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Note

A note on the least number of edges of 3-uniform hypergraphs with upper chromatic number 2[☆]

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Abstract

The upper chromatic number $\overline{\chi}(\mathscr{H})$ of a hypergraph $\mathscr{H}=(X,\mathscr{E})$ is the maximum number k for which there exists a partition of X into non-empty subsets $X=X_1\cup X_2\cup\cdots\cup X_k$ such that for each edge at least two vertices lie in one of the partite sets. We prove that for every $n\geqslant 3$ there exists a 3-uniform hypergraph with n vertices, upper chromatic number 2 and $\lceil n(n-2)/3\rceil$ edges which implies that a corresponding bound proved in [K. Diao, P. Zhao, H. Zhou, About the upper chromatic number of a co-hypergraph, Discrete Math. 220 (2000) 67–73] is best-possible.

Keywords: Hypergraph; Upper chromatic number; Coloring

1. Introduction

The classical coloring theory of graphs and hypergraphs deals with the problem of partitioning the vertex set subject to the restriction that every edge has vertices in at least two partite sets. The *chromatic number* of a hypergraph $\mathscr{H}=(X,\mathscr{E})$ is the minimum number k for which there exists a partition of X into k non-empty sets meeting this restriction. Since the maximum such number is trivially equal to the number of vertices, the main concern of classical coloring theory is to find the minimum such number.

It is very natural to consider partitions of the vertex set X of a hypergraph into non-empty subsets such that for each edge at least two vertices lie in one of the partite sets. Obviously, this leads to maximum vertex set partitions and many results on related colorings of hypergraphs can be found in [1–8]. We now repeat the exact definition of the upper chromatic number of a hypergraph.

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Definition 1.1. Let $\mathscr{H} = (X, \mathscr{E})$ be a hypergraph. The upper chromatic number $\overline{\chi}(\mathscr{H})$ of $\mathscr{H}$ is the maximum number k for which there exists a partition of X into non-empty sets $X = X_1 \cup X_2 \cup \cdots \cup X_k$ such that for each edge at least two vertices lie in one of the partite sets.

A hypergraph $\mathscr{H} = (X, \mathscr{E})$ is r-uniform if $|E| = r \geqslant 2$ for every $E \in \mathscr{E}$. Trivially, $\overline{\chi}(\mathscr{H}) \geqslant r - 1$ for every r-uniform hypergraph $\mathscr{H}$. In [3] it was proved that a 3-uniform hypergraph $\mathscr{H} = (X, \mathscr{E})$ with $|X| = n$ and $\overline{\chi}(\mathscr{H}) = 2$ has at least $\lceil n(n-2)/3 \rceil$ edges. In the present paper we prove that this lower bound is best-possible.

2. Result

We immediately proceed to our main result.

Theorem 2.1. *For each $n \geqslant 3$ there exists a 3-uniform hypergraph $\mathscr{H}_n = (X_n, \mathscr{E}_n)$ such that $\overline{\chi}(\mathscr{H}_n) = 2$, $|X_n| = n$ and $|\mathscr{E}_n| = \lceil n(n-2)/3 \rceil$.*

Proof. Let $X_n = \{1, 2, \ldots, n\}$. If $\mathscr{E}_3 = \{\{1, 2, 3\}\}$, $\mathscr{E}_4 = \{\{1, 2, 3\}, \{2, 3, 4\}, \{1, 3, 4\}\}$ and $\mathscr{E}_5 = \{\{1, 2, 3\}, \{1, 3, 4\}, \{1, 4, 5\}, \{2, 3, 5\}, \{2, 4, 5\}\}$, then it is easy to see that $\mathscr{H}_3$, $\mathscr{H}_4$ and $\mathscr{H}_5$ satisfy the desired properties of the theorem. We will recursively define $\mathscr{H}_{n+3}$ using $\mathscr{H}_n$ and consider two distinct cases.

Case 1: $n \geqslant 4$ and n is even.

