma\left(1-\frac{1}{r}\right)\right)^n} \right)^{\alpha} \times \frac{\left(\Gamma\left(\frac{1-\alpha}{r} + \alpha\left(1-\frac{1}{r}\right)\right)\right)^n}{\Gamma\left(n\left(\frac{1-\alpha}{r} + \alpha\left(1-\frac{1}{r}\right)\right)\right)} \right]$$

and

$$D_\alpha(Q_{B_r^n} \| P_{B_r^n}) = \frac{1}{\alpha - 1} \log \left[\left(\frac{\Gamma(\frac{n}{r})}{\left(\Gamma\left(\frac{1}{r}\right)\right)^n} \right)^{\alpha} \left(\frac{\Gamma\left(n\left(1-\frac{1}{r}\right)\right)}{\left(\Gamma\left(1-\frac{1}{r}\right)\right)^n} \right)^{1-\alpha} \times \frac{\left(\Gamma\left(\frac{\alpha}{r} + (1-\alpha)\left(1-\frac{1}{r}\right)\right)\right)^n}{\Gamma\left(n\left(\frac{\alpha}{r} + (1-\alpha)\left(1-\frac{1}{r}\right)\right)\right)} \right].$$

Now we introduce L_p-affine surface areas for a convex body K in $\mathbb{R}^n$. The L_p-affine surface area, an extension of the affine surface area, was introduced by Lutwak in the ground breaking paper [23] for $p > 1$ and for general p by Schütt and Werner [40]. For real $p \ne -n$, we define the L_p-affine surface area $as_p(K)$ of K as in [23] ($p > 1$) and [40] ($p < 1$, $p \ne -n$) by

$$as_p(K) = \int_{\partial K} \frac{\kappa_K(x)^{\frac{p}{n+p}}}{\langle x, N_K(x)\rangle^{\frac{n(p-1)}{n+p}}} d\mu_K(x) \tag{22}$$

and

$$as_{\pm\infty}(K)=\int_{\partial K}\frac{\kappa_K(x)}{\langle x,N_K(x)\rangle^n}d\mu_K(x),\tag{23}$$

provided the above integrals exist. In particular, for $p=0$

$$as_0(K)=\int_{\partial K}\langle x,N_K(x)\rangle\,d\mu_K(x)=n|K|.$$

The case $p=1$ is the classical affine surface area which goes back to Blaschke. It is independent of the position of K in space

$$as_1(K)=\int_{\partial K}\kappa_K(x)^{\frac{1}{n+1}}\,d\mu_K(x).$$

Originally a basic affine invariant from the field of affine differential geometry, it has recently attracted increased attention too (e.g. [18,23,31,38,44]).

If K is in C^2_+, then $d\mu_K=f_K d\sigma$ and then the L_p-affine surface areas, for all $p\neq -n$, can be written as

$$as_p(K)=\int_{S^{n-1}}\frac{f_K(u)^{\frac{n}{n+p}}}{h_K(u)^{\frac{n(p-1)}{n+p}}}d\sigma(u).\tag{24}$$

In particular,

$$as_{\pm\infty}(K)=\int_{S^{n-1}}\frac{d\sigma(u)}{h_K(u)^n}=n|K^\circ|.$$

Recall that $f_K(u)$ is the curvature function of K at u, i.e., the reciprocal of the Gauss curvature $\kappa_K(x)$ at this point $x\in\partial K$, the boundary of K, that has u as its outer normal.

The mixed p-affine surface area, $as_p(K_1,\dots,K_n)$, of n convex bodies $K_i\in C^2_+$ was introduced – for $p\geq 1$ in [22] and extended to all p in [47] – as

$$as_p(K_1,\dots,K_n)=\int_{S^{n-1}}\Big[h_{K_1}(u)^{1-p}f_{K_1}(u)\cdots h_{K_n}^{1-p}f_{K_n}(u)\Big]^{\frac{1}{n+p}}\,d\sigma(u).\tag{25}$$

Then we observe the following remarkable fact which connects the L_p-Brunn–Minkowski theory and information theory:

L_p-affine surface areas of a convex body are Hellinger integrals – or exponentials of Rényi divergences – of the cone measures of K and K°. For $\alpha=1$, such a connection was already observed in [34], namely

$$\frac{|K|}{|K^\circ|}e^{-D_{KL}(P_K\|Q_K)}=\lim_{p\to\infty}\left(\frac{as_p(K)}{n|K^\circ|}\right)^{\frac{n+p}{n}}.\tag{26}$$

Now we have more generally the following theorem.

Theorem 2.4. *Let K be a convex body in $\mathbb{R}^n$. Let $-\infty<\alpha<\infty$. $\alpha\neq 1$. Then*

$$D_\alpha(P_K\|Q_K)=\frac{1}{\alpha-1}\log\left(\frac{as_{n\frac{\alpha}{1-\alpha}}(K)}{n|K|^{1-\alpha}|K^\circ|^\alpha}\right).$$

$$D_\alpha(Q_K\|P_K)=\frac{1}{\alpha-1}\log\left(\frac{as_{n\frac{1-\alpha}{\alpha}}(K)}{n|K|^\alpha|K^\circ|^{1-\alpha}}\right).$$

Equivalently, for all $-\infty\le p\le\infty$, $p\neq -n$,

$$\begin{aligned}\frac{as_p(K)}{n|K|^{\frac{n}{n+p}}|K^\circ|^{\frac{p}{n+p}}}&=\operatorname{Exp}\left(-\frac{n}{n+p}D_{\frac{p}{n+p}}(P_K\|Q_K)\right)\\&=\operatorname{Exp}\left(-\frac{p}{n+p}D_{\frac{n}{n+p}}(Q_K\|P_K)\right).\end{aligned}$$

In particular,

$$\begin{aligned}\frac{as_1(K)}{n|K|^{\frac{n}{n+1}}|K^\circ|^{\frac{1}{n+1}}}&=\operatorname{Exp}\left(-\frac{n}{n+1}D_{\frac{1}{n+1}}(P_K\|Q_K)\right)\\&=\operatorname{Exp}\left(-\frac{1}{n+1}D_{\frac{n}{n+1}}(Q_K\|P_K)\right).\end{aligned}$$

Remarks.

