Rényi divergence and $L_p$-affine surface area for convex bodies

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Abstract

We show that the fundamental objects of the $L_p$-Brunn–Minkowski theory, namely the $L_p$-affine surface areas for a convex body, are closely related to information theory: they are exponentials of Rényi divergences of the cone measures of a convex body and its polar.

We give geometric interpretations for all Rényi divergences $D_\alpha$, not just for the previously treated special case of relative entropy which is the case $\alpha = 1$. Now, no symmetry assumptions are needed and, if at all, only very weak regularity assumptions are required.

Previously, the relative entropies appeared only after performing second order expansions of certain expressions. Now already first order expansions make them appear. Thus, in the new approach we detect “faster” details about the boundary of a convex body.

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1. Introduction

There exists a fascinating connection between convex geometric analysis and information theory. An example is the close parallel between geometric inequalities for convex bodies and...
inequalities for probability densities. For instance, the Brunn–Minkowski inequality and the entropy power inequality follow both in a very similar way from the sharp Young inequality (see, e.g., [2]).

In several recent papers, Lutwak et al. [24,26,27,29] established further connections between convexity and information theory. For example, they showed in [26] that the Cramer–Rao inequality corresponds to an inclusion of the Legendre ellipsoid and the polar $L_2$-projection body. The latter is a basic notion from the $L_p$-Brunn–Minkowski theory. This $L_p$-Brunn–Minkowski theory has its origins in the 1960s when Firey introduced his $L_p$-addition of convex bodies. It evolved rapidly over the past years and due to a number of highly influential works (see, e.g., [4,6–8,10,9,12,13,16–20,22,23,21,25,28,30–33,36–42,44–46,50]), is now a central part of modern convex geometry. In fact, this theory redirected much of the research about convex bodies from the Euclidean aspects to the study of the affine geometry of these bodies, and some questions that had been considered Euclidean in nature turned out to be affine problems. For example, the famous Busemann–Petty Problem (finally laid to rest in [3,5,48,49]), was shown to be an affine problem with the introduction of intersection bodies by Lutwak in [21].

Two fundamental notions within the $L_p$-Brunn–Minkowski theory are $L_p$-affine surface areas, introduced by Lutwak in the ground breaking paper [23] and $L_p$-centroid bodies introduced by Lutwak and Zhang in [30]. See Section 3 for the definition of those quantities.

Based on these quantities, Paouris and Werner [34] established yet another relation between affine convex geometry and information theory. They proved that the exponential of the relative entropy of the cone measure of a symmetric convex body and its polar equals a limit of normalized $L_p$-affine surface areas. Moreover, also in [34], Paouris and Werner gave geometric interpretations of the relative entropy of the cone measures of a sufficiently smooth, symmetric convex body and its polar.

In this paper we show that the very core of the $L_p$-Brunn–Minkowski theory, namely the $L_p$-affine surface areas themselves are concepts of information theory: they are exponentials of Rényi divergences of the cone measures of a convex body and its polar. This identification allows to translate known properties from one theory to the other.

Even more is gained. Geometric interpretations for all Rényi divergences $D_\alpha$ of cone measures of a convex body and its polar are given for all $\alpha$, not just for the special case of relative entropy which corresponds to the case $\alpha = 1$. We refer to Sections 2 and 3 for the definition of $D_\alpha$. No symmetry assumptions on $K$ are needed. Nor do these new geometric interpretations require the strong smoothness assumptions of [34].

In the context of the $L_p$-centroid bodies, the relative entropies appeared only after performing second order expansions of certain expressions. The remarkable fact now is that in our approach here, already first order expansions make them appear. Thus, these bodies detect “faster” details of the boundary of a convex body than the $L_p$-centroid bodies.

The paper is organized as follows. In Section 2 we introduce Rényi divergences for convex bodies and describe some of their properties. We also introduce $L_p$-affine surface areas and mixed $p$-affine surface areas.

The main observations are Theorems 2.4 and 2.5 which show that $L_p$-affine surface areas and mixed $p$-affine surface areas are exponentials of Rényi divergences. These identifications allow to translate known properties from one theory to the other—this is done in the rest of Section 2 and in Section 3. Also, in Section 3, we give geometric interpretations for Rényi divergences $D_\alpha$ of cone measures of convex bodies for all $\alpha$, including new ones for the relative entropy not requiring the (previously necessary) strong smoothness and symmetry assumptions on the body.
Further notation.

Throughout the paper, we will assume that the centroid of a convex body \( K \) in \( \mathbb{R}^n \) is at the origin. We work in \( \mathbb{R}^n \), which is equipped with a Euclidean structure \( \langle \cdot, \cdot \rangle \). We denote by \( \| \cdot \|_2 \) the corresponding Euclidean norm. \( B_2^n(x, r) \) is the ball centered at \( x \) with radius \( r \). We write \( B_2^n = B_2^n(0, 1) \) for the Euclidean unit ball centered at 0 and \( S^{n-1} \) for the unit sphere. The volume is denoted by \( | \cdot | \) or, if we want to emphasize the dimension, by \( \text{vol}_d(A) \) for a \( d \)-dimensional set \( A \). \( K^0 = \{ y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in K \} \) is the polar body of \( K \).

For a point \( x \in \partial K \), the boundary of \( K \), \( N_K(x) \) is the outer unit normal in \( x \) to \( K \) and \( \kappa_K(x) \) is the (generalized) Gauss curvature in \( x \). We write \( K \in C^2_+ \), if \( K \) has \( C^2 \) boundary \( \partial K \) with everywhere strictly positive Gaussian curvature \( \kappa_K \). \( \mu_K \) is the usual surface area measure on \( \partial K \). \( \sigma \) is the usual surface area measure on \( S^{n-1} \).

Let \( K \) be a convex body in \( \mathbb{R}^n \) and let \( u \in S^{n-1} \). Then \( h_K(u) \) is the support function of \( K \) in direction \( u \in S^{n-1} \), and \( f_K(u) \) is the curvature function, i.e. the reciprocal of the Gaussian curvature \( \kappa_K(x) \) at the point \( x \in \partial K \) that has \( u \) as an outer normal.

2. Rényi divergences for convex bodies

Let \((X, \mu)\) be a measure space and let \(dP = pd\mu\) and \(dQ = qd\mu\) be probability measures on \(X\) that are absolutely continuous with respect to the measure \(\mu\). Then the Rényi divergence of order \(\alpha\), introduced by Rényi [35] for \(\alpha > 0\) and \(\alpha \neq 1\), is defined as

\[
D_\alpha(P \parallel Q) = \frac{1}{\alpha - 1} \log \int_X p^\alpha q^{1-\alpha} d\mu. 
\] (1)

It is the convention to put \( p^\alpha q^{1-\alpha} = 0 \), if \( p = 0 \) or \( q = 0 \), even if \(\alpha < 0\) and \(\alpha > 1\). The integrals

\[
\int_X p^\alpha q^{1-\alpha} d\mu
\] (2)

are also called Hellinger integrals. See e.g. [15] for those integrals and additional information.

Usually, in the literature, \( \alpha \geq 0 \). However, we will also consider \( \alpha < 0 \), provided the expressions exist. Also, usually in the literature, the measures are probability measures. Therefore we normalize the measures.

Special cases.

(i) The case \(\alpha = 1\) is also called the Kullback–Leibler divergence or relative entropy from \(P\) to \(Q\) (see [1]). It is obtained as the limit as \(\alpha \uparrow 1\) in (1) and one gets

\[
D_{KL}(P \parallel Q) = D_1(P \parallel Q) = \lim_{\alpha \uparrow 1} D_\alpha(P \parallel Q) = \int_X p \log \frac{p}{q} d\mu. 
\] (3)

(The limit \(\alpha \to 1\) may not exist but limit \(\alpha \uparrow 1\) exists [14].)

(ii) The case \(\alpha = 0\) gives for \(q \neq 0\) (with the convention that \(0^0 = 1\)) that

\[
D_0(P \parallel Q) = 0, \quad \text{as } dQ = qd\mu \text{ is a probability measure on } X. \text{ If } q = 0, \text{ then } D_0(P \parallel Q) = -\infty. \text{ Note however, that the case } q = 0 \text{ is an exceptional case as it does not give rise to a probability measure.} 
\] (4)
(iii) The case $\alpha = \frac{1}{2}$ gives
\[ D_{\frac{1}{2}}(P \parallel Q) = D_{\frac{1}{2}}(Q \parallel P) = -2 \log \int_X p^{\frac{1}{2}} q^{\frac{1}{2}} d\mu. \] (5)

The expression $\int_X p^{\frac{1}{2}} q^{\frac{1}{2}} d\mu$ is also called the Bhattacharyya coefficient or Bhattacharyya distance of $p$ and $q$.

