Resolvable even cycle decompositions of the tensor product of complete graphs

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In this paper, we consider resolvable k-cycle decompositions (for short, k-RCD) of $K_m \times K_n$, where $\times$ denotes the tensor product of graphs. It has been proved that the standard necessary conditions for the existence of a k-RCD of $K_m \times K_n$ are sufficient when k is even.

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1. Introduction

All graphs considered here are simple and finite. Let $C_k$ (resp. $P_k$), denote the cycle (resp. path) on k vertices. For two graphs $G$ and $H$ their wreath product $G \ast H$ has vertex set $V(G) \times V(H)$ in which $(g_1, h_1)$ and $(g_2, h_2)$ are adjacent whenever $g_1g_2 \in E(G)$ or $g_1 = g_2$ and $h_1h_2 \in E(H)$. Similarly, $G \times H$, the tensor product of the graph $G$ and $H$ has vertex set $V(G) \times V(H)$ in which two vertices $(g_1, h_1)$ and $(g_2, h_2)$ are adjacent whenever $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$; see Fig. 1. It is clear that $(K_m \ast K_n) - nK_m \cong K_m \times K_n$, where $nK_m$ denotes n disjoint copies of $K_m$. Clearly, the tensor product is commutative and distributive over edge disjoint union of graphs, that is, if $G = H_1 \oplus H_2 \oplus \cdots \oplus H_n$, then $G \times H = (H_1 \times H) \oplus (H_2 \times H) \oplus \cdots \oplus (H_n \times H)$. If $G$ is a bipartite graph with bipartition $(X, Y)$, where $X = \{x_0, x_1, \ldots, x_{m-1}\}$, $Y = \{y_0, y_1, \ldots, y_{n-1}\}$ and if $G$ contains the set of edges $F_i(X, Y) = \{xy_j | 0 \leq j \leq n-1\}$, where addition in the subscript is taken modulo n), then $G$ has the 1-factor of jump $j$ from $X$ to $Y$. Clearly, if $G = K_{m,n}$, then $E(G) = \bigcup_{i=0}^{n-1} F_i(X, Y)$. A word of caution! Note that $F_i(X, Y) = F_{m-i}(Y, X)$, $0 \leq i \leq n-1$, where we assume $F_{n}(Y, X) = F_{0}(Y, X)$.

Let $G$ and $H$ be simple graphs with vertex sets $V(G) = \{x_0, x_1, \ldots, x_{m-1}\}$ and $V(H) = \{y_0, y_1, \ldots, y_{n-1}\}$. Then $V(G \times H) = V(G) \times V(H)$. For our convenience, we write $V(G) \times V(H) = \bigcup_{i=0}^{m-1} X_i$, where $X_i$ stands for $\{x_i\} \times V(H)$. Further, in what follows, we shall denote the vertices of $X_i$, $0 \leq i \leq m - 1$, by $\{x_i^j | 0 \leq j \leq n-1\}$, where $x_i^j$ stands for the vertex $(x_i, y_j)$. We shall call $X_i$, the $i$th layer of $G \times H$: see Fig. 1. It is clear that $G \times H$ is an $m$-partite graph with parts $X_0, X_1, \ldots, X_{m-1}$; it can also be considered as an $n$-partite graph with parts $Y_0, Y_1, \ldots, Y_{n-1}$, where $Y_i = V(G \times y_i)$. Further, we shall call $Y_i = \{y_i^j | 0 \leq i \leq m - 1, 0 \leq j \leq n - 1\}$, the $j$th column of $G \times H$: see Fig. 1. For terms not defined here, see [5, 6]. The subgraph induced by $S \subseteq V(G)$ is denoted by $\langle S \rangle$. Similarly, the subgraph induced by $E_i \subseteq E(G)$ is denoted by $\langle E_i \rangle$.

For a graph $G$, if $E(G)$ can be partitioned into $E_1, E_2, \ldots, E_k$ such that $\langle E_i \rangle \cong H$, for all $i, 1 \leq i \leq k$, then we say that $H$ decomposes $G$, or that an $H$-decomposition of $G$, denoted by $H \mid G$, takes place. If the edge set of $G$ can be partitioned into edge disjoint cycles of length $k$, then we write $C_k \mid G$, and in this case we say that $G$ has a $C_k$-decomposition. A k-factor of $G$ is a k-regular spanning subgraph. A k-factorization of a graph $G$ is a partition of the edge set of $G$ into k-factors. A $C_k$-factor of a graph is a 2-factor in which each component is a cycle of length $k$. A resolvable k-cycle decomposition (for short,
For any odd integer \( m \geq 3 \) and any even integer \( k \geq 4 \), \( C_k \parallel C_k \times K_m \).

2. Resolvable even cycle decompositions of \( K_m \times K_n \)

In this section we prove that the obvious necessary conditions for the existence of a \( k \)-RCD are sufficient when \( k \) is even.

**Lemma 2.1.** For any odd integer \( m \geq 3 \) and any even integer \( k \geq 4 \), \( C_k \parallel C_k \times K_m \).
For the construction of $C^{k+2}$ from $C^k$, bold edges of Figure 2(a) are deleted and in the resulting graph we add two vertices $x_k$, $x_{k+1}$ and the five bold edges in Figure 2(b).

**Theorem 2.1** ([2] Walecki's Hamilton Cycle Decomposition). The complete graph $K_n$ is Hamilton cycle decomposable for all $n \geq 3$.

**Proof.** Let $V(C_k) = \{x_0, x_1, \ldots, x_{k-1}\}$ and let $X_i = x_i \times V(K_m) = \{x_i^0, x_i^1, \ldots, x_i^{m-1}\}$, $0 \leq i \leq k - 1$, be the vertices of $C_k \times K_m$.

For $1 \leq i \leq m - 1$, we obtain a $C_k$-factor of $C_k \times K_m$ as follows:

$$G_i = \bigcup_{j=0}^{k-1} \{F_j(x_{2j}, x_{2j+1})\} \cup \bigcup_{j=0}^{k-1} \{F_{m-j}(x_{2j+1}, x_{2j+2})\},$$

where the subscripts of $X$ are taken modulo $k$. Clearly, $G_i$, $1 \leq i \leq m - 1$, is a $C_k$-factor of $C_k \times K_m$. Thus $C_k \| C_k \times K_m$. □

For the sake of completeness we give the proof of the following **Theorem 2.1**, which can be seen in [2].