(i) Theorem 2.4 can also be written as

$$\left(\frac{as_p(K)}{n|K^\circ|}\right)^{\frac{n+p}{n}}=\frac{|K|}{|K^\circ|}\,e^{-D_{\frac{p}{n+p}}(P_K\|Q_K)}.$$

If we now let $p\to\infty$, we recover (26). Also from Theorem 2.4

$$\left(\frac{as_p(K)}{n|K|}\right)^{\frac{n+p}{p}}=\frac{|K^\circ|}{|K|}\,e^{-D_{\frac{n}{n+p}}(Q_K\|P_K)}.$$

If we let $p\to 0$, then we get

$$\lim_{p\to 0}\left(\frac{as_p(K)}{n|K|}\right)^{\frac{n+p}{p}}=\frac{|K^\circ|}{|K|}\,e^{-D_{KL}(Q_K\|P_K)}.\tag{27}$$

We will comment on these expressions in Section 3.

(ii) If $-\infty<\alpha\le 0$, then $-\infty\le p=n\frac{1-\alpha}{\alpha}<-n$. Thus, for this range of α, we get the L_p-affine surface area in the range smaller than $-n$. If $0\le\alpha<\infty$, then $-n<p=n\frac{1-\alpha}{\alpha}\le\infty$. Thus, for this range of α, we get the L_p-affine surface area in the range greater than $-n$. In particular, for $0\le\alpha\le 1$, we get the L_p-affine surface area for $0\le p\le\infty$.

If $-\infty\le\alpha<1$, then $-n<p=n\frac{\alpha}{1-\alpha}\le\infty$. Thus, for this range of α, we get the L_p-affine surface area in the range greater $-n$. If $1<\alpha\le\infty$, then $-\infty\le p=n\frac{\alpha}{1-\alpha}<-n$. Thus, for this range of α, we get the L_p-affine surface area in the range smaller than $-n$.

Theorem 2.5. *Let* $K_1,\dots,K_n$ *be convex bodies in* C^2_+. *Then, for all* $\alpha\neq 1$

$$D_\alpha(P_{K_1}\times\cdots\times P_{K_n}\|Q_{K_1}\times\cdots\times Q_{K_n})=\frac{1}{\alpha-1}\log\left(\frac{as_{n\frac{\alpha}{1-\alpha}}(K_1,\dots,K_n)}{n\prod_{i=1}^n|K_i|^{\frac{1-\alpha}{n}}|K_i^\circ|^{\frac{\alpha}{n}}}\right)$$

and

$$D_\alpha(Q_{K_1} \times \cdots \times Q_{K_n} \| P_{K_1} \times \cdots \times P_{K_n}) = \frac{1}{\alpha - 1} \log \left(\frac{as_{n\frac{1-\alpha}{\alpha}}(K_1, \ldots, K_n)}{n \prod_{i=1}^{n} |K_i|^{\frac{\alpha}{n}} |K_i^\circ|^{\frac{1-\alpha}{n}}} \right).$$

Remark. The expressions in Theorem 2.5 can also be written as

$$\left(\frac{as_{n\frac{\alpha}{1-\alpha}}(K_1, \ldots, K_n)}{n \prod_{i=1}^{n} |K_i^\circ|^{\frac{1}{n}}} \right)^{\frac{1}{1-\alpha}} = \prod_{i=1}^{n} \left(\frac{|K_i|}{|K_i^\circ|} \right)^{\frac{1}{n}} e^{-D_\alpha(P_{K_1} \times \cdots \times P_{K_n} \| Q_{K_1} \times \cdots \times Q_{K_n})}$$

and

$$\left(\frac{as_{n\frac{1-\alpha}{\alpha}}(K_1, \ldots, K_n)}{n \prod_{i=1}^{n} |K_i|^{\frac{1}{n}}} \right)^{\frac{1}{1-\alpha}} = \prod_{i=1}^{n} \left(\frac{|K_i^\circ|}{|K_i|} \right)^{\frac{1}{n}} e^{-D_\alpha(Q_{K_1} \times \cdots \times Q_{K_n} \| P_{K_1} \times \cdots \times P_{K_n})}.$$

If we now let in the first expression $\alpha \uparrow 1$ respectively, putting $p = n\frac{\alpha}{1-\alpha}$, $p \to \infty$, we get

$$\begin{aligned} \prod_{i=1}^{n} \left(\frac{|K_i|}{|K_i^\circ|} \right)^{\frac{1}{n}} e^{-D_1(P_{K_1} \times \cdots \times P_{K_n} \| Q_{K_1} \times \cdots \times Q_{K_n})} &= \lim_{\alpha \to 1} \left(\frac{as_{n\frac{\alpha}{1-\alpha}}(K_1, \ldots, K_n)}{n \prod_{i=1}^{n} |K_i^\circ|^{\frac{1}{n}}} \right)^{\frac{1}{1-\alpha}} \\ &= \lim_{p \to \infty} \left(\frac{as_p(K_1, \ldots, K_n)}{n \prod_{i=1}^{n} |K_i^\circ|^{\frac{1}{n}}} \right)^{\frac{n+p}{n}}. \end{aligned} \tag{28}$$