(iv) The cases $\alpha = \infty$ and $\alpha = -\infty$.

\[ D_\infty(P \parallel Q) = \log \sup_{x} p(x), \] (6)

and
\[ D_{-\infty}(P \parallel Q) = -\log \sup_{x} q(x) = -D_\infty(Q \parallel P). \] (7)

Note that for all $-\infty \leq \alpha \leq \infty$, $\alpha \neq 1$,

\[ D_\alpha(Q \parallel P) = \frac{\alpha}{1-\alpha} D_{1-\alpha}(P \parallel Q). \] (8)

As $\alpha \uparrow 1$, the limit on the left and the limit on the right of (8) exist and are both equal to $D_1(Q \parallel P) = \int_X q \log \frac{q}{p} d\mu$. Thus (8) holds for all $-\infty \leq \alpha \leq \infty$.

We will now consider Rényi divergence for convex bodies $K$ in $\mathbb{R}^n$. Let

\begin{align*}
  p_K(x) &= \frac{\kappa_K(x)}{\langle x, N_K(x) \rangle^n |K^\circ|}, \\
  q_K(x) &= \frac{\langle x, N_K(x) \rangle}{n |K|}.
\end{align*} (9)

Then

\begin{align*}
  P_K &= p_K \mu_K \quad \text{and} \quad Q_K = q_K \mu_K \quad \text{(10)}
\end{align*}

are probability measures on $\partial K$ that are absolutely continuous with respect to $\mu_K$.

Recall that the normalized cone measure $cm_K$ on $\partial K$ is defined as follows: for every measurable set $A \subseteq \partial K$

\[ cm_K(A) = \frac{1}{|K|} \left| \{ ta : a \in A, \ t \in [0, 1] \} \right|. \] (11)

The next proposition is well known. See e.g. [34] for a proof. It shows that the measures $P_K$ and $Q_K$ defined in (10) are the cone measures of $K$ and $K^\circ$. $N_K : \partial K \to S^{n-1}, x \mapsto N_K(x)$ is the Gauss map.

**Proposition 2.1.** Let $K$ be a convex body in $\mathbb{R}^n$. Let $P_K$ and $Q_K$ be the probability measures on $\partial K$ defined by (10). Then

\[ Q_K = cm_K, \]

or, equivalently, for every measurable subset $A$ in $\partial K$

\[ Q_K(A) = cm_K(A). \]

If $K$ is in addition in $C^2_+$, then

\[ P_K = N_K^{-1} N_K \circ cm_K. \]
or, equivalently, for every measurable subset A in \( \partial K \)

\[
P_K(A) = cm_K \left( N_K^{-1}(N_K(A)) \right).
\]

For \( \alpha = 1 \), the relative entropy of a convex body \( K \) in \( \mathbb{R}^n \) was considered in [34], namely

\[
D_1(P_K \parallel Q_K) = D_{KL}(P_K \parallel Q_K) = \int_{\partial K} \frac{\kappa_K(x)}{n|K^\circ|\langle x, N_K(x)\rangle^n} \log \left( \frac{|K|\kappa_K(x)}{|K^\circ|\langle x, N_K(x)\rangle^{n+1}} \right) d\mu_K(x)
\]

\[
D_1(Q_K \parallel P_K) = D_{KL}(Q_K \parallel P_K) = \int_{\partial K} \frac{\langle x, N_K(x)\rangle}{n|K|} \log \left( \frac{|K^\circ|\langle x, N_K(x)\rangle^{n+1}}{|K|\kappa_K(x)} \right) d\mu_K(x),
\]

provided the expressions exist.

We now define the Rényi divergence of \( K \) of order \( \alpha \) for all other \( \alpha, -\infty \leq \alpha \leq \infty, \alpha \neq 1 \).

**Definition 2.2.** Let \( K \) be a convex body in \( \mathbb{R}^n \) and let \( -\infty < \alpha < \infty, \alpha \neq 1 \). Then the Rényi divergences of order \( \alpha \) of \( K \) are

\[
D_\alpha(Q_K \parallel P_K) = \frac{1}{\alpha - 1} \log \left( \frac{\int_{\partial K} \kappa_K^{1-\alpha}(x) \, d\mu_K}{n|K^\circ|\langle x, N_K(x)\rangle^{n+1}} \right)
\]

\[
D_\alpha(P_K \parallel Q_K) = \frac{1}{\alpha - 1} \log \left( \frac{\int_{\partial K} \kappa_K^\alpha(x) \, d\mu_K}{n|K|\langle x, N_K(x)\rangle^{n+1}} \right)
\]

\[
D_\infty(Q_K \parallel P_K) = \log \left( \sup_{x \in \partial K} \text{ess sup} \frac{|K^\circ|\langle x, N_K(x)\rangle^{n+1}}{|K|\kappa_K(x)} \right)
\]

\[
D_\infty(P_K \parallel Q_K) = \log \left( \sup_{x \in \partial K} \text{ess sup} \frac{|K|\kappa_K(x)}{|K^\circ|\langle x, N_K(x)\rangle^{n+1}} \right)
\]

and

\[
D_{-\infty}(Q_K \parallel P_K) = -D_\infty(P_K \parallel Q_K), \quad D_{-\infty}(P_K \parallel Q_K) = -D_\infty(Q_K \parallel P_K),
\]

provided the expressions exist.

**Remarks.**

(i) By (8) for all \( -\infty \leq \alpha \leq \infty, \alpha \neq 1 \),

\[
D_\alpha(Q_K \parallel P_K) = \frac{\alpha}{1 - \alpha} D_{1-\alpha}(P_K \parallel Q_K).
\]

This identity also holds for \( \alpha \uparrow 1 \). Therefore, it is enough to consider only one of the two, \( D_\alpha(Q_K \parallel P_K) \) or \( D_\alpha(P_K \parallel Q_K) \).

(ii) If we put \( N_K(x) = u \in S^{n-1} \), then \( \langle x, N_K(x)\rangle = h_K(u) \). If \( K \) is in \( C_+^2 \), then, with the inverse \( N_K^{-1} \) of the Gauss map, \( d\mu_K(N_K^{-1}(u)) = f_K(u) \, d\sigma(u) \). Hence, in that case, we can express the
Rényi divergences also as

\[
D_\alpha(Q_K \parallel P_K) = \frac{1}{\alpha - 1} \log \left( \int_{S^{n-1}} f_{K_i}^\alpha \frac{d\sigma(u)}{|K|^{\alpha} |K^\circ|^{1-\alpha}} \right)
\]

(17)

\[
D_\alpha(P_K \parallel Q_K) = \frac{1}{\alpha - 1} \log \left( \int_{S^{n-1}} f_{K_i}^{1-\alpha} \frac{d\sigma(u)}{|K|^{1-\alpha} |K^\circ|^{\alpha}} \right).
\]

(18)

accordingly for \(D_{KL}(Q_K \parallel P_K)\) and \(D_{KL}(P_K \parallel Q_K)\).

Let \(K_1, \ldots, K_n\) be convex bodies in \(\mathbb{R}^n\). Let \(u \in S^{n-1}\). For \(1 \leq i \leq n\), define

\[
q_{K_i}(u) = \frac{1}{n^{\frac{1}{n}} |K_i^\circ|^{\frac{1}{n}} h_{K_i}(u)^{\frac{1}{n}}}, \quad q_{K_i}(u) = \frac{f_{K_i}(u)^{\frac{1}{n}} h_{K_i}(u)^{\frac{1}{n}}}{n^{\frac{1}{n}} |K_i^\circ|^{\frac{1}{n}}}
\]

(19)

and measures on \(S^{n-1}\) by

\[
P_{K_i} = p_{K_i} \sigma \quad \text{and} \quad Q_{K_i} = q_{K_i} \sigma.
\]

(20)

Then we define the Rényi divergences of order \(\alpha\) for convex bodies \(K_1, \ldots, K_n\) by the following.