**Remark 2.1.** For an even integer $k \geq 4$, we define $G^k$ and explain how to construct $G^{k+2}$ from $G^k$.

**Remark 2.1.** For an even integer $k \geq 4$, we define a cubic graph $G^k$; let $G^4 = K_4$ and for $k \geq 6$ it is defined as follows: let $V(G^k) = \{x_0, x_1, x_2, \ldots, x_{k-1}\}$ and $E(G^k) = F_1^k \cup F_2^k \cup F_3^k$, where $F_1^k = \bigcup_{i=0}^{k-1} \{x_{2i}x_{2i+1}\}$, $F_2^k = \bigcup_{i=0}^{k-1} \{x_{2i+1}x_{2i+2}\}$, $F_3^k = \bigcup_{i=0}^{k-2} \{x_{2i+1}x_{2i+4}\} \cup \{x_{3k-3x_{k-1}}\}$, where the subscripts of $x$ are taken modulo $k$. Clearly $F_1^k$, $F_2^k$ and $F_3^k$ are 1-factors of $G^k$. If $k = 4$, then $F_1^k = \{x_0x_1, x_2x_3\}$, $F_2^k = \{x_1x_2, x_3x_0\}$ and $F_3^k = \{x_0x_2, x_1x_3\}$.

It is easy to check that $G^k$ is a cubic graph which admits a perfect 1-factorization with 1-factors $F_1^k$, $F_2^k$ and $F_3^k$. Also, it is not difficult to check that $G^k$ is isomorphic to the union of the last Hamilton cycle $H_{k-2}$ and the 1-factor $F$ in the Walecki’s Hamilton cycle decomposition of $K_4$; see the proof of Theorem 2.1.

The graph $G^{k+2}$ can be constructed from $G^k$ by deleting two of its edges and adding two new vertices and five new edges as in Fig. 2. Define namely $V(G^{k+2}) = V(G^k) \cup \{x_k, x_{k+1}\}$ and $E(G^{k+2}) = F_1^{k+2} \cup F_2^{k+2} \cup F_3^{k+2}$, where

$$F_1^{k+2} = F_1^k \cup \{x_kx_{k+1}\}, \quad F_2^{k+2} = (F_2^k - \{x_kx_0\}) \cup \{x_{k+1}x_0, x_{k-1}x_k\} \quad \text{and}$$

$$F_3^{k+2} = (F_3^k - \{x_{k-3}x_{k-1}\}) \cup \{x_{k-3}x_k, x_{k-1}x_{k+1}\}$$

are 1-factors of $G^{k+2}$; see Fig. 2. Let the graph $G^k_j$, $1 \leq i < j \leq 3$, denote the 2-factor $F_i^k \cup F_j^k$ of $G^k$. The graph $G^{k+2}$ can also be defined from $G^k_j$ by adding some vertices and edges; see Fig. 3. □

As our proofs of the results in this section rely heavily on $G^k_j$, we often invoke the above **Remark 2.1**.
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\[ F \times K \]

Correspondingtothe2-factor

\[ G \]

Nextwedescribetheconstructionofa6-RCDof

\[ G \]

Let

\[ \text{Weinitiallyconstructa4-RCDanda6-RCDof} \]

and even integer k

\[ \text{and possibly for} \ (k, m) = (4, 3), \text{where} \]

\[ G \]

is the cubic graph defined in Remark 2.1.

**Proof.** We prove this lemma in two cases.

Case 1: \( m \equiv 1 \pmod{4} \).

We initially construct a 4-RCD and a 6-RCD of \( G^4 \times K_m \) and \( G^6 \times K_m \), respectively. Then for \( k \geq 6 \), we obtain, recursively, a \((k + 2)\)-RCD of \( G^{k+2} \times K_m \) from the \( k\)-RCD of \( G^k \times K_m \).

Let \( m = 4 \ell + 1 \) for some integer \( \ell \geq 1 \). First we prove this result when \( k = 4 \), that is, we find a 4-RCD of the graph \( G^4 \times K_m \). In this case, by definition, \( G^4 \cong K_4 \). In what follows, \( F^4_{2}(X_i, X_j) \) denotes the 1-factor of jump \( t \) from \( X_i \) to \( X_j \) in the subgraph \( (X_i \cup X_j) \) of the graph \( G^4 \times K_m \), where \( X_i = (X_i \times V(K_m)) \) is the \( i \)th layer of the graph.

Corresponding to the 2-factor \( G_{12i}^4 \) of \( G^4 \), we obtain \( 2\ell \) edge disjoint \( C_4 \)-factors, namely, \( G_{12i-1}^{4} \) and \( G_{12i}^{4} \), \( 1 \leq i \leq \ell \), of \( G^4 \times K_m \) as follows:

\[
G_{12i-1}^{4} = F_{4}^{i}(X_0, X_1) \oplus F_{m-i}^{4}(X_1, X_2) \oplus F_{4}^{4}(X_2, X_3) \oplus F_{m-i}^{4}(X_3, X_0), \\
G_{12i}^{4} = F_{m-i}^{4}(X_0, X_1) \oplus F_{4}^{i}(X_1, X_2) \oplus F_{m-i}^{4}(X_2, X_3) \oplus F_{i}^{4}(X_3, X_0).
\]

The graphs \( G_{12i-1}^{4} \) and \( G_{12i}^{4} \) are \( C_4 \)-factors of \( G^4 \times K_m \), since the g.c.d. of \( \sum j \) and \( m \) is \( m \), that is, \( (\sum j, m) = m \), where \( \sum j \) stands for the sum of the jumps of the 1-factors (between the layers, that is, \( F_{j}(X_i, X_s) \) for some \( r \) and \( s \)) that we have chosen for the construction of the 2-factors.