If we let in the second expression $\alpha \uparrow 1$, respectively, putting $p = n\frac{1-\alpha}{\alpha}$, $p \to 0$, we get

$$\begin{aligned} \prod_{i=1}^{n} \left(\frac{|K_i^\circ|}{|K_i|} \right)^{\frac{1}{n}} e^{-D_1(Q_{K_1} \times \cdots \times Q_{K_n} \| P_{K_1} \times \cdots \times P_{K_n})} &= \lim_{\alpha \to 1} \left(\frac{as_{n\frac{1-\alpha}{\alpha}}(K_1, \ldots, K_n)}{n \prod_{i=1}^{n} |K_i|^{\frac{1}{n}}} \right)^{\frac{1}{1-\alpha}} \\ &= \lim_{p \to 0} \left(\frac{as_p(K_1, \ldots, K_n)}{n \prod_{i=1}^{n} |K_i|^{\frac{1}{n}}} \right)^{\frac{n+p}{p}}. \end{aligned} \tag{29}$$

We will comment on these quantities in Section 3.

Special cases.

(i) If $\alpha = \frac{1}{2}$, then

$$D_{\frac{1}{2}}(Q_K \| P_K) = D_{\frac{1}{2}}(P_K \| Q_K) = -2\log\left(\frac{as_n(K)}{n|K|^{\frac{1}{2}}|K^\circ|^{\frac{1}{2}}}\right),$$

and $\frac{as_n(K)}{n|K|^{\frac{1}{2}}|K^\circ|^{\frac{1}{2}}}$ is the Bhattacharyya coefficient of p_K and q_K.

$$\begin{aligned} &D_{\frac{1}{2}}(Q_{K_1} \times \cdots \times Q_{K_n} \| P_{K_1} \times \cdots \times P_{K_n}) \\ &\quad = D_{\frac{1}{2}}(P_{K_1} \times \cdots \times P_{K_n} \| Q_{K_1} \times \cdots \times Q_{K_n}) \\ &\quad = -2\log\left(\frac{as_n(K_1, \ldots, K_n)}{n\prod_{i=1}^{n}|K_i|^{\frac{1}{2n}}|K_i^\circ|^{\frac{1}{2n}}}\right). \end{aligned}$$

(ii) If $\alpha = 0$, then $D_0(P_K \| Q_K) = 0$. Likewise,

$$D_0(Q_K \| P_K) = -\log\left(\frac{as_\infty(K)}{n|K^\circ|}\right) \tag{30}$$

which, if K is sufficiently smooth, is equal to

$$-\log\left(\frac{as_\infty(K)}{n|K^\circ|}\right) = -\log\left(\frac{\int_{\partial K}\frac{\kappa_K(x)d\mu(x)}{\langle x, N_K(x)\rangle^n}}{n|K^\circ|}\right) = -\log 1 = 0$$

and equal to ∞ if K is a polytope.

$$D_0(P_{K_1} \times \cdots \times P_{K_n} \| Q_{K_1} \times \cdots \times Q_{K_n}) = -\log\left(\frac{as_0(K_1, \ldots, K_n)}{n\prod_{i=1}^{n}|K_i|^{\frac{1}{n}}}\right)$$

and

$$\begin{aligned} D_0(Q_{K_1} \times \cdots \times Q_{K_n} \| P_{K_1} \times \cdots \times P_{K_n}) &= -\log\left(\frac{as_\infty(K_1, \ldots, K_n)}{n\prod_{i=1}^{n}|K_i^\circ|^{\frac{1}{n}}}\right) \\ &= -\log\left(\frac{\tilde{V}(K_1, \ldots, K_n)}{\prod_{i=1}^{n}|K_i^\circ|^{\frac{1}{n}}}\right), \end{aligned}$$

where $\tilde{V}(K_1, \ldots, K_n)$ is the dual mixed volume introduced by Lutwak in [20].

(iii) If $\alpha \to \infty$, then $p = n\frac{1-\alpha}{\alpha} \to -n$ from the right. Therefore, by definition, $D_\infty(Q_K\|P_K) = \log\left(\sup_x \text{ess} \frac{q_K(x)}{p_K(x)}\right) = \log\left(\sup_x \text{ess} \frac{\langle x, N_K(x)\rangle^{n+1}|K^\circ|}{\kappa_K(x)|K|}\right)$. On the other hand

$$\lim_{\alpha\to\infty}\left(\frac{as_{n\frac{1-\alpha}{\alpha}}(K)}{n|K|^\alpha|K^\circ|^{1-\alpha}}\right)^{\frac{1}{\alpha-1}} = \frac{|K^\circ|}{|K|}\lim_{\alpha\to\infty}\left\|\frac{\langle x, N_K(x)\rangle^{n+\frac{\alpha}{1-\alpha}}}{\kappa_K(x)}\right\|_{L_{\alpha-1}}$$
$$= \frac{|K^\circ|}{|K|}\left\|\frac{\langle x, N_K(x)\rangle^{n-1}}{\kappa_K(x)}\right\|_{L_\infty},$$