**Definition 2.3.** Let \(K_1, \ldots, K_n\) be convex bodies in \(\mathbb{R}^n\). Then for \(-\infty < \alpha < \infty, \alpha \neq 1\)

\[
D_\alpha(Q_{K_1} \times \cdots \times Q_{K_n} \parallel P_{K_1} \times \cdots \times P_{K_n}) = \frac{\log \left( \int_{S^{n-1}} \prod_{i=1}^n f_{K_i}^{\frac{1}{n}} h_{K_i}^{\frac{1}{n}} \frac{d\sigma}{|K_i^\circ|^{\frac{1}{n}} |K_i^\circ|^{1-\frac{1}{n}}} \right)}{\alpha - 1}
\]

provided the expressions exist.

For \(\alpha = 1\) the definitions were given in [34]:

\[
D_1(Q_{K_1} \times \cdots \times Q_{K_n} \parallel P_{K_1} \times \cdots \times P_{K_n}) = \int_{S^{n-1}} \prod_{i=1}^n f_{K_i}^{\frac{1}{n}} h_{K_i}^{\frac{1}{n}} \log \left( \prod_{i=1}^n \frac{|K_i^\circ|^{\frac{1}{n}} f_{K_i}^{\frac{1}{n}} h_{K_i}^{\frac{1}{n}}}{|K_i^\circ|^{\frac{1}{n}} f_{K_i}^{\frac{1}{n}} h_{K_i}^{\frac{1}{n}}} \right) d\sigma
\]

\[
D_1(P_{K_1} \times \cdots \times P_{K_n} \parallel Q_{K_1} \times \cdots \times Q_{K_n}) = \int_{S^{n-1}} \prod_{i=1}^n h_{K_i}^{\frac{1}{n}} \log \left( \prod_{i=1}^n \frac{|K_i^\circ|^{\frac{1}{n}} f_{K_i}^{\frac{1}{n}} h_{K_i}^{\frac{1}{n}}}{|K_i^\circ|^{\frac{1}{n}} f_{K_i}^{\frac{1}{n}} h_{K_i}^{\frac{1}{n}}} \right) d\sigma,
\]

provided the expressions exist.

**Remark.** For \(-\infty < \alpha < \infty, \alpha \neq 1,\)

\[
D_\alpha(P_{K_1} \times \cdots \times P_{K_n} \parallel Q_{K_1} \times \cdots \times Q_{K_n}) = \frac{\alpha}{1 - \alpha} D_{1-\alpha}(Q_{K_1} \times \cdots \times Q_{K_n} \parallel P_{K_1} \times \cdots \times P_{K_n}),
\]

(21)
and, again, for $\alpha \uparrow 1$, the limits on both sides exist and coincide. Therefore it is enough to consider either $D_\alpha(P_{K_1} \times \cdots \times P_{K_n} \| Q_{K_1} \times \cdots \times Q_{K_n})$ or $D_\alpha(Q_{K_1} \times \cdots \times Q_{K_n} \| P_{K_1} \times \cdots \times P_{K_n})$.

We first present some examples and look at special cases below. In particular, $D_{\pm \infty}(Q_{K_1} \times \cdots \times Q_{K_n} \| P_{K_1} \times \cdots \times P_{K_n})$ will be considered below.

**Examples.**

(i) If $K = \rho B^n_2$, then $D_\alpha(Q_K \| P_K) = D_\alpha(P_K \| Q_K) = 0$ for all $-\infty \leq \alpha \leq \infty$.

(ii) If $K$ is a polytope, then $\kappa_K = 0$ a.e. on $\partial K$. Thus, for $\alpha = 1$, $D_1(Q_K \| P_K) = \infty$. For $-\infty < \alpha < 1$, $\int_{\partial K} \frac{\kappa_{\alpha}^{1-\alpha} d\mu_K}{(x, N_K(x))^{n-\alpha(n+1)}} = 0$ and for $\alpha > 1$, $\int_{\partial K} \frac{\kappa_{\alpha}^{1-\alpha} d\mu_K}{(x, N_K(x))^{n-\alpha(n+1)}} = \infty$. Hence $D_\alpha(Q_K \| P_K) = \infty$ for all $-\infty < \alpha < \infty$, and $K$ a polytope.

Similarly, $D_1(P_K \| Q_K) = 0$ (with the convention that $0 \cdot \infty = 0$).

$D_\alpha(P_K \| Q_K) = -\infty$, for $0 < \alpha < 1$ and $-\infty < \alpha < 0$ and $K$ a polytope and $D_\alpha(Q_K \| P_K) = \infty$, for $0 < \alpha < 1$ and $K$ a polytope.

This also shows that $D_\alpha$ need not be continuous at $\alpha = 1$.

For $\alpha = 0$ and $\alpha = \pm \infty$, see below.

(iii) For $1 < r < \infty$, let $K = B_r^n = \{ x \in \mathbb{R}^n : \sum_{i=1}^n |x_i|^r \leq 1 \}$ be the unit ball of $l_r^n$. We will compute $D_\alpha(Q_K \| P_K)$ and $D_\alpha(P_K \| Q_K)$ for all $-\infty < \alpha < \infty$, $\alpha \neq 1$. The case $\alpha = 1$ was considered in [34]. The cases $\alpha = 0$ and $\alpha = \pm \infty$ are treated below.

If $1 < r < 2$ and $\alpha \geq \frac{1}{r-1}$, then $D_\alpha(P_{B_r^n} \| Q_{B_r^n}) = \infty$. If $1 < r < 2$ and $\alpha \leq -\frac{r-1}{2-r}$, then $D_\alpha(Q_{B_r^n} \| P_{B_r^n}) = -\infty$. If $2 < r < \infty$ and $\alpha \leq \frac{1}{r-2}$, then $D_\alpha(Q_{B_r^n} \| P_{B_r^n}) = -\infty$. If $2 < r < \infty$ and $\alpha \geq \frac{1}{r-2}$, then $D_\alpha(Q_{B_r^n} \| P_{B_r^n}) = \infty$. In all other cases we have

\[
D_{\alpha}(P_{B_r^n} \| Q_{B_r^n}) = \frac{1}{\alpha - 1} \log \left[ \left( \frac{\Gamma \left( \frac{n}{r} \right)}{\Gamma \left( \frac{1}{r} \right)^n} \right)^{1-\alpha} \left( \frac{\Gamma \left( n \left( \frac{1}{r} - \frac{1}{\alpha} \right) \right)}{\Gamma \left( n \left( \frac{1}{\alpha} + \frac{1}{r} \right) \right)} \right)^{\alpha} \right] \\
\times \left( \frac{\Gamma \left( \frac{1-\alpha}{\alpha} \frac{1}{r} + \frac{1-\alpha}{1-\alpha} \frac{1}{r} \right)}{\Gamma \left( n \left( \frac{1-\alpha}{\alpha} \frac{1}{r} + \frac{1-\alpha}{1-\alpha} \frac{1}{r} \right) \right)} \right)
\]

and

\[
D_{\alpha}(Q_{B_r^n} \| P_{B_r^n}) = \frac{1}{\alpha - 1} \log \left[ \left( \frac{\Gamma \left( \frac{n}{r} \right)}{\Gamma \left( \frac{1}{r} \right)^n} \right)^{\alpha} \left( \frac{\Gamma \left( n \left( \frac{1}{r} - \frac{1}{\alpha} \right) \right)}{\Gamma \left( n \left( \frac{1}{\alpha} + \frac{1}{r} \right) \right)} \right)^{1-\alpha} \right] \\
\times \left( \frac{\Gamma \left( \frac{\alpha}{r} + (1-\alpha) \frac{1}{r} \right)}{\Gamma \left( n \left( \frac{\alpha}{r} + (1-\alpha) \frac{1}{r} \right) \right)} \right). 
\]

Now we introduce $L_p$-affine surface areas for a convex body $K$ in $\mathbb{R}^n$. The $L_p$-affine surface area, an extension of the affine surface area, was introduced by Lutwak in the groundbreaking paper [23] for $p > 1$ and for general $p$ by Schütz and Werner [40]. For real $p \neq -n$, we define the $L_p$-affine surface area $as_p(K)$ of $K$ as in [23] ($p > 1$) and [40] ($p < 1$, $p \neq -n$) by

\[
as_p(K) = \int_{\partial K} \frac{\kappa_K(x)^{p/n}}{|a_{p-1}(\mu_K)}^{1/p} d\mu_K(x) \quad (22)
\]
and
\[
as_{\pm\infty}(K) = \int_{\partial K} \frac{\kappa_K(x)}{\langle x, N_K(x) \rangle^n} d\mu_K(x),
\]
provided the above integrals exist. In particular, for \(p = 0\)
\[
as_0(K) = \int_{\partial K} \langle x, N_K(x) \rangle d\mu_K(x) = n|K|.
\]
The case \(p = 1\) is the classical affine surface area which goes back to Blaschke. It is independent of the position of \(K\) in space
\[
as_1(K) = \int_{\partial K} \kappa_K(x)^{\frac{1}{n+1}} d\mu_K(x).
\]
Originally a basic affine invariant from the field of affine differential geometry, it has recently attracted increased attention too (e.g. [18,23,31,38,44]).