Corresponding to the 2-factor \( G_{23}^{4} \) of \( G^4 \), we describe \( 2\ell \) edge disjoint \( C_4 \)-factors, \( G_{23i-1}^{4} \) and \( G_{23i}^{4} \), \( 1 \leq i \leq \ell \), of \( G^4 \times K_m \) as follows:

\[
G_{23i-1}^{4} = F_{4}^{i}(X_0, X_2) \oplus F_{m-i}^{4}(X_2, X_1) \oplus F_{4}^{4}(X_1, X_3) \oplus F_{m-i}^{4}(X_3, X_0), \\
G_{23i}^{4} = F_{m-i}^{4}(X_0, X_2) \oplus F_{4}^{i}(X_2, X_1) \oplus F_{m-i}^{4}(X_1, X_3) \oplus F_{i}^{4}(X_3, X_0).
\]

Corresponding to the 2-factor \( G_{13}^{4} \) of \( G^4 \), we obtain \( 2\ell \) edge disjoint \( C_4 \)-factors, \( G_{13i-1}^{4} \) and \( G_{13i}^{4} \), \( 1 \leq i \leq \ell \), of \( G^4 \times K_m \) as follows:

\[
G_{13i-1}^{4} = F_{4}^{i}(X_0, X_1) \oplus F_{4}^{i}(X_1, X_3) \oplus F_{m-i}^{4}(X_3, X_2) \oplus F_{m-i}^{4}(X_2, X_0), \\
G_{13i}^{4} = F_{m-i}^{4}(X_0, X_1) \oplus F_{m-i}^{4}(X_1, X_3) \oplus F_{i}^{4}(X_3, X_2) \oplus F_{i}^{4}(X_2, X_0).
\]

Thus \( G_{12i}^{4} \), \( G_{23i}^{4} \), \( 1 \leq i \leq 2\ell \), together yields a 4-RCD of \( G^4 \times K_m \).

Next we describe the construction of a 6-RCD of \( G^6 \times K_m \).

Corresponding to the 2-factor \( G_{12}^{6} \) of \( G^6 \), we describe \( 2\ell \) edge disjoint \( C_6 \)-factors, \( G_{12i-1}^{6} \) and \( G_{12i}^{6} \), \( 1 \leq i \leq \ell \), of \( G^6 \times K_m \) as follows:

\[
G_{12i-1}^{6} = F_{6}^{i}(X_0, X_1) \oplus F_{m-i}^{6}(X_1, X_2) \oplus F_{6}^{i}(X_2, X_3) \oplus F_{m-i}^{6}(X_3, X_4) \oplus F_{m-i}^{6}(X_4, X_5) \oplus F_{m-i}^{6}(X_5, X_0), \\
G_{12i}^{6} = F_{m-i}^{6}(X_0, X_1) \oplus F_{i}^{6}(X_1, X_2) \oplus F_{m-i}^{6}(X_2, X_3) \oplus F_{i}^{6}(X_3, X_4) \oplus F_{m-i}^{6}(X_4, X_5) \oplus F_{i}^{6}(X_5, X_0).
\]
Corresponding to the 2-factor $G^k_{13}$, we obtain $2\ell$ edge disjoint $C$-factors, $G^k_{13,2i-1}$ and $G^k_{13,2i}$, $1 \leq i \leq \ell$, of $G^k \times K_m$ as follows:

$$G^k_{13,2i-1} = F^k_{m-\ell}(X_0, X_1) \oplus F^k_{m-\ell+1}(X_0, X_1) \oplus F^k_{m-\ell+2}(X_0, X_1) \oplus F^k_{m-\ell-1}(X_0, X_1) \oplus F^k_{m-\ell+1}(X_0, X_1) \oplus F^k_{m-\ell-1}(X_0, X_1),$$

$$G^k_{13,2i} = F^k_{m-\ell}(X_0, X_1) \oplus F^k_{m-\ell+1}(X_0, X_1) \oplus F^k_{m-\ell+2}(X_0, X_1) \oplus F^k_{m-\ell-1}(X_0, X_1) \oplus F^k_{m-\ell+1}(X_0, X_1) \oplus F^k_{m-\ell-1}(X_0, X_1).$$

Corresponding to the 2-factor $G^k_{23}$ of $G^6$, we obtain $2\ell$ edge disjoint $C$-factors, $G^k_{23,2i-1}$ and $G^k_{23,2i}$, $1 \leq i \leq \ell$, of $G^k \times K_m$ as follows:

$$G^k_{23,2i-1} = F^k_{m-\ell}(X_0, X_1) \oplus F^k_{m-\ell+1}(X_0, X_1) \oplus F^k_{m-\ell+2}(X_0, X_1) \oplus F^k_{m-\ell+1}(X_0, X_1) \oplus F^k_{m-\ell+1}(X_0, X_1) \oplus F^k_{m-\ell+1}(X_0, X_1).$$

The basic idea behind the proof of this lemma is to obtain a $(k + 2)$-RCD of $G^{k+2} \times K_m$ out of the $k$-RCD of the graph $G^k \times K_m$ for even $k \geq 6$. We start with the $k$-RCD of the graph $G^k \times K_m$ obtained above for $k = 6$. In what follows, we suppose that if the edge $x_i$ is common to both $G^k$ and $G^{k+2}$, then $F^k_i(x_i, Y_i) = F^{k+2}_i(x_i, Y_i), 1 \leq i \leq m - 1$, in the common subgraph $(X_i \cup Y_i)$ of the graphs $G^k \times K_m$ and $G^{k+2} \times K_m$. Using the following recursive construction we obtain a $(k + 2)$-RCD of $G^{k+2} \times K_m$ for all $k \geq 6$.

