Approximate solution for quadratic Riccati differential equation

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Abstract

The quadratic Riccati differential equations are a class of nonlinear differential equations of much importance, and play a significant role in many fields of applied science. This paper introduces an efficient method for solving the quadratic Riccati differential equation and the Riccati differential-difference equation. In this technique, the Bezier curves method is considered as an algorithm to find the approximate solution of the nonlinear Riccati equation. Some examples in different cases are given to demonstrate simplicity and efficiency of the proposed method.

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Keywords: Riccati differential equation; Approximate solutions; Bezier curve; Riccati differential-difference equation

1. Introduction

Riccati differential equations are a class of nonlinear differential equations of much importance, and play a significant role in many fields of applied science [1]. For example, a one-dimensional static Schrödinger equation [2] is closely related to the Riccati differential equation. The Riccati differential equation is named after the Italian nobleman Count Jacopo Francesco Riccati (1676–1754) [3]. The applications of this equation may be found not only in random processes, optimal control, and diffusion problems [1] but also in stochastic realization theory, optimal control, network synthesis and financial mathematics. Solitary wave solutions of a nonlinear partial differential equation can be expressed as a polynomial in two elementary functions satisfying a projective Riccati equation [4]. Such type of problems also arises in optimal control. Therefore, the problem has attracted much attention and has been studied by many authors. Recently, various iterative methods are employed for the numerical and analytical solution of functional equations such as Adomian’s decomposition method (ADM) [5,6], homotopy perturbation method (HPM) [7], variational iteration method (VIM) [8], and differential transform method (DTM) [4]. Liao [9] has shown that HPM equations are equivalent to HAM equations.

Two numerical techniques were presented for solving the solution of Riccati differential equation (see [10]). These methods used the cubic B-spline scaling functions and Chebyshev cardinal functions. The methods consisted of expanding the required approximate solution as the elements of cubic B-spline scaling function or Chebyshev cardinal functions. Using the operational matrix of derivative, they reduced the problem to a set of algebraic equations (see [10]).

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There are many papers and books dealing with the Bezier curves. Harada and Nakamae [11] and Nürnberger and Zeilfelder [12] used the Bezier control points in approximating data and functions. Zheng et al. [13] proposed the use of control points of the Bernstein-Bezier form for solving differential equations numerically and also Evrenosoglu and Somali [14] used this approach for solving singular perturbed two points boundary value problems. Also the Bezier control points method is used for solving delay differential equation (see [15]). Some other applications of the Bezier functions and control points are found in (see [16]). In the present work, we suggest a technique similar to the one which was used in [16] for solving Riccati differential equations with delay.

This study is organized as follows: In Section 2, the problem is described. In Section 3, the method is applied to a variety of examples to show efficiency and simplicity of the method. Finally, Section 4 will give a conclusion briefly.

2. Problem statement

Consider the nonlinear Riccati differential equation as the following form

\[ u'(t) = p(t) + q(t)u(t) + r(t)u^2(t), \quad t_0 \leq t \leq t_f, \]

\[ u(t_0) = \alpha, \]

where \( p(t), q(t) \) and \( r(t) \) are continuous, \( t_0, t_f \) and \( \alpha \) are arbitrary constants, and \( u(t) \) is unknown function.

Our strategy is using Bezier curves to approximate the solutions \( u(t) \) by \( v(t) \) and where \( v(t) \) is as given below. Define the Bezier polynomials of degree \( n \) that approximate respectively the actions of \( u(t) \) over the interval \([t_0, t_f]\) as follows:

\[ v(t) = \sum_{i=0}^{n} a_r B_r(t) \left( \frac{t - t_0}{h} \right), \]

where \( h = t_f - t_0 \), and

\[ B_r(t) = \binom{n}{r} \frac{1}{h^n} (t - t_0)^n (t_f - t)^n, \]

is the Bernstein polynomial of degree \( n \) over the interval \([t_0, t_f]\); \( a_r \) is the control points. By substituting \( v(t) \) in (2), for \( u(t) \) in (1), one may define \( R_1(t) \) for \( t \in [t_0, t_f] \) as follows:

\[ R_1(t) = v'(t) - \left( p(t) + q(t)v(t) + r(t)v^2(t) \right), \]

Ghomanjani et al. [16] proved the convergence of this method where \( n \to \infty \).

Now, we define the residual function over the interval \([t_0, t_f]\) as follows:

\[ R = \int_{t_0}^{t_f} \left( ||R_1(t)||^2 \right) dt, \]

where \( || \cdot || \) is the Euclidean norm. Our aim is to solve the following optimization problem over the interval \([t_0, t_f]\) to find the values \( a_r \), for \( r = 0, 1, \ldots, n \)

\[ \min R \]

\[ \text{s.t. } v(t_0) = \alpha. \]

The mathematical programming problem (5) can be solved by many subroutine algorithms. Here, we used Maple 12 to solve this optimization problem.

Remark 2.1. Now, the Bezier curves method is used for solving the Riccati type differential-difference equation

\[ s(t)u'(\beta_1 t + \mu_1) = p(t) + q(t)u(\beta_2 t + \mu_2) + r(t)u^2(\beta_3 t + \mu_3), \quad t_0 \leq t \leq t_f, \]

with the mixed condition

\[ \beta_4 u(t_0) + \beta_5 u(t_f) = \lambda, \]

where \( u(t) \) is an unknown function, \( s(t), p(t), q(t) \) and \( r(t) \) are the known functions defined on the interval \([t_0, t_f]\) and \( \beta_i \), for \( i = 1, 2, \ldots, 5, \mu_i \), for \( i = 1, 2, 3, t_0 \) and \( t_f \) are real constants.