If \(K\) is in \(C^2_+\), then \(d\mu_K = f_K d\sigma\) and then the \(L_p\)-affine surface areas, for all \(p \neq -n\), can be written as
\[
as_p(K) = \int_{S^{n-1}} \frac{f_K(u)^{\frac{n}{p}}}{h_K(u)^{\frac{n(p-1)}{n+p}}} d\sigma(u).
\]
In particular,
\[
as_{\pm\infty}(K) = \int_{S^{n-1}} \frac{d\sigma(u)}{h_K(u)^n} = n|K^\circ|.
\]
Recall that \(f_K(u)\) is the curvature function of \(K\) at \(u\), i.e., the reciprocal of the Gauss curvature \(\kappa_K(x)\) at this point \(x \in \partial K\), the boundary of \(K\), that has \(u\) as its outer normal.

The mixed \(p\)-affine surface area, \(as_p(K_1, \ldots, K_n)\), of \(n\) convex bodies \(K_i \in C^2_+\) was introduced – for \(p \geq 1\) in [22] and extended to all \(p\) in [47] – as
\[
as_p(K_1, \ldots, K_n) = \int_{S^{n-1}} \left[ h_{K_1}(u)^{1-p} f_{K_1}(u) \cdots h_{K_n}(u)^{1-p} f_{K_n}(u) \right]^{\frac{1}{n+p}} d\sigma(u).
\]

Then we observe the following remarkable fact which connects the \(L_p\)-Brunn–Minkowski theory and information theory:

\(L_p\)-affine surface areas of a convex body are Hellinger integrals – or exponentials of Rényi divergences – of the cone measures of \(K\) and \(K^\circ\). For \(\alpha = 1\), such a connection was already observed in [34], namely
\[
\frac{|K|}{|K^\circ|} e^{-D_{KL}(P_K\|Q_K)} = \lim_{p \to \infty} \left( \frac{as_p(K)}{n|K^\circ|} \right)^{\frac{n+p}{n}}.
\]
Now we have more generally the following theorem.

**Theorem 2.4.** Let \(K\) be a convex body in \(\mathbb{R}^n\). Let \(-\infty < \alpha < \infty\), \(\alpha \neq 1\). Then
\[
D_{\alpha}(P_K\|Q_K) = \frac{1}{\alpha - 1} \log \left( \frac{as_{\frac{\alpha}{1-\alpha}}(K)}{n|K|^{1-\alpha}|K^\circ|^{\alpha}} \right).
\]
\[
D_\alpha(Q_K \| P_K) = \frac{1}{\alpha - 1} \log \left( \frac{a_s \frac{n}{\alpha}(K)}{n |K^{\alpha}| |K^\alpha|^{1-\alpha}} \right).
\]

Equivalently, for all \(-\infty \leq p \leq \infty, p \neq -n,\)
\[
\frac{a_s_p(K)}{n |K^{\alpha}| |K^\alpha|^{1-\alpha}} = \text{Exp} \left( -\frac{n}{n + p} D_{\frac{p}{\alpha}}(P_K \| Q_K) \right)
\]
\[
= \text{Exp} \left( -\frac{p}{n + p} D_{\frac{p}{\alpha}}(Q_K \| P_K) \right).
\]

In particular,
\[
\frac{a_s_1(K)}{n |K^{\alpha}| |K^\alpha|^{1-\alpha}} = \text{Exp} \left( -\frac{n}{n + 1} D_{\frac{1}{\alpha}}(P_K \| Q_K) \right)
\]
\[
= \text{Exp} \left( -\frac{1}{n + 1} D_{\frac{1}{\alpha}}(Q_K \| P_K) \right).
\]

Remarks.
(i) Theorem 2.4 can also be written as
\[
\left( \frac{a_s_p(K)}{n |K^{\alpha}|} \right)^{\frac{n+p}{p}} = \frac{|K|}{|K^\alpha|} e^{-D_{\frac{p}{\alpha}}(P_K \| Q_K)}.
\]

If we now let \(p \to \infty,\) we recover (26). Also from Theorem 2.4
\[
\left( \frac{a_s_p(K)}{n |K^{\alpha}|} \right)^{\frac{n+p}{p}} = \frac{|K^\alpha|}{|K|} e^{-D_{\frac{n}{\alpha}}(Q_K \| P_K)}.
\]

If we let \(p \to 0,\) then we get
\[
\lim_{p \to 0} \left( \frac{a_s_p(K)}{n |K^{\alpha}|} \right)^{\frac{n+p}{p}} = \frac{|K^\alpha|}{|K|} e^{-D_{KL}(Q_K \| P_K)}.
\]

We will comment on these expressions in Section 3.
(ii) If \(-\infty < \alpha \leq 0,\) then \(-\infty \leq p = n \frac{1-\alpha}{\alpha} < -n.\) Thus, for this range of \(\alpha,\) we get the \(L_p\)-affine surface area in the range smaller than \(-n).\] If \(0 \leq \alpha < \infty,\) then \(-n < p = n \frac{1-\alpha}{\alpha} \leq \infty.\) Thus, for this range of \(\alpha,\) we get the \(L_p\)-affine surface area in the range greater than \(-n). \) In particular, for \(0 \leq \alpha \leq 1,\) we get the \(L_p\)-affine surface area for \(0 \leq p \leq \infty.\)

If \(-\infty \leq \alpha < 1,\) then \(-n < p = n \frac{1-\alpha}{\alpha} \leq \infty.\) Thus, for this range of \(\alpha,\) we get the \(L_p\)-affine surface area in the range greater than \(-n). \) If \(1 < \alpha \leq \infty,\) then \(-\infty \leq p = n \frac{\alpha}{1-\alpha} < -n.\) Thus, for this range of \(\alpha,\) we get the \(L_p\)-affine surface area in the range smaller than \(-n). \)

**Theorem 2.5.** Let \(K_1, \ldots, K_n\) be convex bodies in \(C_+^2.\) Then, for all \(\alpha \neq 1\)
\[
D_\alpha(P_{K_1} \times \cdots \times P_{K_n} \| Q_{K_1} \times \cdots \times Q_{K_n}) = \frac{1}{\alpha - 1} \log \left( \frac{a_s \frac{n}{\alpha}(K_1, \ldots, K_n)}{n \prod_{i=1}^n |K_i^{\alpha}| |K_i^\alpha|^{\frac{n}{\alpha}}} \right)
\]
and

\[ D_\alpha(Q_{K_1} \times \cdots \times Q_{K_n} \| P_{K_1} \times \cdots \times P_{K_n}) = \frac{1}{\alpha - 1} \log \left( \frac{as_n^{1-\alpha}(K_1, \ldots, K_n)}{n \prod_{i=1}^{n} |K_i|^{\frac{\alpha}{n}}} \right). \]

**Remark.** The expressions in Theorem 2.5 can also be written as

\[ \left( \frac{as_n^{1-\alpha}(K_1, \ldots, K_n)}{n \prod_{i=1}^{n} |K_i|^{\frac{\alpha}{n}}} \right)^{\frac{1}{1-\alpha}} = \prod_{i=1}^{n} \left( \frac{|K_i|^{\frac{1}{n}}}{|K_i^\circ|^{\frac{\alpha}{n}}} \right)^{\frac{1}{\alpha}} e^{-D_\alpha(P_{K_1} \times \cdots \times P_{K_n} \| Q_{K_1} \times \cdots \times Q_{K_n})} \]

and

\[ \left( \frac{as_n^{1-\alpha}(K_1, \ldots, K_n)}{n \prod_{i=1}^{n} |K_i|^{\frac{1}{n}}} \right)^{\frac{1}{1-\alpha}} = \prod_{i=1}^{n} \left( \frac{|K_i^\circ|^{\frac{1}{n}}}{|K_i|^{\frac{\alpha}{n}}} \right)^{\frac{1}{\alpha}} e^{-D_\alpha(Q_{K_1} \times \cdots \times Q_{K_n} \| P_{K_1} \times \cdots \times P_{K_n})}. \]