Corresponding to the 2-factor $G^k_{12}$ of $G^{k+2}$, we obtain $2\ell$ edge disjoint $C$-factors, namely, $G^{k+2}_{12,2i-1}$ and $G^{k+2}_{12,2i}$, $1 \leq i \leq \ell$, of $G^{k+2} \times K_m$ as follows:

$$G^{k+2}_{12,2i-1} = \{G^k_{12,2i-1} - F^k_{m-\ell}(X_0, X_1) \oplus F^{k+2}_{i}(X_0, X_1) \oplus F^{k+2}_{m-\ell}(X_0, X_1) \oplus F^{k+2}_{m-\ell+1}(X_0, X_1),$$

$$G^{k+2}_{12,2i} = \{G^k_{12,2i} - F^k_{m-\ell}(X_0, X_1) \oplus F^{k+2}_{m-\ell}(X_0, X_1) \oplus F^{k+2}_{m-\ell+1}(X_0, X_1) \oplus F^{k+2}_{m-\ell+1}(X_0, X_1).$$

Corresponding to the 2-factor $G^k_{23}$ of $G^{k+2}$, we obtain $2\ell$ edge disjoint $C$-factors, $G^{k+2}_{23,2i-1}$ and $G^{k+2}_{23,2i}$, $1 \leq i \leq \ell$, of $G^{k+2} \times K_m$ as follows:

$$G^{k+2}_{23,2i-1} = \{G^k_{23,2i-1} - F^k_{m-\ell}(X_0, X_1) \oplus F^{k+2}_{m-\ell}(X_0, X_1) \oplus F^{k+2}_{m-\ell+1}(X_0, X_1) \oplus F^{k+2}_{m-\ell+1}(X_0, X_1),$$

$$G^{k+2}_{23,2i} = \{G^k_{23,2i} - F^k_{m-\ell}(X_0, X_1) \oplus F^{k+2}_{m-\ell}(X_0, X_1) \oplus F^{k+2}_{m-\ell+1}(X_0, X_1) \oplus F^{k+2}_{m-\ell+1}(X_0, X_1).$$

Corresponding to the 2-factor $G^k_{13}$ of $G^{k+2}$, we obtain $2\ell$ edge disjoint $C$-factors, $G^{k+2}_{13,2i-1}$ and $G^{k+2}_{13,2i}$, $1 \leq i \leq \ell$, of $G^{k+2} \times K_m$ as described below:

$$G^{k+2}_{13,2i-1} = \{G^k_{13,2i-1} - H^k_{13,2i-1} \oplus H^{k+2}_{13,2i-1},$$

$$G^{k+2}_{13,2i} = \{G^k_{13,2i} - H^k_{13,2i} \oplus H^{k+2}_{13,2i},$$

where

$$H^k_{13,2i-1} = \{F^k_{m-\ell}(X_0, X_1) \oplus F^{k+2}_{m-\ell}(X_0, X_1) \oplus F^{k+2}_{m-\ell+1}(X_0, X_1) \oplus F^{k+2}_{m-\ell+1}(X_0, X_1),$$

$$H^k_{13,2i} = \{F^k_{m-\ell}(X_0, X_1) \oplus F^{k+2}_{m-\ell}(X_0, X_1) \oplus F^{k+2}_{m-\ell+1}(X_0, X_1) \oplus F^{k+2}_{m-\ell+1}(X_0, X_1).$$

Since the g.c.d. of $\sum j$ and $m$ is $m$, that is, $(\sum j, m) = m$, where $\sum j$ stands for the sum of the jumps of the 1-factors (between the layers, that is, $F^i(X_0, X_1)$ for some $r$ and $s$) that we have chosen for the construction of the 2-factors, it is straightforward to check that the 2-factors described above yield a $(k + 2)$-RCD of the graph $G^{k+2} \times K_m$ consisting of the 2-factors $G^{k+2}_{12j, i}$, $G^{k+2}_{23j, i}$ and $G^{k+2}_{13j, i}$, $1 \leq i \leq 2\ell$.

Case 2: $m \equiv 3 \pmod{4}$.

Subcase 2.1: $m = 4\ell + 3$ for some $\ell \geq 1$. 

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First we consider the case \( k = 4 \). Corresponding to the 2-factor \( G^4_{12} \) of \( G^4 \), we obtain \( 2\ell + 1 \) edge disjoint \( C_4 \)-factors, namely, \( G^4_{12,i}, 1 \leq i \leq 2\ell + 1 \), of the graph \( G^4 \times K_m \) as described below:

\[
G^4_{12,i} = F^4_i(X_0, X_1) \oplus F^4_{m-i}(X_1, X_2) \oplus F^4_i(X_2, X_3) \oplus F^4_{m-i}(X_3, X_0).
\]

Corresponding to the 2-factor \( G^4_{23} \) of \( G^4 \), we obtain \( 2\ell + 1 \) edge disjoint \( C_4 \)-factors of the graph \( G^4 \times K_m \) as follows:

For each \( i, 1 \leq i \leq \ell \), we obtain two edge disjoint \( C_4 \)-factors, namely, \( G^4_{23,2i-1}, G^4_{23,2i} \) and after getting these \( 2\ell \) \( C_4 \)-factors, finally we get one more \( C_4 \)-factor \( G^4_{23,2\ell+1} \) of the graph \( G^4 \times K_m \):

\[
G^4_{23,2i-1} = F^4_{i+i+1}(X_0, X_2) \oplus F^4_{m-i-1}(X_1, X_2) \oplus F^4_{m-i-1}(X_1, X_3) \oplus F^4_i(X_3, X_0)
\]

\[
G^4_{23,2i} = F^4_{m-i}(X_0, X_2) \oplus F^4_{m-i-1}(X_1, X_2) \oplus F^4_i(X_1, X_3) \oplus F^4_{m-i}(X_3, X_0)
\]

and

\[
G^4_{23,2\ell+1} = F^4_{m-i}(X_0, X_2) \oplus F^4_{m-i}(X_1, X_3) \oplus F^4_{m-i}(X_3, X_0).
\]

For \( 1 \leq i \leq \ell + 1 \), we define

\[
 j_i = \begin{cases} 
 i & \text{if } 1 \leq i \leq \ell + 1, \\
 i + 2\ell + 2 & \text{if } \ell + 2 \leq i \leq 2\ell, \\
 2\ell + 2 & \text{if } i = 2\ell + 1.
\end{cases}
\]

Corresponding to the 2-factor \( G^4_{13} \) of \( G^4 \), we obtain \( 2\ell + 1 \) edge disjoint \( C_4 \)-factors, \( G^4_{13,i}, 1 \leq i \leq 2\ell + 1 \), of \( G^4 \times K_m \) as described below:

\[
G^4_{13,i} = F^4_{m-i}(X_0, X_1) \oplus F^4_{m-i}(X_1, X_3) \oplus F^4_{m-i}(X_3, X_0).
\]

Thus \( G^4_{12,i}, G^4_{13,i} \) and \( G^4_{13,i} \), \( 1 \leq i \leq 2\ell + 1 \), constitute a 4-RCD of \( G^4 \times K_m \).