which is thus consistent with the definition of $D_\infty(Q_K\|P_K)$. Similarly, one shows that, if $\alpha \to \infty$, then $p = n\frac{\alpha}{1-\alpha} \to -n$ from the left. Hence, by definition, $D_\infty(P_K\|Q_K) = \log\left(\sup_x \text{ess} \frac{q_K(x)}{p_K(x)}\right) = \log\left(\sup_x \text{ess} \frac{\kappa_K(x)|K|}{\langle x, N_K(x)\rangle^{n+1}|K^\circ|}\right)$, which is consistent with $\lim_{\alpha\to\infty}\left(\frac{as_{n\frac{\alpha}{1-\alpha}}(K)}{n|K|^{1-\alpha}|K^\circ|^\alpha}\right)^{\frac{1}{\alpha-1}}$.
Thus, also it would make sense to define

$$\lim_{p\to -n^+} as_p(K) = \sup_{x\in\partial K} \text{ess} \frac{\langle x, N_K(x)\rangle^{n+1}}{\kappa_K(x)} \tag{31}$$

and

$$\lim_{p\to -n^-} as_p(K) = \sup_{x\in\partial K} \text{ess} \frac{\kappa_K(x)}{\langle x, N_K(x)\rangle^{n+1}}, \tag{32}$$

which would imply that $\lim_{p\to -n} as_p(K)$ does not exist.
If $\alpha \to -\infty$, then $p = n\frac{1-\alpha}{\alpha} \to -n$ from the left and by (7), $D_{-\infty}(Q_K\|P_K) = -D_\infty(P_K\|Q_K)$. On the other hand,

$$\lim_{\alpha\to\infty}\log\left(\frac{as_{n\frac{1-\alpha}{\alpha}}(K)}{n|K|^\alpha|K^\circ|^{1-\alpha}}\right)^{\frac{1}{\alpha-1}} = \log\left(\frac{1}{\sup_x \frac{\kappa_K(x)|K|}{\langle x, N_K(x)\rangle^{n+1}|K^\circ|}}\right)$$
$$= -\log\left(\sup_x \frac{\kappa_K(x)|K|}{\langle x, N_K(x)\rangle^{n+1}|K^\circ|}\right)$$
$$= -D_\infty(P_K\|Q_K),$$

hence this is also consistent with the definitions. Similar considerations hold for $D_{-\infty}(P_K\|Q_K)$, $D_\alpha(P_{K_1}\times\cdots\times P_{K_n}\|Q_{K_1}\times\cdots\times Q_{K_n})$ and $D_\alpha(Q_{K_1}\times\cdots\times Q_{K_n}\|P_{K_1}\times\cdots\times P_{K_n})$.

Having identified L_p-affine surface areas as Rényi divergences, we can now translate known results from one theory to the other.

Affine invariance of L_p-affine surface areas translates into affine invariance of Rényi divergences: for all $p \neq -n$, $as_p(T(K)) = |\det\ T|^{\frac{n-p}{n+p}} as_p(K)$ (see [40]). Theorem 2.4 then implies that for all linear maps T with $\det\ T \neq 0$, for all $-\infty < \alpha < \infty$, $\alpha \neq 1$,

$$D_\alpha(P_{T(K)}\|Q_{T(K)}) = D_\alpha(P_K\|Q_K)$$

and

$$D_\alpha(Q_{T(K)}\|P_{T(K)}) = D_\alpha(Q_K\|P_K).$$

The case $\alpha = 1$ was treated in [34].

As $as_p(T(K_1), \dots, T(K_n)) = |\det T|^{\frac{n-p}{n+p}} as_p(K_1, \dots, K_n)$ (see [47]), it follows from Theorem 2.5 that for all linear maps T with $\det T \neq 0$, for all $-\infty < \alpha < \infty$, $\alpha \neq 1$,

$$\begin{aligned} &D_\alpha(P_{T(K_1)} \times \cdots \times P_{T(K_n)} \| Q_{T(K_1)} \times \cdots \times Q_{T(K_n)}) \\ &\quad = D_\alpha(P_{K_1} \times \cdots \times P_{K_n} \| Q_{K_1} \times \cdots \times Q_{K_n}) \end{aligned}$$

and

$$\begin{aligned} &D_\alpha(Q_{T(K_1)} \times \cdots \times Q_{T(K_n)} \| P_{T(K_1)} \times \cdots \times P_{T(K_n)}) \\ &\quad = D_\alpha(Q_{K_1} \times \cdots \times Q_{K_n} \| P_{K_1} \times \cdots \times P_{K_n}). \end{aligned}$$

The case $\alpha = 1$ is in [34].

Moreover, all inequalities and results mentioned in e.g. [46] about L_p-affine surface area and in e.g. [47] about mixed L_p-affine surface area can be translated into the corresponding inequalities and results about Rényi divergences. Conversely, results about Rényi divergences from e.g. [43] have consequences for L_p-affine surface areas. We mention only a few.

Proposition 2.6. (i) *Let K be a convex body in C^2_+. For all $-\infty \le \alpha \le \infty$,*

$$(1-\alpha) D_\alpha(Q_{K^\circ} \| P_{K^\circ}) = \alpha D_{1-\alpha}(Q_K \| P_K)$$

and

$$(1-\alpha) D_\alpha(P_{K^\circ} \| Q_{K^\circ}) = \alpha D_{1-\alpha}(P_K \| Q_K).$$

The equalities hold trivially if $\alpha = 0$ or $\alpha = 1$.