If we now let in the first expression \( \alpha \uparrow 1 \) respectively, putting \( p = n^{\frac{\alpha}{1-\alpha}}, \; p \to \infty \), we get

\[ \prod_{i=1}^{n} \left( \frac{|K_i|^{\frac{1}{n}}}{|K_i^\circ|^{\frac{\alpha}{n}}} \right)^{\frac{1}{\alpha}} e^{-D_1(P_{K_1} \times \cdots \times P_{K_n} \| Q_{K_1} \times \cdots \times Q_{K_n})} = \lim_{\alpha \to 1} \left( \frac{as_n^{1-\alpha}(K_1, \ldots, K_n)}{n \prod_{i=1}^{n} |K_i|^{\frac{\alpha}{n}}} \right)^{\frac{1}{1-\alpha}} \]

\[ = \lim_{p \to \infty} \left( \frac{as_p(K_1, \ldots, K_n)}{n \prod_{i=1}^{n} |K_i^\circ|^{\frac{1}{n}}} \right)^{\frac{n+p}{n}}. \]  \hspace{1cm} (28)

If we let in the second expression \( \alpha \uparrow 1 \), respectively, putting \( p = n^{\frac{1-\alpha}{\alpha}}, \; p \to 0 \), we get

\[ \prod_{i=1}^{n} \left( \frac{|K_i^\circ|^{\frac{1}{n}}}{|K_i|^{\frac{\alpha}{n}}} \right)^{\frac{1}{\alpha}} e^{-D_1(Q_{K_1} \times \cdots \times Q_{K_n} \| P_{K_1} \times \cdots \times P_{K_n})} = \lim_{\alpha \to 1} \left( \frac{as_n^{1-\alpha}(K_1, \ldots, K_n)}{n \prod_{i=1}^{n} |K_i|^{\frac{\alpha}{n}}} \right)^{\frac{1}{1-\alpha}} \]

\[ = \lim_{p \to 0} \left( \frac{as_p(K_1, \ldots, K_n)}{n \prod_{i=1}^{n} |K_i|^{\frac{1}{n}}} \right)^{\frac{n+p}{p}}. \]  \hspace{1cm} (29)

We will comment on these quantities in Section 3.
Special cases.

(i) If $\alpha = \frac{1}{2}$, then
\[
D_{\frac{1}{2}}(Q_K \parallel P_K) = D_{\frac{1}{2}}(P_K \parallel Q_K) = -2 \log \left( \frac{a_{sn}(K)}{n|K|^{\frac{1}{2}}|K^o|^{\frac{1}{2}}} \right),
\]
and $\frac{a_{sn}(K)}{n|K|^{\frac{1}{2}}|K^o|^{\frac{1}{2}}}$ is the Bhattacharyya coefficient of $p_K$ and $q_K$.

\[
D_{\frac{1}{2}}(Q_{K_1} \times \cdots \times Q_{K_n} \parallel P_{K_1} \times \cdots \times P_{K_n}) = -2 \log \left( \frac{a_{sn}(K_1, \ldots, K_n)}{n \prod_{i=1}^{n} |K_i|^{\frac{1}{n^\alpha}}|K_i^o|^{\frac{1}{n^\alpha}}} \right).
\]

(ii) If $\alpha = 0$, then $D_0(P_K \parallel Q_K) = 0$. Likewise,
\[
D_0(Q_{K_1} \times \cdots \times Q_{K_n} \parallel P_{K_1} \times \cdots \times P_{K_n}) = -\log \left( \frac{a_{s\infty}(K_1, \ldots, K_n)}{n|K^o|} \right)
\]
which, if $K$ is sufficiently smooth, is equal to
\[
-\log \left( \frac{a_{s\infty}(K)}{n|K^o|} \right) = -\log \left( \frac{\int_{\partial K} \frac{K_k(x) d\mu(x)}{\langle x, N_K(x) \rangle^n}}{n|K^o|} \right) = -\log 1 = 0
\]
and equal to $\infty$ if $K$ is a polytope.

\[
D_0(P_{K_1} \times \cdots \times P_{K_n} \parallel Q_{K_1} \times \cdots \times Q_{K_n}) = -\log \left( \frac{a_{s0}(K_1, \ldots, K_n)}{n \prod_{i=1}^{n} |K_i|^{\frac{1}{n}}} \right)
\]
and
\[
D_0(Q_{K_1} \times \cdots \times Q_{K_n} \parallel P_{K_1} \times \cdots \times P_{K_n}) = -\log \left( \frac{a_{s\infty}(K_1, \ldots, K_n)}{n \prod_{i=1}^{n} |K_i^o|^{\frac{1}{n}}} \right)
\]
\[
= -\log \left( \frac{\tilde{V}(K_1, \ldots, K_n)}{\prod_{i=1}^{n} |K_i^o|^{\frac{1}{n}}} \right),
\]
where $\tilde{V}(K_1, \ldots, K_n)$ is the dual mixed volume introduced by Lutwak in [20].
(iii) If \( \alpha \to \infty \), then \( p = n \frac{1-\alpha}{\alpha} \to -n \) from the right. Therefore, by definition, 
\[
D_{\infty}(Q_K \| P_K) = \log \left( \sup_{x} \text{ess} \frac{q_K(x)}{p_K(x)} \right) = \log \left( \sup_{x} \text{ess} \frac{(x, N_K(x))^{n+1}|K^\circ|}{\kappa_K(x)|K|} \right).
\]
On the other hand
\[
\lim_{\alpha \to \infty} \left( \frac{a_{n \frac{1-\alpha}{\alpha}}(K)}{n|K|^{\alpha}|K^\circ|^{1-\alpha}} \right)^{\frac{1}{\alpha-1}} = \frac{|K^\circ|}{|K|} \lim_{\alpha \to \infty} \left( \frac{x, N_K(x))^{n+1|K^\circ|}}{\kappa_K(x)|K|} \right)_{L_{\infty}^1},
\]
which is thus consistent with the definition of \( D_{\infty}(Q_K \| P_K) \). Similarly, one shows that, if \( \alpha \to \infty \), then \( p = n \frac{1-\alpha}{\alpha} \to -n \) from the left. Hence, by definition, 
\[
D_{\infty}(P_K \| Q_K) = \log \left( \sup_{x} \text{ess} \frac{q_K(x)}{p_K(x)} \right) = \log \left( \sup_{x} \text{ess} \frac{\kappa_K(x)|K|}{(x, N_K(x))^{n+1|K^\circ|}} \right),
\]
which is consistent with \( \lim_{\alpha \to \infty} \left( \frac{a_{n \frac{1-\alpha}{\alpha}}(K)}{n|K|^{\alpha}|K^\circ|^{1-\alpha}} \right)^{\frac{1}{\alpha-1}} \).
Thus, also it would make sense to define
\[
\lim_{p \to -n^\alpha} a_s P(K) = \sup_{x \in \partial K} \frac{(x, N_K(x))^{n+1}}{\kappa_K(x)} \tag{31}
\]
and
\[
\lim_{p \to -n^-} a_s P(K) = \sup_{x \in \partial K} \frac{\kappa_K(x)(x, N_K(x))^{n+1}}{(x, N_K(x))^{n+1}}, \tag{32}
\]
which would imply that \( \lim_{p \to -n} a_s P(K) \) does not exist.
If \( \alpha \to -\infty \), then \( p = n \frac{1-\alpha}{\alpha} \to -n \) from the left and by (7), 
\[
D_{-\infty}(Q_K \| P_K) = -D_{\infty}(P_K \| Q_K).
\]
On the other hand,
\[
\lim_{\alpha \to \infty} \log \left( \frac{a_{n \frac{1-\alpha}{\alpha}}(K)}{n|K|^{\alpha}|K^\circ|^{1-\alpha}} \right)^{\frac{1}{\alpha-1}} = \log \left( \frac{1}{\sup_x \frac{\kappa_K(x)|K|}{(x, N_K(x))^{n+1|K^\circ|}}} \right) \]
\[
= -\log \left( \frac{\kappa_K(x)|K|}{(x, N_K(x))^{n+1|K^\circ|}} \right) \]
\[
= -D_{\infty}(P_K \| Q_K),
\]
hence this is also consistent with the definitions. Similar considerations hold for \( D_{-\infty}(P_K \| Q_K) \), 
\[
D_{\alpha}(P_{K_1} \times \cdots \times P_{K_n} \| Q_{K_1} \times \cdots \times Q_{K_n}) \quad \text{and} \quad D_{\alpha}(Q_{K_1} \times \cdots \times Q_{K_n} \| P_{K_1} \times \cdots \times P_{K_n}).
\]

Having identified \( L_{p^-} \)-affine surface areas as Rényi divergences, we can now translate known results from one theory to the other.