We proceed as in Case 1 and begin by constructing a 6-RCD of \( G^6 \times K_m \).

Corresponding to the 2-factor \( G^6_{12} \) of \( G^6 \), we obtain \( 2\ell + 1 \) edge disjoint \( C_6 \)-factors, namely, \( G^6_{12,i}, 1 \leq i \leq 2\ell + 1 \), of \( G^6 \times K_m \) as follows:

\[
G^6_{12,i} = F^6_i(X_0, X_1) \oplus F^6_{m-i}(X_1, X_2) \oplus F^6_{m-i}(X_2, X_3) \oplus F^6_{m-i}(X_3, X_4) \oplus F^6_{m-i}(X_4, X_5) \oplus F^6_{m-i}(X_5, X_0).
\]

Corresponding to the 2-factor \( G^6_{23} \) of \( G^6 \), we obtain \( 2\ell + 1 \) edge disjoint \( C_6 \)-factors, \( G^6_{23,2i-1} \) and \( G^6_{23,2i}, 1 \leq i \leq \ell \), and after getting these \( 2\ell \) \( C_6 \)-factors we get one more \( C_6 \)-factor \( G^6_{23,2\ell+1} \) of \( G^6 \times K_m \) as described below: for each \( i, 1 \leq i \leq \ell \),

\[
G^6_{23,2i-1} = F^6_{i+i+1}(X_0, X_2) \oplus F^6_{m-i}(X_1, X_2) \oplus F^6_{m-i}(X_1, X_4) \oplus F^6_{m-i}(X_4, X_5) \oplus F^6_{m-i}(X_3, X_5) \oplus F^6_{m-i}(X_5, X_0)
\]

\[
G^6_{23,2i} = F^6_{m-i}(X_0, X_2) \oplus F^6_{m-i}(X_1, X_2) \oplus F^6_{m-i}(X_1, X_4) \oplus F^6_{m-i}(X_4, X_5) \oplus F^6_{m-i}(X_3, X_5) \oplus F^6_{m-i}(X_5, X_0)
\]

\[
G^6_{23,2\ell+1} = F^6_{m-i}(X_0, X_2) \oplus F^6_{m-i}(X_1, X_2) \oplus F^6_{m-i}(X_1, X_4) \oplus F^6_{m-i}(X_5, X_3) \oplus F^6_{m-i}(X_3, X_5) \oplus F^6_{m-i}(X_5, X_0)
\]

Corresponding to the 2-factor \( G^6_{13} \) of \( G^6 \), we obtain \( 2\ell + 1 \) edge disjoint \( C_6 \)-factors, \( G^6_{13,i}, 1 \leq i \leq 2\ell + 1 \), of \( G^6 \times K_m \) as follows:

\[
G^6_{13,i} = F^6_{m-i}(X_0, X_1) \oplus F^6_{m-i}(X_1, X_4) \oplus F^6_{m-i}(X_4, X_5) \oplus F^6_{m-i}(X_3, X_2) \oplus F^6_{m-i}(X_2, X_0),
\]

where

\[
 j_i = \begin{cases} 
 i & \text{if } 1 \leq i \leq \ell + 1, \\
 i + 2\ell + 2 & \text{if } \ell + 2 \leq i \leq 2\ell, \\
 2\ell + 2 & \text{if } i = 2\ell + 1.
\end{cases}
\]

As in Case 1, we recursively construct a \((k+2)\)-RCD of \( G^{k+2} \times K_m \) out of a \( k \)-RCD of the graph \( G^k \times K_m \) for even \( k \). We start with the 6-RCD of the graph \( G^6 \times K_m \) obtained above. Having constructed a \( k \)-RCD of \( G^{k+2} \times K_m \), using the following recursive construction we obtain a \((k+2)\)-RCD of \( G^{k+2} \times K_m \) for all \( k \geq 6 \).

Corresponding to the 2-factor \( G^{k+2}_{12} \) of \( G^{k+2} \), we obtain \( 2\ell + 1 \) edge disjoint \( C_{k+2} \)-factors, namely, \( G^{k+2}_{12,i}, 1 \leq i \leq 2\ell + 1 \), of \( G^{k+2} \times K_m \) as described below:

\[
G^{k+2}_{12,i} = \{ F^k_{m-i}(X_{k-1}, X_0) \} \oplus F^{k+2}_{m-i}(X_{k+1}, X_0) \oplus F^{k+2}_{m-i}(X_{k-1}, X_k) \oplus F^{k+2}_{m-i}(X_k, X_{k+1}).
\]

Corresponding to the 2-factor \( G^{k+2}_{23} \) of \( G^{k+2} \), we obtain \( 2\ell + 1 \) edge disjoint \( C_{k+2} \)-factors of \( G^{k+2} \times K_m \) as follows:
For each $i$, $1 \leq i \leq \ell$, we obtain two edge disjoint $C_{k+2}$-factors, $G_{23,2i-1}^{k+2}$ and $G_{23,2i}$ of $G^{k+2} \times K_m$:

$$G_{23,2i-1}^{k+2} = \{G_{23,2i-1}^k - F_{m-2i-1}(X_{k-3}, X_{k-1}) - F_i(X_{k-1}, X_0) \} \oplus F_{m-2i+1}(X_{k-3}, X_k)$$

$$G_{23,2i}^{k+2} = \{G_{23,2i-1}^k - F_{m-2i}(X_{k-3}, X_{k-1}) - F_i(X_{k-1}, X_0) \} \oplus F_{m-2i+1}(X_{k-3}, X_k)$$

and

$$G_{23,2i+1}^{k+2} = \{G_{23,2i+1}^k - F_{m-2i}(X_{k-3}, X_{k-1}) - F_i(X_{k-1}, X_0) \} \oplus F_{m-2i-1}(X_{k-3}, X_k)$$