3. Numerical examples

In this section, we give some computational results of numerical experiments with methods based on two preceding sections, to support our theoretical discussion.

Example 1. Consider the following quadratic Riccati differential equation taken from

\[ u'(t) = 16t^2 - 5 + 8tu(t) + u^2(t), \quad 0 \leq t \leq 1, \]

\[ u(0) = 1, \]

where the exact solution of above equation is \( u(t) = 1 - 4t \) (see [3]). With \( n = 7 \), the following approximated solution can be found

\[ u(t) = 1 - 4t - 1.029983988r^7 + 0.08239871800r^2 - 0.844526868r^3 + 3.986953553r^4 - 4.79972534t^5 + 3.60494394r^6. \]
The graphs of approximated and exact solution are plotted in Fig. 1. From Table 1 we can see that, if we choose \( n = 7 \), the errors of the present method are less than \( 10^{-4} \).

**Example 2.** Consider the following quadratic Riccati differential equation taken from (see [3])

\[
 u'(t) = 1 + 2u(t) - u^2(t), \quad 0 \leq t \leq 1,
\]

\[
u(0) = 0,
\]

where the exact solution of above equation was found to be of the form

\[
u(t) = 1 + \sqrt{2} \tanh \left( \sqrt{2}t + \frac{1}{2} \log \left( \frac{\sqrt{2} - 1}{\sqrt{2} + 1} \right) \right).
\]

With \( n = 7 \), the following approximated solution can be found

\[
u(t) = 0.8795413035t^7 - 3.611374466t^6
+ 0.4836486195 + 6.005471116t^5
- 4.458236528t^4 + 0.2583047925t^3
+ 1.733381250t + 0.8045502275t^2,
\]

if we choose \( n = 7 \), the errors of the presented method are less than \( 5 \times 10^{-4} \). The graphs of approximated and exact solution are plotted in Fig. 2. From Table 2 we can see that, if we choose \( n = 7 \), the errors of the present method are less than \( 10^{-4} \).

**Example 3.** Consider the following quadratic Riccati differential equation taken from

\[
u'(t) = e^t - e^{3t} + 2e^{2t}u(t) - e^tu^2(t), \quad 0 \leq t \leq 1,
\]

\[
u(0) = 1,
\]

where the exact solution of this equation is \( u(t) = e^t \) (see [3]). With \( n = 7 \), the following approximated solution can be found

\[
u(t) = 1 - 1.528850280t^7 + 5.35328989t^6
- 1.08724193 + 4.47620190t^4
- 7.117450800t^5 + 0.6223328200t^2 + t.
\]

The graphs of approximated and exact solution are plotted in Fig. 3. From Table 3 we can see that, if we choose \( n = 7 \), the errors of the present method are less than \( 10^{-4} \).

### Table 1
Error values of \( u(t) \) for **Example 1**.

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<thead>
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<th>( t )</th>
<th>( \text{Error of present method} )</th>
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<tr>
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</table>

### Table 2
Error values of \( u(t) \) for **Example 2**.

<table>
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<td>3.2516 \times 10^{-10}</td>
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</table>
**Example 4.** Consider the following quadratic Riccati differential equation taken from
\[ u'(t) = -u(t) + u^2(t), \quad 0 \leq t \leq 1, \]
\[ u(0) = \frac{1}{2}, \]
where the exact solution of above equation is \( u(t) = \frac{e^{-t}}{1 + e^{-t}} \) (see [3]). With \( n = 7 \), the following approximated solution can be found
\[ u(t) = 0.5000000003 - 0.25000000001t \\
+ 0.000508520726t^5 - 0.0026216025t^4 \\
+ 0.680799498 \times 10^{-5}t^2 + 0.0207594402t^3 \\
+ 0.288239261 \times 10^{-3}t^4, \]
if we choose \( n = 7 \), the errors of the presented method are less than \( 3 \times 10^{-8} \). The graphs of approximated and exact solution are plotted in Fig. 4.

**Example 5.** Let us consider the Riccati differential-difference equation given by
\[ u'(t + 2) - r^2u(2t - 3) - u^2(t - 1) \\
= 3 + 2t - 19r^2 + 20r^3 - 5r^4, \quad 0 \leq t \leq 1, \]
\[ u(0) = -2, \]

<table>
<thead>
<tr>
<th>( t )</th>
<th>Error of present method</th>
</tr>
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<tbody>
<tr>
<td>0.0</td>
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Table 3

Error values of \( u(t) \) for Example 3.

**Example 6.** Now, consider the Riccati differential-difference equation given by
\[ u'(t) + \cos t \quad u(0.5t) - \sin tu^2(t + 1) \\
= \cos t (1 + \sin(0.5t)) - \sin t \sin^2(t + 1), \quad 0 \leq t \leq 1, \]
\[ u(0) = 0, \]
where the exact solution of above equation is \( u(t) = \sin t \) (see [17]). With \( n = 5 \), the following approximated solution can be found
\[ u(t) = t + 0.249314 \times 10^{-3}t^2 - 0.167903624t^3 \\
+ 0.002111924t^4 + 0.00701337085t^5. \]

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if we choose \( n = 5 \), the errors of the presented method are less than \( 3 \times 10^{-6} \). The graphs of approximated and exact solution are plotted in Fig. 6.

4. Conclusions

In this paper, we presented Bezier curves method for solving the Riccati differential equations and Riccati differential-difference equations. The method is computationally attractive, also reduces the CPU time and the computer memory and at the same time keeps the accuracy of the solution. The algorithm has been successfully applied to the Riccati differential equations. Comparing with other methods, the results of numerical examples demonstrate that this method is more accurate than some existing methods.

References