Affine invariance of \( L_{p^-} \)-affine surface areas translates into affine invariance of Rényi divergences: for all \( p \neq -n, a_s P(T(K)) = |\det T|^{\frac{p-\alpha}{p-\alpha}} a_s P(K) \) (see [40]). Theorem 2.4 then implies that for all linear maps \( T \) with \( \det T \neq 0 \), for all \( -\infty < \alpha < \infty, \alpha \neq 1 \),
\[
D_{\alpha}(P_{T(K)} \| Q_{T(K)}) = D_{\alpha}(P_K \| Q_K)
\]
and
\[
D_{\alpha}(Q_{T(K)} \| P_{T(K)}) = D_{\alpha}(Q_K \| P_K).
\]
The case \( \alpha = 1 \) was treated in [34].

As \( as_p(T(K_1), \ldots, T(K_n)) = |\det T|^\frac{n-p}{p+\alpha} as_p(K_1, \ldots, K_n) \) (see [47]), it follows from Theorem 2.5 that for all linear maps \( T \) with \( |\det T| \neq 0 \), for all \(-\infty < \alpha < \infty, \alpha \neq 1\)

\[
D_\alpha(P_T(K_1) \times \cdots \times P_T(K_n)) \| Q_T(K_1) \times \cdots \times Q_T(K_n))
\]

\[
= D_\alpha(P_{K_1} \times \cdots \times P_{K_n} \| Q_{K_1} \times \cdots \times Q_{K_n})
\]

and

\[
D_\alpha(Q_T(K_1) \times \cdots \times Q_T(K_n)) \| P_T(K_1) \times \cdots \times P_T(K_n))
\]

\[
= D_\alpha(Q_{K_1} \times \cdots \times Q_{K_n} \| P_{K_1} \times \cdots \times P_{K_n}).
\]

The case \( \alpha = 1 \) is in [34].

Moreover, all inequalities and results mentioned in e.g. [46] about \( L_p \)-affine surface area and in e.g. [47] about mixed \( L_p \)-affine surface area can be translated into the corresponding inequalities and results about Rényi divergences. Conversely, results about Rényi divergences from e.g. [43] have consequences for \( L_p \)-affine surface areas. We mention only a few.

**Proposition 2.6.** (i) Let \( K \) be a convex body in \( C^2_+ \). For all \(-\infty \leq \alpha \leq \infty\)

\[
(1 - \alpha)D_\alpha(Q_{K^\circ} \| P_{K^\circ}) = \alpha D_{1-\alpha}(Q_K \| P_K)
\]

and

\[
(1 - \alpha)D_\alpha(P_{K^\circ} \| Q_{K^\circ}) = \alpha D_{1-\alpha}(P_K \| Q_K).
\]

The equalities hold trivially if \( \alpha = 0 \) or \( \alpha = 1 \).

(ii) Let \( K_i, 1 \leq i \leq n, \) be convex bodies in \( C^2_+ \). Then for all \( 0 \leq \alpha \)

\[
as_p \left[ \frac{\alpha}{n} (K_1, \ldots, K_n) \right] = \int_{S^{n-1}} \prod_{i=1}^n \left[ f_{K_i} h_{K_i}^{\frac{1-\alpha}{\alpha}} \right]^{\frac{1-\alpha}{\alpha}} d\sigma
\]

\[
= \prod_{i=1}^n \int_{S^{n-1}} \left[ f_{K_i} h_{K_i}^{\frac{1-\alpha}{\alpha}} \right]^{\frac{1-\alpha}{\alpha}} d\sigma,
\]

i.e. we can interchange integration and product.

(iii) Let \( K \) and \( L \) be convex bodies in \( C^2_+ \). Let \( 0 \leq p \leq \infty \). Let \( 0 \leq \lambda \leq 1 \). Then

\[
\int_{S^{n-1}} \left[ \frac{f_K h_K}{|K|} + (1 - \lambda) \frac{f_L h_L}{|L|} \right]^\frac{n}{\lambda(\frac{n}{p} + \frac{n}{\lambda})} \left[ \frac{\lambda}{h_K |K^\circ|} + \frac{1 - \lambda}{h_L |L^\circ|} \right]^\frac{p}{\lambda(\frac{n}{p} + \frac{n}{\lambda})} d\sigma
\]

\[
\geq \left( \frac{as_p(K)}{|K| \left[ \frac{n}{\lambda(\frac{n}{p} + \frac{n}{\lambda})} \right]} \right)^\lambda \left( \frac{as_p(L)}{|L| \left[ \frac{n}{\lambda(\frac{n}{p} + \frac{n}{\lambda})} \right]} \right)^{1-\lambda}
\]

with equality iff \( K = L \). Equality holds trivially if \( p = 0 \) or \( p = \infty \) or \( \lambda = 0 \) or \( \lambda = 1 \).

**Proof.**

(i) For \(-\infty < \alpha < \infty\), (i) follows from the duality formula \( as_p(K) = as_{\frac{p}{n+p}}(K^\circ) \), or, formulated in a more symmetric way, using the parameter \( \alpha = \frac{p}{n+p} \).
This identity was proved for $p > 0$ in [11] and – with a different proof – for all other $p$ in [46].

Let now $\alpha = \infty$. Then, on the one hand
\[
\lim_{\alpha \to \infty} \frac{1 - \alpha}{\alpha} D_\alpha(Q_K \parallel P_K) = -D_\infty(Q_K \parallel P_K) = - \log \sup_{x \in \partial K^\circ} \ess_{q \in P_K} q_K(x).
\]

On the other hand, by (16),
\[
D_{\infty}(Q_K \parallel P_K) = -D_\infty(P_K \parallel Q_K) = - \log \sup_{x \in \partial K} \ess_{q \in P_K} \frac{p_K(x)}{q_K(x)}.
\]

(33) equals (34), as (see [11]) for $x \in \partial K$, $y \in \partial K^\circ$ such that $\langle x, y \rangle = 1$,
\[
\langle y, N_{K^\circ}(y) \rangle \langle x, N_K(x) \rangle = (\kappa_{K^\circ}(y)\kappa_K(x))^{\frac{1}{\alpha+1}}.
\]

Similarly, for $\alpha = -\infty$.

(ii) Follows from Theorem 2.5 and the fact that [43]
\[
D_\alpha(Q_{K_1} \times \cdots \times Q_{K_n} \parallel P_{K_1} \times \cdots \times P_{K_n}) = \sum_{i=1}^n D_\alpha(Q_{K_i} \parallel P_{K_i}),
\]
respectively the corresponding equation for $D_\alpha(P_{K_1} \times \cdots \times P_{K_n} \parallel Q_{K_1} \times \cdots \times Q_{K_n})$.

(iii) For $0 \leq \alpha \leq 1$, $D_\alpha(Q_K \parallel P_K)$, respectively $D_\alpha(P_K \parallel Q_K)$, are jointly convex [43]. We put $p = n \frac{1 - \alpha}{\alpha}$ respectively $p = n \frac{\alpha}{1 - \alpha}$ and use the joint convexity together with Theorem 2.4.

If $p \neq 0, \infty$ and $\lambda \neq 0, 1$, then equality implies that $K = L$ as the logarithm is strictly concave. \(\square\)

3. Geometric interpretations of Rényi divergences

In this section we present geometric interpretations of Rényi divergences $D_\alpha$ of convex bodies, for all $\alpha$. Geometric interpretations for the case $\alpha = 1$, the relative entropy, were given first in [34] in terms of $L_p$-centroid bodies. Recall that for a convex body $K$ in $\mathbb{R}^n$ of volume 1 and $1 \leq p \leq \infty$, the $L_p$-centroid body $Z_p(K)$ is this convex body that has support function
\[
h_{Z_p(K)}(\theta) = \left( \int_K |\langle x, \theta \rangle|^p dx \right)^{1/p}.
\]

Now that we observed that Rényi divergences are logarithms of $L_p$-affine surface areas, we can use their geometric characterizations to obtain the ones for Rényi divergences. We will mostly concentrate on the geometric characterization of $L_p$-affine surface areas via the surface bodies [40] and illumination surface bodies [47], though there are many more available (see e.g. [32,39,45,46]).