(ii) *Let K_i, $1 \le i \le n$, be convex bodies in C^2_+. Then for all $0 \le \alpha$*

$$\begin{aligned} as_{n\frac{\alpha}{1-\alpha}}(K_1, \dots, , K_n) &= \int_{S^{n-1}} \prod_{i=1}^n \left[f_{K_i} h_{K_i}^{1-\frac{n\alpha}{1-\alpha}} \right]^{\frac{1-\alpha}{n}} d\sigma \\ &= \prod_{i=1}^n \int_{S^{n-1}} \left[f_{K_i} h_{K_i}^{1-\frac{n\alpha}{1-\alpha}} \right]^{\frac{1-\alpha}{n}} d\sigma, \end{aligned}$$

i.e. we can interchange integration and product.

(iii) *Let K and L be convex bodies in C^2_+. Let $0 \le p \le \infty$. Let $0 \le \lambda \le 1$. Then*

$$\begin{aligned} &\int_{S^{n-1}} \left[\lambda \frac{f_K h_K}{|K|} + (1-\lambda) \frac{f_L h_L}{|L|} \right]^{\frac{n}{n+p}} \left[\frac{\lambda}{h_K^n |K^\circ|} + \frac{1-\lambda}{h_L^n |L^\circ|} \right]^{\frac{p}{n+p}} d\sigma \\ &\quad \ge \left(\frac{as_p(K)}{|K|^{\frac{n}{n+p}} |K^\circ|^{\frac{p}{n+p}}} \right)^\lambda \left(\frac{as_p(L)}{|L|^{\frac{n}{n+p}} |L^\circ|^{\frac{p}{n+p}}} \right)^{1-\lambda} \end{aligned}$$

with equality iff $K = L$. Equality holds trivially if $p = 0$ or $p = \infty$ or $\lambda = 0$ or $\lambda = 1$.

Proof.

(i) For $-\infty < \alpha < \infty$, (i) follows from the duality formula $as_p(K) = as_{\frac{n^2}{p}}(K^\circ)$, or, formulated in a more symmetric way, using the parameter $\alpha = \frac{p}{n+p}$

$$as_{n\frac{\alpha}{1-\alpha}}(K) = as_{n\frac{1-\alpha}{\alpha}}(K^\circ).$$

This identity was proved for $p > 0$ in [11] and – with a different proof – for all other p in [46].

Let now $\alpha = \infty$. Then, on the one hand

$$\lim_{\alpha\to\infty} \frac{1-\alpha}{\alpha} D_\alpha(Q_{K^\circ} \| P_{K^\circ}) = -D_\infty(Q_{K^\circ} \| P_{K^\circ}) = -\log \sup_{x\in\partial K^\circ} \text{ess } \frac{q_{K^\circ}(x)}{p_{K^\circ}(x)}. \tag{33}$$

On the other hand, by (16),

$$D_{-\infty}(Q_K \| P_K) = -D_{-\infty}(P_K \| Q_K) = -\log \sup_{x\in\partial K} \text{ess } \frac{p_K(x)}{q_K(x)}. \tag{34}$$

(33) equals (34), as (see [11]) for $x \in \partial K$, $y \in \partial K^\circ$ such that $\langle x, y\rangle = 1$,

$$\langle y, N_{K^\circ}(y)\rangle\langle x, N_K(x)\rangle = (\kappa_{K^\circ}(y)\kappa_K(x))^{\frac{1}{n+1}}.$$

Similarly, for $\alpha = -\infty$.

(ii) Follows from Theorem 2.5 and the fact that [43]

$$D_\alpha(Q_{K_1} \times \cdots \times Q_{K_n} \| P_{K_1} \times \cdots \times P_{K_n}) = \sum_{i=1}^n D_\alpha(Q_{K_i} \| P_{K_i}),$$

respectively the corresponding equation for $D_\alpha(P_{K_1} \times \cdots \times P_{K_n} \| Q_{K_1} \times \cdots \times Q_{K_n})$.

(iii) For $0 \le \alpha \le 1$, $D_\alpha(Q_K \| P_K)$, respectively $D_\alpha(P_K \| Q_K)$, are jointly convex [43]. We put $p = n\frac{1-\alpha}{\alpha}$ respectively $p = n\frac{\alpha}{1-\alpha}$ and use the joint convexity together with Theorem 2.4.

If $p \neq 0, \infty$ and $\lambda \neq 0, 1$, then equality implies that $K = L$ as the logarithm is strictly concave. □

3. Geometric interpretations of Rényi divergences

In this section we present geometric interpretations of Rényi divergences D_α of convex bodies, for all α. Geometric interpretations for the case $\alpha = 1$, the relative entropy, were given first in [34] in terms of L_p-centroid bodies. Recall that for a convex body K in $\mathbb{R}^n$ of volume 1 and $1 \le p \le \infty$, the L_p-centroid body $Z_p(K)$ is this convex body that has support function

$$h_{Z_p(K)}(\theta) = \left(\int_K |\langle x, \theta\rangle|^p dx\right)^{1/p}.$$

Now that we observed that Rényi divergences are logarithms of L_p-affine surface areas, we can use their geometric characterizations to obtain the ones for Rényi divergences. We will mostly concentrate on the geometric characterization of L_p-affine surface areas via the *surface bodies* [40] and *illumination surface bodies* [47], though there are many more available (see e.g. [32,39,45,46]).

Even more is gained. First, we need not assume that the body is symmetric as in [34] nor that it has C^2_+ boundary as it was needed in [34], to obtain the desired geometric interpretation for D_α for all α. Weaker regularity assumptions on the boundary suffice.