Let $n = 2k$ and let $\mathscr{E}^1 = \bigcup_{i=1}^{k} \{\{2i-1, 2i, n+1\}\}$, $\mathscr{E}^2 = \bigcup_{i=1}^{k} \{\{2i-1, n+1, n+3\}\}$, $\mathscr{E}^3 = \bigcup_{i=1}^{k-1} \{\{2i, 2i+1, n+2\}\} \cup \{\{2k, 1, n+2\}\}$, $\mathscr{E}^4 = \bigcup_{i=1}^{k} \{\{2i, n+2, n+3\}\}$ and $\mathscr{E}_{n+3} = \mathscr{E}_n \cup \{\{n+1, n+2, n+3\}\} \cup \mathscr{E}^1 \cup \mathscr{E}^2 \cup \mathscr{E}^3 \cup \mathscr{E}^4$. Clearly, $|\mathscr{E}_{n+3}| = |\mathscr{E}_n| + 4k + 1 = \lceil n(n-2)/3 \rceil + 2n + 1 = \lceil (n+3)(n+1)/3 \rceil$. Now we assume that there is a partition of X_{n+3} into non-empty subsets A, B and C such that for each edge at least two vertices lie in one of the partite sets. Since $\overline{\chi}(\mathscr{H}_n) = 2$, we can assume that $A \subseteq \{n+1, n+2, n+3\}$.

If $A = \{n+1\}$, then the edge $\{n+1, n+2, n+3\}$ implies that $n+2$ and $n+3$ are in the same partite set, say B. Now the edges in $\mathscr{E}^2$ imply that $1, 3, \ldots, n-1 \in B$. Now the edges in $\mathscr{E}^1$ imply that $2, 4, \ldots, n \in B$. Hence $C = \emptyset$, which is a contradiction. Similarly, the two assumptions $A = \{n+2\}$ or $A = \{n+3\}$ lead to a contradiction.

If $A = \{n+1, n+2\}$, then the edges in $\mathscr{E}^2$ imply that $1, 3, \ldots, n-1$ and $n+3$ are in the same partite set, say B. Now the edges in $\mathscr{E}^4$ imply that $2, 4, \ldots, n \in B$. Hence $C = \emptyset$, which is a contradiction. Similarly, the two assumptions $A = \{n+1, n+3\}$ or $A = \{n+2, n+3\}$ lead to a contradiction.

If $A = \{n+1, n+2, n+3\}$, then the edges in $\mathscr{E}^1$ imply that for $1 \leqslant i \leqslant k$ the vertices $2i-1$ and $2i$ are in the same partite set and the edges in $\mathscr{E}^3$ imply that for $1 \leqslant i \leqslant k-1$ the vertices $2i+1$ and $2i$ are in the same partite set and the vertices 1 and $2k$ are in the same partite set. Hence B or C is empty, which is a contradiction.

Case 2: $n \geqslant 3$ and n is odd.

Let $n = 2k+1$ and let $\mathscr{E}^1 = \bigcup_{i=1}^{k} \{\{2i-1, 2i, n+1\}\}$, $\mathscr{E}^2 = \bigcup_{i=2}^{k+1} \{\{2i-1, n+1, n+3\}\}$, $\mathscr{E}^3 = \bigcup_{i=1}^{k} \{\{2i, 2i+1, n+2\}\}$, $\mathscr{E}^4 = \bigcup_{i=1}^{k} \{\{2i, n+2, n+3\}\}$ and $\mathscr{E}_{n+3} = \mathscr{E}_n \cup \{\{n+1, n+2, n+3\}, \{1, n+1, n+2\}, \{1, n, n+3\}\} \cup \mathscr{E}^1 \cup \mathscr{E}^2 \cup \mathscr{E}^3 \cup \mathscr{E}^4$. Clearly, $|\mathscr{E}_{n+3}| = |\mathscr{E}_n| + 4k + 3 = \lceil n(n-2)/3 \rceil + 2n + 1 = \lceil (n+3)(n+1)/3 \rceil$. Now we assume that there is a partition of X_{n+3} into non-empty subsets A, B and C such that for each edge at least two vertices lie in one of the partite sets. As above we can assume that $A \subseteq \{n+1, n+2, n+3\}$. Similar arguments as in the first case imply that either B or C is empty. This completes the proof. $\square$

In [3] the authors actually prove their lower bound on the number of edges for r-uniform hypergraphs with $r \geqslant 3$. For $r \geqslant 4$ it is still open whether this bound is best-possible.

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