Even more is gained. First, we need not assume that the body is symmetric as in [34] nor that it has $C^2_+$ boundary as it was needed in [34], to obtain the desired geometric interpretation for $D_\alpha$ for all $\alpha$. Weaker regularity assumptions on the boundary suffice.

Second, in the context of the $L_p$-centroid bodies, the relative entropies appeared only after performing a second order expansion of certain expressions. Now, using the surface bodies or illumination surface bodies, already a first order expansion makes them appear. Thus, these bodies detect “faster” details of the boundary of a convex body than the $L_p$-centroid bodies.
Let $K$ be a convex body in $\mathbb{R}^n$. Let $f : \partial K \rightarrow \mathbb{R}$ be a nonnegative, integrable, function. Let $s \geq 0$.

The surface body $K_{f,s}$, introduced in [40], is the intersection of all closed half-spaces $H^+$ whose defining hyperplanes $H$ cut off a set of $f \mu_K$-measure less than or equal to $s$ from $\partial K$. More precisely,

$$K_{f,s} = \bigcap_{f \mu_K \leq s} H^+.$$  

The illumination surface body $K^{f,s}$ [47] is defined as

$$K^{f,s} = \{ x : \mu_f(\partial K \cap [x, K] \setminus K) \leq s \},$$

where $\mu_f$ is the measure defined for measurable sets $O$ of $\partial K$ by $\mu_f(O) = \int_O f \, d\mu_K$ and where for sets $A$ and $B$ (respectively points $x$ and $y$) in $\mathbb{R}^n$, $[A, B] = \{ \lambda a + (1 - \lambda)b : a \in A, b \in B, 0 \leq \lambda \leq 1 \}$ (respectively $[x, y] = \{ \lambda x + (1 - \lambda)y : 0 \leq \lambda \leq 1 \}$) is the convex hull of $A$ and $B$ (respectively $x$ and $y$).

For $x \in \partial K$ and $s > 0$ and $f$ and $K_{f,s}$ as above, we put

$$x_s = [0, x] \cap \partial K_{f,s}.$$  

The minimal function $M_f : \partial K \rightarrow \mathbb{R}$

$$M_f(x) = \inf_{0 < s} \frac{\int_{\partial K \cap H^-(x_s, N_{K_{f,s}}(x_s))} f \, d\mu_K}{\text{vol}_{n-1}(\partial K \cap H^-(x_s, N_{K_{f,s}}(x_s)))}$$

was introduced in [40]. $H(x, \xi)$ is the hyperplane through $x$ and orthogonal to $\xi$. $H^-(x, \xi)$ is the closed halfspace containing the point $x + \xi$. $H^+(x, \xi)$ the other halfspace.

For $x \in \partial K$, we define $r(x)$ as the maximum of all real numbers $\rho$ so that $B_2^n(x - \rho N_K(x), \rho) \subseteq K$. Then we formulate an integrability condition for the minimal function

$$\int_{\partial K} \frac{d\mu_K(x)}{(M_f(x))^{\frac{n-1}{2}}} r(x) < \infty. \quad (36)$$

The following theorem was proved in [40].

**Theorem 3.1.** Let $K$ be a convex body in $\mathbb{R}^n$. Suppose that $f : \partial K \rightarrow \mathbb{R}$ is an integrable, almost everywhere strictly positive function that satisfies the integrability condition (36). Then

$$c_n \lim_{s \rightarrow 0} \frac{|K| - |K_{f,s}|}{s^{\frac{n-1}{2}}} = \int_{\partial K} \frac{\kappa_{K}^{\frac{n-1}{2}}}{f^{\frac{n-1}{2}}} d\mu_K,$$

where $c_n = 2|B_2^n|^{\frac{n-1}{2}}$.

**Theorem 3.1** was used in [40] to give geometric interpretations of the $L_p$-affine surface area. Now we use this theorem to give geometric interpretations of Rényi divergence of order $\alpha$ for all $\alpha$ for cone measures of convex bodies. First we treat the case $\alpha \neq 1$.

**Corollary 3.2.** Let $K$ be a convex body in $\mathbb{R}^n$. 

For $-\infty \leq p \leq \infty$, $p \neq -n$, let $f_p : \partial K \to \mathbb{R}$ be defined as

$$f_p(x) = \frac{\langle x, N_K(x) \rangle}{\kappa_K(x)^{\frac{n-p}{2(n+p)}}}. $$

If $f_p$ is almost everywhere strictly positive and satisfies the integrability condition (36), then

$$\frac{c_n}{n|K|^{\frac{n-p}{2(n+p)}}|K^o|^p} \lim_{s \to 0} \frac{|K| - |K_{f_p,s}|}{s^\frac{2}{n-1}} = \exp \left( -\frac{p}{n+p} D_{\frac{n}{n+p}}(Q_K \| P_K) \right),$$

and, provided $p \neq \pm \infty$,

$$\frac{c_n}{n|K|^{\frac{n-p}{2(n+p)}}|K^o|^p} \lim_{s \to 0} \frac{|K| - |K_{f_p,s}|}{s^\frac{2}{n-1}} = \exp \left( -\frac{n}{n+p} D_{\frac{p}{n+p}}(P_K \| Q_K) \right).$$

If $K$ is in $C_2^2$, the last equation also holds for $p = \pm \infty$. 

**Proof.** The proof of the corollary follows immediately from Theorems 3.1 and 2.4.

The next corollary treats the case $\alpha = 1$. There, we need to make additional regularity assumptions on the boundary of $K$. Those are weaker though than $C_2^2$. \(\square\)

**Corollary 3.3.** Let $K$ be a convex body in $\mathbb{R}^n$. Assume that $K$ is such that there are $0 < r \leq R < \infty$ so that for all $x \in \partial K$

$$B_2^n(x - rN_K(x), r) \subset K \subset B_2^n(x - RN_K(x), R). \quad (37)$$

Let $f_{PQ} : \partial K \to \mathbb{R}$ and $f_{QP} : \partial K \to \mathbb{R}$ be defined by

$$f_{PQ}(x) = \frac{(n|K^o|\langle x, N_K(x) \rangle)^{\frac{n-1}{2}}}{\kappa_K(x)^{\frac{n-2}{2}}} \left( \log \left( \frac{R^{2n}|K| \kappa_K(x)}{r^{2n}|K^o| \langle x, N_K(x) \rangle^{n+1}} \right) \right)^{-\frac{n-1}{2}},$$

$$f_{QP}(x) = \left( \frac{n|K|}{\langle x, N_K(x) \rangle} \right)^{\frac{n-1}{2}} \kappa_K(x)^\frac{1}{2} \left( \log \left( \frac{R^{2n}|K^o| \langle x, N_K(x) \rangle^{n+1}}{r^{2n}|K| \kappa_K(x)} \right) \right)^{-\frac{n-1}{2}}.$$

Then $f_{PQ}$ and $f_{QP}$ are almost everywhere strictly positive, satisfy the integrability condition (36) and

$$c_n \lim_{s \to 0} \frac{|K| - |K_{f_{PQ},s}|}{s^\frac{2}{n-1}} = D_{KL}(P_K \| Q_K) + 2 \log \left( \frac{R}{r} \right) \frac{a_{s+\infty}(K)}{|K^o|}.$$

If $K$ is in $C_2^2$, then this equals $D_{KL}(N_K N_K^{-1}(cm_K \| cm_K^o) + 2n \log \left( \frac{R}{r} \right))$. 

$$c_n \lim_{s \to 0} \frac{|K| - |K_{f_{QP},s}|}{s^\frac{2}{n-1}} = D_{KL}(Q_K \| P_K) + 2n \log \left( \frac{R}{r} \right).$$

If $K$ is in $C_2^2$, then this is equal to $D_{KL}(N_K N_K^{-1}(cm_K \| cm_K^o) + 2n \log \left( \frac{R}{r} \right))$.