Second, in the context of the L_p-centroid bodies, the relative entropies appeared only after performing a second order expansion of certain expressions. Now, using the surface bodies or illumination surface bodies, already a first order expansion makes them appear. Thus, these bodies detect "faster" details of the boundary of a convex body than the L_p-centroid bodies.

Let K be a convex body in $\mathbb{R}^n$. Let $f : \partial K \to \mathbb{R}$ be a nonnegative, integrable, function. Let $s \geq 0$.

The *surface body* $K_{f,s}$, introduced in [40], is the intersection of all closed half-spaces H^+ whose defining hyperplanes H cut off a set of $f\mu_K$-measure less than or equal to s from ∂K. More precisely,

$$K_{f,s} = \bigcap_{\int_{\partial K \cap H^-} f d\mu_K \leq s} H^+.$$

The *illumination surface body* $K^{f,s}$ [47] is defined as

$$K^{f,s} = \left\{x : \mu_f(\partial K \cap \overline{[x, K] \setminus K}) \leq s\right\},$$

where μ_f is the measure defined for measurable sets O of ∂K by $\mu_f(O) = \int_O f d\mu_K$ and where for sets A and B (respectively points x and y) in $\mathbb{R}^n$, $[A, B] = \{\lambda a + (1-\lambda)b : a \in A, b \in B, 0 \leq \lambda \leq 1\}$ (respectively $[x, y] = \{\lambda x + (1-\lambda)y : 0 \leq \lambda \leq 1\}$)is the convex hull of A and B (respectively x and y).

For $x \in \partial K$ and $s > 0$ and f and $K_{f,s}$ as above, we put

$$x_s = [0, x] \cap \partial K_{f,s}.$$

The *minimal function* $M_f : \partial K \to \mathbb{R}$

$$M_f(x) = \inf_{0<s} \frac{\int_{\partial K \cap H^-(x_s, N_{K_{f,s}}(x_s))} f \, d\mu_K}{\mathrm{vol}_{n-1}\left(\partial K \cap H^-(x_s, N_{K_{f,s}}(x_s))\right)} \tag{35}$$

was introduced in [40]. $H(x, \xi)$ is the hyperplane through x and orthogonal to ξ. $H^-(x, \xi)$ is the closed halfspace containing the point $x + \xi$, $H^+(x, \xi)$ the other halfspace.

For $x \in \partial K$, we define $r(x)$ as the maximum of all real numbers ρ so that $B_2^n(x - \rho N_K(x), \rho) \subseteq K$. Then we formulate an integrability condition for the minimal function

$$\int_{\partial K} \frac{d\mu_K(x)}{\left(M_f(x)\right)^{\frac{2}{n-1}} r(x)} < \infty. \tag{36}$$

The following theorem was proved in [40].

Theorem 3.1. *Let K be a convex body in $\mathbb{R}^n$. Suppose that $f : \partial K \to \mathbb{R}$ is an integrable, almost everywhere strictly positive function that satisfies the integrability condition (36). Then*

$$c_n \lim_{s \to 0} \frac{|K| - |K_{f,s}|}{s^{\frac{2}{n-1}}} = \int_{\partial K} \frac{\kappa_K^{\frac{1}{n-1}}}{f^{\frac{2}{n-1}}} d\mu_K,$$

where $c_n = 2|B_2^{n-1}|^{\frac{2}{n-1}}$.

Theorem 3.1 was used in [40] to give geometric interpretations of the L_p-affine surface area. Now we use this theorem to give geometric interpretations of Rényi divergence of order α for all α for cone measures of convex bodies. First we treat the case $\alpha \neq 1$.

Corollary 3.2. *Let K be a convex body in $\mathbb{R}^n$.*

For $-\infty \le p \le \infty$, $p \neq -n$, let $f_p : \partial K \to \mathbb{R}$ be defined as

$$f_p(x) = \frac{\langle x, N_K(x)\rangle^{\frac{(n-1)n(p-1)}{2(n+p)}}}{\kappa_K(x)^{\frac{n(p-1)-2p}{2(n+p)}}}.$$

If f_p is almost everywhere strictly positive and satisfies the integrability condition (36), then

$$\frac{c_n}{n|K|^{\frac{n}{n+p}}|K^\circ|^{\frac{p}{n+p}}} \lim_{s\to 0} \frac{|K| - |K_{f_p,s}|}{s^{\frac{2}{n-1}}} = \operatorname{Exp}\left(-\frac{p}{n+p} D_{\frac{n}{n+p}}(Q_K \| P_K)\right),$$

and, provided $p \neq \pm\infty$,

$$\frac{c_n}{n|K|^{\frac{n}{n+p}}|K^\circ|^{\frac{p}{n+p}}} \lim_{s\to 0} \frac{|K| - |K_{f_p,s}|}{s^{\frac{2}{n-1}}} = \operatorname{Exp}\left(-\frac{n}{n+p} D_{\frac{p}{n+p}}(P_K \| Q_K)\right).$$

If K is in C^2_+, the last equation also holds for $p = \pm\infty$.

Proof. The proof of the corollary follows immediately from Theorems 3.1 and 2.4.