Corresponding to the 2-factor $G_{13}^{k+2}$ of $G^{k+2}$, we obtain $2\ell+1$ edge disjoint $C_{k+2}$-factors, $G_{13,i}^{k+2}$, $1 \leq i \leq 2\ell+1$, of $G^{k+2} \times K_m$ as follows:

$$G_{13,i}^{k+2} = \{G_{13,i}^k - H_{13,i}^k \} \oplus H_{13,i}^{k+2},$$

where

$$H_{13,i}^k = \begin{cases} F_k(X_{k-3}, X_{k-1}) & \text{if } k \equiv 0 \pmod{4} \\ F_{m-1}(X_{k-1}, X_{k-3}) & \text{if } k \equiv 2 \pmod{4} \end{cases}$$

and

$$H_{13,i}^{k+2} = \begin{cases} F_i(X_{k-3}, X_{k-1}) \oplus F_{m-1}(X_{k-1}, X_{k-3}) & \text{if } k \equiv 0 \pmod{4} \\ F_{m-1}(X_{k-1}, X_{k-3}) \oplus F_i(X_{k-3}, X_{k-1}) & \text{if } k \equiv 2 \pmod{4} \end{cases}$$

For the same reason given in Case 1, it is straightforward to check that the constructions of the 2-factors described above yield a $(k+2)$-RCD of the graph $G^{k+2} \times K_m$ consisting of the 2-factors $G_{12,i}^{k+2}, G_{23,i}^{k+2}, G_{13,i}^{k+2}$, $1 \leq i \leq 2\ell+1$.

**Subcase 2.2:** $m = 3$ and $k \geq 6$.

First we prove this result for $k = 6$. Let $G^6$ be the cubic graph defined in Remark 2.1.

$$G_{12}^6 = F_6(X_0, X_1) \oplus F_6(X_1, X_2) \oplus F_6(X_2, X_3) \oplus F_6(X_3, X_4) \oplus F_6(X_4, X_5) \oplus F_6(X_5, X_0)$$

$$G_{23}^6 = F_6(X_0, X_2) \oplus F_6(X_2, X_3) \oplus F_6(X_3, X_4) \oplus F_6(X_4, X_5) \oplus F_6(X_5, X_0)$$

$$G_{13}^6 = F_6(X_0, X_1) \oplus F_6(X_1, X_4) \oplus F_6(X_4, X_5) \oplus F_6(X_5, X_2) \oplus F_6(X_3, X_2) \oplus F_6(X_2, X_0)$$

Clearly, $G_{12}^6, G_{23}^6$ and $G_{13}^6$ form a 6-RCD of the graph $G^6 \times K_m$.

Having constructed a $k$-RCD of $G^k \times K_3$ for $k = 6$, we proceed as in Subcase 2.1 to obtain a $(k+2)$-RCD of $G^{k+2} \times K_3$ for all $k \geq 6$.

This completes the proof of the lemma. □

**Theorem 2.2.** For any odd integer $m \geq 3$ and for any even integer $k \geq 4$, we have $C_k \parallel K_m \times K_m$ except possibly for $(k, m) = (4, 3)$.

**Proof.** By Walecki’s Hamilton cycle decomposition, $K_k = C_k \oplus C_k \oplus \cdots \oplus C_k \oplus G^4$, where $G^4$ is the cubic graph defined in

$$\frac{1}{2} - \text{times}$$

**Remark 2.1.** Clearly, $K_k \times K_m = (C_k \times K_m) \oplus (C_k \times K_m) \oplus \cdots \oplus (C_k \times K_m) \oplus (G^6 \times K_m)$. Now apply Lemmas 2.1 and 2.2 to complete the proof. □

A resolvable modified group divisible design with block size 3 (for short, 3-RMGDD) [see [8]] is nothing but a 3-RCD of $K_m \times K_n$. We state the result in [8] about 3-RMGDD’s in terms of 3-RCD’s as follows.

**Theorem 2.3** ([8]). There exists a 3-RCD of $K_m \times K_n$ if and only if $m, n \geq 3$, $mn \equiv 0 \pmod{3}$ and either $m$ or $n$ is odd except when $(m, n) = (3, 6)$ or $(6, 3)$. □

**Lemma 2.3.** If $n \geq 3$ is any integer and $k \geq 3$, $m \geq 3$ are odd integers such that $k \mid m$, then $C_k \parallel K_m \times K_n$ except when $(k, m, n) = (3, 3, 6)$.

**Proof.** We assume $k \geq 5$, since the case $k = 3$ follows from Theorem 2.3. Since $m \equiv 0 \pmod{k}$ and $m$ is odd, we have $m \equiv k \pmod{2k}$. Hence $C_k \parallel K_m$, by Theorem 1.1, that is, $K_m = F_1 \oplus F_2 \oplus \cdots \oplus F_{m-1}$, where each $F_i$ is a $C_k$-factor. As the tensor product is distributive over edge disjoint subgraphs, $K_m \times K_n = (F_1 \times K_n) \oplus (F_2 \times K_n) \oplus \cdots \oplus (F_{m-1} \times K_n)$; further, as each $F_i$ is a $C_k$-factor, $F_i \times K_n = (C_k \times K_n) \oplus \cdots \oplus (C_k \times K_n)$. But $C_k \parallel (C_k \times K_n)$ for all odd integer $k \geq 5$, by Theorem 1.2. Hence $C_k \parallel (F_i \times K_n)$, $1 \leq i \leq \frac{m-1}{2}$, and therefore $C_k \parallel K_m \times K_n$. □
The following theorem is used in the proof of Theorem 2.5 given below.

**Theorem 2.4** ([16]). The graph $C_{2n+1} \times K_{2m}$ has a Hamilton cycle decomposition. □

**Theorem 2.5.** For $m$, $n \geq 3$ and even integer $k \geq 4$, $C_k \mid K_n \times K_m$ if and only if (1) either $m$ or $n$ is odd and (2) $k \mid mn$ except possibly for $(k, m) = (4, 3)$.