**Proof.** Note that $r = R$ iff $K$ is a Euclidean ball with radius $r$. Then the right hand sides of the identities in the corollary are equal to $0$ and $f_{PQ}$ and $f_{QP}$ are identically equal to $\infty$. Therefore, for all $s \geq 0$, $K_{f_{PQ},s} = K$ and $K_{f_{QP},s} = K$ and hence for all $s \geq 0$, $|K| - |K_{f_{PQ},s}| = 0$ and $|K| - |K_{f_{QP},s}| = 0$. Therefore, the corollary holds trivially in this case.
Assume now that \( r < R \), then
\[
1 \leq \frac{R^{2n} |K| \kappa_K(x)}{r^{2n} |K^\circ| (x, N_K(x))^{n+1}} \leq \left( \frac{R}{r} \right)^{4n},
\]
and we get for all \( x \in \partial K \) that
\[
f_{PQ}(x) \geq \left( \frac{|K^\circ|r^{n-1}}{2 \log \left( \frac{R}{r} \right)} \right)^{\frac{n-1}{2}} > 0.
\]
Also, for all \( x \in \partial K \),
\[
\left| K^\circ \right| r^{n-1} 2 \log \left( \frac{R}{r} \right)^{\frac{n-1}{2}} \leq M_{f_{PQ}}(x) \leq \infty
\]
and therefore \( f_{PQ} \) satisfies the integrability condition (36). The proof of the corollary then follows immediately from Theorem 3.1.

Similarly for \( f_{QP} \).

If \( K \) is in \( C^2_+ \), condition (37), holds. We can take
\[
r = \inf_{x \in \partial K} \min_{1 \leq i \leq n-1} r_i(x) \quad \text{and} \quad R = \sup_{x \in \partial K} \max_{1 \leq i \leq n-1} r_i(x),
\]
where for \( x \in \partial K, r_i(x), 1 \leq i \leq n-1 \) are the principal radii of curvature.

For convex bodies \( K \) and \( K_i, i = 1, \ldots, n \), define
\[
f_r(N_K^{-1}(u)) = f_K(u) \left( \frac{|K^\circ|}{2 \log \left( \frac{R}{r} \right)} \right)^{\frac{n-1}{2}} \left[ f_p(K_1, u) \cdots f_p(K_n, u) \right]^{\frac{1-p}{2p(n+1)}},
\]
where \( f_p(K, u) = h_K(u)^{1-p} f_K(u) \). \( \square \)

**Corollary 3.4.** Let \( K \) and \( K_i, i = 1, \ldots, n \), be convex bodies in \( C^2_+ \). Then
\[
\frac{c_n}{n} \lim_{s \to 0} \left( \frac{\|K| - |K_{\bar{f}_s}\|}{s^{\frac{n}{2}}} \right) = \text{Exp} \left( -\frac{n}{n + p} D_{\frac{n}{n + p}} (P_{K_1} \times \cdots \times P_{K_n} || Q_{K_1} \times \cdots \times Q_{K_n}) \right),
\]
and
\[
\frac{c_n}{n} \lim_{s \to 0} \left( \frac{\|K| - |K_{\bar{f}_s}\|}{s^{\frac{n}{2}}} \right) = \text{Exp} \left( -\frac{p}{n + p} D_{\frac{n}{n + p}} (Q_{K_1} \times \cdots \times Q_{K_n} || P_{K_1} \times \cdots \times P_{K_n}) \right).
\]

**Proof.** Again, the proof follows immediately from Theorems 2.5 and 3.1. \( \square \)

**Remark.** It was shown in [47] that for a convex body \( K \) in \( \mathbb{R}^n \) with \( C^2_+ \)-boundary
\[
\lim_{s \to 0} \frac{c_n |K_{\bar{f}_s}| - |K|}{s^{\frac{n}{2}}} = \int_{\partial K} \frac{\kappa_K(x)^{\frac{1}{n-1}}}{f(x)^{\frac{n}{2}}} d\mu_K(x),
\]
(39)
where \( c_n = 2|B_2^{n-1}|^{\frac{2}{n-1}} \) and \( f : \partial K \to \mathbb{R} \) is an integrable function such that \( f \geq c \mu_K \) almost everywhere. \( c > 0 \) is a constant. Using (39), similar geometric interpretations of Rényi divergence can be obtained via the illumination surface body instead of the surface body. We can use the same functions as in Corollaries 3.2–3.4. We will also have to assume that \( K \) is in \( C^2_+ \).

In [34], the following new affine invariants \( \Omega_K \) were introduced and their relation to the relative entropies were established. Let \( K, K_1, \ldots, K_n \) be convex bodies in \( \mathbb{R}^n \), all with centroid at the origin. Then

\[
\Omega_K = \lim_{p \to \infty} \left( \frac{as_p(K)}{n|K^\circ|} \right)^{n+p}
\]

and

\[
\Omega_{K_1,\ldots,K_n} = \lim_{p \to \infty} \left( \frac{as_p(K_1,\ldots,K_n)}{as_\infty(K_1,\ldots,K_n)} \right)^{n+p}.
\]

It was proved in [34] that for a convex body \( K \) in \( \mathbb{R}^n \) that is \( C^2_+ \)

\[
D_{KL}(P_K \| Q_K) = \log \left( \frac{|K|}{|K^\circ|} \Omega_K \right)^{\frac{1}{n}}
\]

and

\[
D_{KL}(Q_K \| P_K) = \log \left( \frac{|K^\circ|}{|K|} \Omega_K \right)^{\frac{1}{n}}.
\]

Note that Eq. (40) also followed from (26). Similar results hold for \( \Omega_{K_1,\ldots,K_n} \). We now concentrate on \( \Omega_K \). As shown in [34], these invariants can also be obtained as

\[
\Omega_K^{\frac{1}{n}} = \lim_{p \to 0} \left( \frac{as_p(K)}{n|K^\circ|} \right)^{\frac{n+p}{p}}
\]

and thus, denoting by \( A_K = \lim_{p \to 0} \left( \frac{as_p(K)}{n|K|} \right)^{\frac{n+p}{p}} \), \( \Omega_K^{\frac{1}{n}} = A_K^\circ \). This implies e.g. that

\[
\lim_{p \to 0} \left( \frac{as_p(K)^n}{as_{\frac{1}{p}}(K)^{\frac{1}{p}}} \frac{1}{n^n|K|^n} \right)^{\frac{1}{p^n}} = 1.
\]

Geometric interpretations in terms of \( L_p \)-centroid bodies were given in [34] for the new affine invariants \( \Omega_K \). These interpretations are in the spirit of Corollaries 3.2–3.4: as \( p \to \infty \), appropriately chosen volume differences of \( K \) and its \( L_p \)-centroid bodies make the quantity \( \Omega_K \) appear.

Again, however, with the \( L_p \)-centroid bodies, only symmetric convex bodies in \( C^2_+ \) could be handled and it was needed to go to a second order expansion for the volume differences.

Now, it follows from Corollary 3.3 that there exist such interpretations for \( \Omega_K \) also for non-symmetric convex bodies and under weaker smoothness assumptions than \( C^2_+ \).

Moreover, again already a first order expansion gives such geometric interpretations if one uses the surface bodies or the illumination surface bodies instead of the \( L_p \)-centroid bodies.
Corollary 3.5. Let $K$ be a convex body in $\mathbb{R}^n$ such that 0 is the center of gravity of $K$ and such that $K$ satisfies (37) of Corollary 3.3. Let $f_{PQ}: \partial K \to \mathbb{R}$ and $f_{QP}: \partial K \to \mathbb{R}$ be as in Corollary 3.3. Then

$$c_n \lim_{s \to 0} \frac{|K| - |K f_{PQ,s}|}{s^{2/n-1}} - 2 \log \left( \frac{R}{r} \right) \frac{a_{s^{1/n}}(K)}{|K^o|} = \log \left( \frac{|K|}{|K^o|} \beta_{K^o} \right)$$

$$= \log \left( \frac{|K|}{|K^o|} A_{K^o}^{-1} \right)$$

and

$$c_n \lim_{s \to 0} \frac{|K| - |K f_{QP,s}|}{s^{2/n-1}} - 2n \log \left( \frac{R}{r} \right) = \log \left( \frac{|K^o|}{|K|} \Omega_{K^o}^{-1/2} \right) = \log \left( \frac{|K^o|}{|K|} A_{K^o}^{-1} \right).$$

**Proof.** The proof of the corollary follows immediately from Corollary 3.3, (40), (41) and the definition of $A_K$. \(\square\)

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**References**


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