The next corollary treats the case $\alpha = 1$. There, we need to make additional regularity assumptions on the boundary of K. Those are weaker though than C^2_+. $\square$

Corollary 3.3. *Let K be a convex body in $\mathbb{R}^n$. Assume that K is such that there are $0 < r \le R < \infty$ so that for all $x \in \partial K$*

$$B^n_2(x - rN_K(x), r) \subset K \subset B^n_2(x - RN_K(x), R). \tag{37}$$

Let $f_{PQ} : \partial K \to \mathbb{R}$ and $f_{QP} : \partial K \to \mathbb{R}$ be defined by

$$f_{PQ}(x) = \frac{(n|K^\circ|\langle x, N_K(x)\rangle)^{\frac{n-1}{2}}}{\kappa_K(x)^{\frac{n-2}{2}}} \left(\log\left(\frac{R^{2n}|K|\,\kappa_K(x)}{r^{2n}|K^\circ|\,\langle x, N_K(x)\rangle^{n+1}}\right)\right)^{-\frac{n-1}{2}},$$

$$f_{QP}(x) = \left(\frac{n|K|}{\langle x, N_K(x)\rangle}\right)^{\frac{n-1}{2}} \kappa_K(x)^{\frac{1}{2}} \left(\log\left(\frac{R^{2n}|K^\circ|\,\langle x, N_K(x)\rangle^{n+1}}{r^{2n}|K|\,\kappa_K(x)}\right)\right)^{-\frac{n-1}{2}}.$$

Then f_{PQ} and f_{QP} are almost everywhere strictly positive, satisfy the integrability condition (36) and

$$c_n \lim_{s\to 0} \frac{|K| - |K_{f_{PQ},s}|}{s^{\frac{2}{n-1}}} = D_{KL}(P_K \| Q_K) + 2\log\left(\frac{R}{r}\right) \frac{as_{\pm\infty}(K)}{|K^\circ|}.$$

If K is in C^2_+, then this equals $D_{KL}(N_K N^{-1}_{K^\circ} cm_{K^\circ} \| cm_K) + 2n\log\left(\frac{R}{r}\right)$.

$$c_n \lim_{s\to 0} \frac{|K| - |K_{f_{QP},s}|}{s^{\frac{2}{n-1}}} = D_{KL}(Q_K \| P_K) + 2n\log\left(\frac{R}{r}\right).$$

If K is in C^2_+, then this is equal to $D_{KL}\left(N_K N^{-1}_{K^\circ} cm_{K^\circ} \| cm_K\right) + 2n\log\left(\frac{R}{r}\right)$.

Proof. Note that $r = R$ iff K is a Euclidean ball with radius r. Then the right hand sides of the identities in the corollary are equal to 0 and f_{PQ} and f_{QP} are identically equal to ∞. Therefore, for all $s \ge 0$, $K_{f_{PQ},s} = K$ and $K_{f_{QP},s} = K$ and hence for all $s \ge 0$, $|K| - |K_{f_{PQ},s}| = 0$ and $|K| - |K_{f_{QP},s}| = 0$. Therefore, the corollary holds trivially in this case.

Assume now that $r < R$, then

$$1 \le \frac{R^{2n}|K|\,\kappa_K(x)}{r^{2n}|K^\circ|\,\langle x, N_K(x)\rangle^{n+1}} \le \left(\frac{R}{r}\right)^{4n},$$

and we get for all $x \in \partial K$ that

$$f_{PQ}(x) \ge \left(\frac{|K^\circ| r^{n-1}}{2\log\left(\frac{R}{r}\right)}\right)^{\frac{n-1}{2}} > 0.$$

Also, for all $x \in \partial K$, $\left(\frac{|K^\circ| r^{n-1}}{2\log\left(\frac{R}{r}\right)}\right)^{\frac{n-1}{2}} \le M_{f_{PQ}}(x) \le \infty$ and therefore f_{PQ} satisfies the integrability condition (36). The proof of the corollary then follows immediately from Theorem 3.1.

Similarly for f_{QP}.

If K is in C^2_+, condition (37), holds. We can take

$$r = \inf_{x\in\partial K} \min_{1\le i\le n-1} r_i(x) \quad \text{and} \quad R = \sup_{x\in\partial K} \max_{1\le i\le n-1} r_i(x), \tag{38}$$

where for $x \in \partial K$, $r_i(x)$, $1 \le i \le n-1$ are the principal radii of curvature.

For convex bodies K and K_i, $i = 1, \dots, n$, define

$$\tilde{f}\left(N_K^{-1}(u)\right) = f_K(u)^{\frac{n-2}{2}}\left[f_p(K_1, u)\cdots f_p(K_n, u)\right]^{\frac{1-n}{2(n+p)}},$$

where $f_p(K, u) = h_K(u)^{1-p} f_K(u)$. $\square$

Corollary 3.4. *Let K and K_i, $i = 1, \dots, n$, be convex bodies in C^2_+. Then*

$$\frac{c_n}{n\left(\prod_{i=1}^n |K_i|\,|K_i^\circ|^{\frac{p}{n}}\right)^{\frac{1}{n+p}}} \lim_{s\to 0} \frac{|K| - |K_{\tilde{f},s}|}{s^{\frac{2}{n-1}}} = \operatorname{Exp}\left(-\frac{n}{n+p} D_{\frac{p}{n+p}}\left(P_{K_1}\times\cdots\times P_{K_n} \| Q_{K_1}\times\cdots\times Q_{K_n}\right)\right),$$

and

$$\frac{c_n}{n\left(\prod_{i=1}^n |K_i|\,|K_i^\circ|^{\frac{p}{n}}\right)^{\frac{1}{n+p}}} \lim_{s\to 0} \frac{|K| - |K_{\tilde{f},s}|}{s^{\frac{2}{n-1}}} = \operatorname{Exp}\left(-\frac{p}{n+p} D_{\frac{n}{n+p}}\left(Q_{K_1}\times\cdots\times Q_{K_n} \| P_{K_1}\times\cdots\times P_{K_n}\right)\right).$$