**Proof.** The necessity of the conditions follows from the standard divisibility relations. We deal with the sufficiency. As the tensor product is commutative, we assume that $m$ is odd. Since $k$ is even, $n$ must be even.

Case 1: $n \equiv 0 \pmod{k}$.

Subcase 1.1: $(k, n) \neq (6, 12)$.

Let $n = kt$ for some $t \in \mathbb{N}$. If $t = 1$, then the proof follows from Theorem 2.2 and hence we assume that $t > 1$. Consequently, $K_n = K_{kt} = H_1 \oplus H_2$, where $H_1$ consists of $t$ vertex disjoint copies of $K_2$ and $H_2 = K_t - E(H_1) = K_t \setminus \hat{K}$.

Hence, $K_n \times K_m = (H_1 \oplus H_2) \times K_m = (H_1 \times K_m) \oplus (H_2 \times K_m)$. The graph $H_1 \times K_m$ is the union of $t$ vertex disjoint copies of $K_2 \times K_m$ and $C_k \mid K_2 \times K_m$ by Theorem 2.2. Also $C_k \mid H_2$, by Theorem 1.3. Let us denote this $k$-RCD of $H_2$ by $\mathcal{S}$. To each $C_k$-factor in $\mathcal{S}$, we get the union of $t$ vertex disjoint copies of $C_k \times K_m$ in the graph $H_2 \times K_m$. By Lemma 2.1, $C_k \mid C_k \times K_m$. This completes the proof of this subcase.

Subcase 1.2: $(k, n) = (6, 12)$.

We have to show that $C_6 \mid (K_{12} \times K_{12})$. For that, we factorize $K_{12}$ into $4$ $C_6$-factors $H_1, H_2, H_3$ and $H_4$ and a cubic subgraph consisting of two components $G^6$ and $\hat{G}^6$ as follows: let the vertex set of $K_{12}$ be $V(K_{12}) = \{x_0, x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5\}$.

$$
H_1 = \{(x_0y_1, x_1x_2y_3y_5), (x_1y_2, x_2x_3y_4y_5)\} \\
H_2 = \{(x_0y_2, x_2x_3y_1y_4), (x_3y_0, x_3x_4y_1y_5)\} \\
H_3 = \{(x_0y_3, x_4x_5y_1y_2), (y_0y_1, y_1y_2y_3y_4)\} \\
H_4 = \{(x_0y_4, x_5x_1y_2y_3), (y_0y_5, y_3y_4y_5y_1)\}.
$$

Clearly, $K_{12} - \{\bigcup_{i=1}^{6} E(H_i)\}$ is a cubic graph whose components are $G^6$ and $\hat{G}^6$ having vertex sets $\{x_0, x_1, x_2, y_0, y_1, y_2\}$ and $\{x_1, x_2, x_3, y_1, y_2, y_3\}$, respectively. Clearly, $G^6 = F^6_1 \oplus F^6_2 \oplus F^6_3$, where $F^6_1 = \{x_0y_0, x_1y_1, x_2y_2, \}$, $F^6_2 = \{y_0x_1, x_3y_1, \}$, and $F^6_3 = \{x_3y_3, x_4y_3, x_5y_3, \}$. It is easy to see that for $H_1 = F^6_1 \oplus F^6_2$ and $\hat{G}^6 = F^6_1 \oplus F^6_3$, $1 \leq i \leq j \leq 3$, are cycles of length six each. Since, each $H_i$, $1 \leq i \leq 4$, contains only cycles of length $6$ and $C_6 \mid C_6 \times K_m$, by Lemma 2.1, $C_6 \mid H_1 \times K_m$. It can be checked that the graph $G^6$ is isomorphic to the graph defined in Remark 2.1 (see Fig. 4a) and $\hat{G}^6 \cong K_{1, 3}$. Therefore, $C_6 \mid G^6 \times K_m$, by (Subcase 2.2) of Lemma 2.2.

Now it remains to show that $C_6 \mid \hat{G}^6 \times K_m$. For that we relabel the vertices $x_3, x_4, y_3, y_4$ and $y_5$ of $V(\hat{G}^6)$ as $z_0, z_2, z_4, z_5$, and $z_2$, respectively; see Fig. 4b. Hence $\hat{F}^6_1 = \{z_0z_1, z_2z_3, z_4z_5\}$, $\hat{F}^6_2 = \{z_0z_5, z_2z_1, z_4z_3\}$, $\hat{F}^6_3 = \{z_0z_3, z_2z_5, z_4z_1\}$. Since $m$ is odd, we assume that $m = 2\ell + 1$.

For each $i$, $1 \leq i \leq \ell$, we obtain three $C_6$-factors of the graph $\hat{G}^6 \times K_m$, namely, $\hat{G}^6_{12, i}$, $\hat{G}^6_{23, i}$, and $\hat{G}^6_{13, i}$ as follows:

$$
\hat{G}^6_{12, i} = F^6_{\ell+i}(Z_0, Z_1) \oplus F^6_{\ell+i}(Z_1, Z_2) \oplus F^6_{m-2\ell+i}(Z_2, Z_3) \oplus F^6_{\ell+i}(Z_3, Z_4) \oplus F^6_{\ell+i}(Z_4, Z_5) \oplus F^6_{m-2\ell+i}(Z_5, Z_0) \\
\hat{G}^6_{23, i} = F^6_{\ell+i}(Z_0, Z_2) \oplus F^6_{\ell+i}(Z_1, Z_4) \oplus F^6_{m-2\ell+i}(Z_1, Z_2) \oplus F^6_{\ell+i}(Z_2, Z_3) \oplus F^6_{\ell+i}(Z_3, Z_4) \oplus F^6_{m-2\ell+i}(Z_4, Z_5) \\
\hat{G}^6_{13, i} = F^6_{\ell+i}(Z_0, Z_1) \oplus F^6_{m-2\ell+i}(Z_1, Z_4) \oplus F^6_{\ell+i}(Z_4, Z_5) \oplus F^6_{\ell+i}(Z_5, Z_2) \oplus F^6_{\ell+i}(Z_2, Z_3) \oplus F^6_{m-2\ell+i}(Z_3, Z_0).
$$

Clearly, $\hat{G}^6_{12, i}$, $\hat{G}^6_{23, i}$, and $\hat{G}^6_{13, i} \leq i \leq \ell$, is a 6-RCD of $\hat{G}^6 \times K_m$.