Proof. Again, the proof follows immediately from Theorems 2.5 and 3.1. $\square$

Remark. It was shown in [47] that for a convex body K in $\mathbb{R}^n$ with C^2_+-boundary

$$\lim_{s\to 0} c_n \frac{|K^{f,s}| - |K|}{s^{\frac{2}{n-1}}} = \int_{\partial K} \frac{\kappa_K(x)^{\frac{1}{n-1}}}{f(x)^{\frac{2}{n-1}}}\, d\mu_K(x), \tag{39}$$

where $c_n = 2|B_2^{n-1}|^{\frac{2}{n-1}}$ and $f : \partial K \to \mathbb{R}$ is an integrable function such that $f \geq c\ \mu_K$-almost everywhere. $c > 0$ is a constant. Using (39), similar geometric interpretations of Rényi divergence can be obtained via the illumination surface body instead of the surface body. We can use the same functions as in Corollaries 3.2–3.4. We will also have to assume that K is in C^2_+.

In [34], the following new affine invariants Ω_K were introduced and their relation to the relative entropies were established.
Let $K, K_1, \dots, K_n$ be convex bodies in $\mathbb{R}^n$, all with centroid at the origin. Then

$$\Omega_K = \lim_{p\to\infty} \left(\frac{as_p(K)}{n|K^\circ|}\right)^{n+p}$$

and

$$\Omega_{K_1,\dots,K_n} = \lim_{p\to\infty} \left(\frac{as_p(K_1,\dots,K_n)}{as_\infty(K_1,\dots,K_n)}\right)^{n+p}.$$

It was proved in [34] that for a convex body K in $\mathbb{R}^n$ that is C^2_+

$$D_{KL}(P_K \| Q_K) = \log\left(\frac{|K|}{|K^\circ|}\Omega_K^{-\frac{1}{n}}\right) \tag{40}$$

and

$$D_{KL}(Q_K \| P_K) = \log\left(\frac{|K^\circ|}{|K|}\Omega_{K^\circ}^{-\frac{1}{n}}\right). \tag{41}$$

Note that Eq. (40) also followed from (26). Similar results hold for $\Omega_{K_1,\dots,K_n}$. We now concentrate on Ω_K. As shown in [34], these invariants can also be obtained as

$$\Omega_K^{\frac{1}{n}} = \lim_{p\to 0} \left(\frac{as_p(K^\circ)}{n|K^\circ|}\right)^{\frac{n+p}{p}}$$

and thus, denoting by $\mathcal{A}_K = \lim_{p\to 0} \left(\frac{as_p(K)}{n|K|}\right)^{\frac{n+p}{p}}$, $\Omega_K^{\frac{1}{n}} = \mathcal{A}_{K^\circ}$. This implies e.g. that

$$\lim_{p\to 0} \left(\frac{as_p(K)^n}{as_{\frac{1}{p}}(K)^{\frac{1}{n}}} \frac{n^{\frac{1}{n}}|K^\circ|^{\frac{1}{n}}}{n^n|K|^n}\right)^{\frac{1}{p}} = 1.$$

Geometric interpretations in terms of L_p-centroid bodies were given in [34] for the new affine invariants Ω_K. These interpretations are in the spirit of Corollaries 3.2–3.4: as $p \to \infty$, appropriately chosen volume differences of K and its L_p-centroid bodies make the quantity Ω_K appear.

Again, however, with the L_p-centroid bodies, only symmetric convex bodies in C^2_+ could be handled and it was needed to go to a second order expansion for the volume differences.

Now, it follows from Corollary 3.3 that there exist such interpretations for Ω_K also for non-symmetric convex bodies and under weaker smoothness assumptions than C^2_+.

Moreover, again already a first order expansion gives such geometric interpretations if one uses the surface bodies or the illumination surface bodies instead of the L_p-centroid bodies.

Corollary 3.5. *Let K be a convex body in $\mathbb{R}^n$ such that 0 is the center of gravity of K and such that K satisfies (37) of Corollary 3.3. Let $f_{PQ} : \partial K \to \mathbb{R}$ and $f_{QP} : \partial K \to \mathbb{R}$ be as in Corollary 3.3. Then*

$$c_n \lim_{s\to 0} \frac{|K| - |K_{f_{PQ},s}|}{s^{\frac{2}{n-1}}} - 2\log\left(\frac{R}{r}\right)\frac{as_{\pm\infty}(K)}{|K^\circ|} = \log\left(\frac{|K|}{|K^\circ|}\Omega_K^{-\frac{1}{n}}\right) = \log\left(\frac{|K|}{|K^\circ|}\mathcal{A}_{K^\circ}^{-1}\right)$$

and

$$c_n \lim_{s\to 0} \frac{|K| - |K_{f_{QP},s}|}{s^{\frac{2}{n-1}}} - 2n\log\left(\frac{R}{r}\right) = \log\left(\frac{|K^\circ|}{|K|}\Omega_{K^\circ}^{-\frac{1}{n}}\right) = \log\left(\frac{|K^\circ|}{|K|}\mathcal{A}_K^{-1}\right).$$

Proof. The proof of the corollary follows immediately from Corollary 3.3, (40), (41) and the definition of $\mathcal{A}_K$. $\square$

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