This completes the proof of this subcase.

Case 2: $n \not\equiv 0 \pmod{k}$.

Let $k = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}$ be the prime factorization of $k$. Since $k$ does not divide both $m$ and $n$, some prime factors of $k$ divide $m$ and some prime factors of $k$ divide $n$. Without loss of generality we may assume that $\beta_1 = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}$ divides $m$ and $\beta_2 = p_1^{a_1-\alpha_1} p_2^{a_2-\alpha_2} \cdots p_t^{a_t-\alpha_t}$ divides $n$; thus $m = \beta_1 r$ and $n = \beta_2 s$, where some $a_i$’s and $(\alpha_i - a_i)$’s may be equal to zero; $\beta_1$ is odd, as $m$ is odd and $\beta_2$ is even as $k = \beta_1 \beta_2$ is even.

First we partition the vertex set of $K_m$, into $s$ subsets each having $\beta_2$ vertices. Each of these $\beta_2$-subsets of $K_m$ gives rise to a copy of $K_m \times K_{\beta_2}$ in $K_m \times K_m$. Let the union of these $s$ vertex disjoint copies of $K_m \times K_{\beta_2}$ be $H_1$. Since $\beta_1$ and $m$ are odd and $\beta_1 \mid m$, we have $C_{\beta_1} \mid K_m$, by Theorem 1.1. Now $K_m \times K_{\beta_2} = \left(\bigoplus_{i=1}^{m-1} (\epsilon_i \times K_{\beta_2})\right) \times K_{\beta_2}$, where each $\epsilon_i$ is a $C_{\beta_1}$-factor of $K_m$. Consequently, $K_m \times K_{\beta_2} = \bigoplus_{i=1}^{m-1} (\epsilon_i \times K_{\beta_2})$; by Theorem 2.4, each of the graphs $\epsilon_i \times K_{\beta_2}$ admits a $C_{\beta_1}\beta_2$-factorization and hence we have $C_{\beta_1}\beta_2 \mid K_m \times K_{\beta_2}$. This proves that $H_1$ admits a $C_{\beta_1}\beta_2$-factorization, as each copy of $K_m \times K_{\beta_2}$ in $H_1$ admits a $C_{\beta_1}\beta_2$-factorization.
After deleting the edges of $H_1$ in $K_m \times K_n$ what remains is $H_2$, say. To each of the $\beta_2$ subsets of $K_m$ we have a $\beta_2$ subset of vertices in each of the layers of $H_2$. In $H_2$, identify each of these $\beta_2$ subsets of vertices in each of the layers of $H_2$ into a single vertex and join two of them by an edge if and only if the corresponding $\beta_2$ subsets induce a $K_{\beta_2,\beta_2}$ in $H_2$. Let the graph thus obtained from $H_2$ be $H'_2$. Then $H'_2 \cong K_m \times K_s$ (Here we can assume $s \geq 2$ since if $s = 1$, $K_m \times K_n \cong K_m \times K_{\beta_2}$ and consequently, $K_m \times K_n - E(H_1)$ is a graph without edges).

Using the graph $H'_2 \cong K_m \times K_s$, we obtain a $C_{\beta_1,\beta_2}$-factorization of $H_2$ to complete the proof of the theorem.

**Subcase 2.1:** $s$ is odd.

Since $m$ is odd and $\beta_1 \mid m$, $C_{\beta_1} \parallel K_m \times K_s$ by Lemma 2.3. When we lift back each $C_{\beta_1}$-factor of $K_m \times K_s$ to $H_2$, we get the union of $\frac{m}{\beta_1}$ vertex disjoint copies of the graph isomorphic to $C_{\beta_1} \ast K_{\beta_2}$ in the graph $H_2$. Now the relation $C_{\beta_1,\beta_2} \parallel C_{\beta_1} \ast K_{\beta_2}$ follows from Theorem 1.5.

**Subcase 2.2:** $s$ is even.

Since $m$ is odd and $\beta_1 \mid m$, $C_{\beta_1} \parallel K_m$ by Theorem 1.1. Let $\varphi_1, \varphi_2, \ldots, \varphi_{m-1}$ be the $C_{\beta_1}$-factorization of $K_m$. As $s$ is even, $K_s$ has a 1-factorization $\mathcal{F} = F_1 \oplus F_2 \oplus \cdots \oplus F_{s-1}$ and hence $K_m \times K_s = K_m \times K_s \cong (F_1 \times K_m) \oplus (F_2 \times K_m) \oplus \cdots \oplus (F_{s-1} \times K_m)$, where each $F_i \times K_m = (K_2 \times K_m) \oplus (K_2 \times K_m) \oplus \cdots \oplus (K_2 \times K_m)$. Now each $K_2 \times K_m = (K_2 \times \varphi_1) \oplus (K_2 \times \varphi_2) \oplus \cdots \oplus (K_2 \times \varphi_{m-1}) = \bigoplus_{i=1}^{m-1} (K_2 \times \varphi_i) = C_{2\beta_1} \oplus C_{2\beta_1} \oplus \cdots \oplus C_{2\beta_1}$, as each component of $\varphi_i$ is a $C_{\beta_1}$. Hence $C_{2\beta_1} \parallel K_m \times K_s$. When we lift back each $C_{2\beta_1}$-factor of $K_m \times K_s$ to $H_2$, we get the union of $\frac{m}{2\beta_1}$ vertex disjoint copies of the graph $C_{2\beta_1} \ast K_{\beta_2}$ in the graph $H_2$. Now by Theorem 1.4 we have $C_k \parallel C_{2\beta_1} \ast K_{\beta_2}$, as $2\beta_1 \mid k$.

This completes the proof of the theorem. □

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