Nonperturbative definition of the pole mass and short distance expansion of the heavy quark potential in QCD

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Abstract

We show that the $\mathcal{O}(\Lambda)$ ambiguity in the pole mass can be fixed in a natural way by introducing a modified non-perturbative V-scheme momentum space coupling $\tilde{\alpha}_V(q)$ where the confining contributions have been subtracted out. The method used is in the spirit of the infrared finite coupling approach to power corrections, and gives a non-perturbative definition of the 'potential subtracted' mass. The short distance expansion of the static potential is derived, taking into account an hypothetical short distance linear term. The magnitude of the standard OPE contributions are estimated in quenched QCD, based on results of Lüscher and Weisz. It is observed that the expansion is not yet reliable at the shortest distances presently measured on the lattice.

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1. Introduction

Historically, the pole mass M and the heavy quark potential $V(r)$ were among the first quantities where renormalons [1] have been discussed in a physical context in QCD. Latter, the connection of the $\mathcal{O}(\Lambda)$ ambiguity in the pole mass [2,3] with a corresponding ambiguity in the coordinate space potential [4] was pointed out. It was observed [5,6] that the leading renormalon contribution cancels in the total static energy $E_{\text{static}} = 2M + V(r)$, a physical quantity which should be free of ambiguities. This cancellation is a non-trivial finding. Indeed, one might have expected that the pole mass and the static potential should be separately well defined: for instance, in the Schrödinger equation, the quark mass normalizes the kinetic energy. Furthermore, although the potential appears to be non-perturbatively defined only up to an arbitrary constant (in particular only the force is the quantity free of ambiguity in lattice calculations), it is difficult to maintain the view that the arbitrary normalization of $V(r)$ implies an arbitrary normalization of M, which nevertheless would follow from the non-ambiguity of the static energy if

there were no independent way to fix the normalization of either the mass or the potential. In this Letter I suggest that there is in fact a natural way to define unambiguously the pole mass at the non-perturbative level (at least as far as the *leading* renormalon ambiguity is concerned) even in a confining theory like QCD, by properly subtracting out the confining contributions to the self-energy, hence to fix also the 'constant term' in the potential. In Section 2, the definition of the $\mathcal{O}(\Lambda)$ term in the pole mass is given, in term of a properly defined non-perturbative momentum space V-scheme coupling $\tilde{\alpha}_V(q)$. The method used is in the spirit of the infrared (IR) finite coupling approach to power corrections [7]. In Section 3, theoretical constraints on $\tilde{\alpha}_V(q)$ are reviewed. In Section 4 the short distance expansion of $V(r)$ is derived, including the effect of an hypothetical linear short distance term, and the standard IR power corrections are estimated on theoretical ground. It is shown that present lattice data are not available at distances short enough for a reliable short distance analysis to be performed yet.

2. The non-perturbative pole mass

To define the pole mass, one has to fix its well-known renormalon ambiguity [2,3]. I start from the result [5,6] that the leading IR contribution $\delta M_{\text{PT|IR}}$ to the perturbative pole mass M_{PT} (when expressed in term of a short distance mass like $m \equiv m_{\overline{\text{MS}}}$), is related (presumably to all orders of perturbation theory [5]) to the leading long distance contribution $\delta V_{\text{PT|IR}}$ to the perturbative coordinate space potential V_{PT} by the relation

$$\delta M_{\text{PT|IR}}(\mu_f) = -\frac{1}{2}\delta V_{\text{PT|IR}}(\mu_f), \tag{2.1}$$

where

$$\delta V_{\text{PT|IR}}(\mu_f) = \int\limits_{|\vec{q}|<\mu_f} \frac{d^3\vec{q}}{(2\pi)^3}\tilde{V}_{\text{PT}}(q). \tag{2.2}$$

$\tilde{V}_{\text{PT}}(q)$ is the momentum space perturbative potential, related to $V_{\text{PT}}(r)$ by Fourier transformation

$$V_{\text{PT}}(r) = \int \frac{d^3\vec{q}}{(2\pi)^3}\exp(i\vec{q}\cdot\vec{r})\tilde{V}_{\text{PT}}(q), \tag{2.3}$$

and μ_f is an IR factorization scale. Defining to all orders of perturbation theory a momentum space potential effective coupling $\alpha_{V|\text{PT}}(q)$ by

$$\tilde{V}_{\text{PT}}(q) \equiv -4\pi C_F \frac{\alpha_{V|\text{PT}}(q)}{q^2}, \tag{2.4}$$

Eq. (2.1) can be rewritten as

$$\delta M_{\text{PT|IR}}(\mu_f) = \frac{C_F}{\pi}\int\limits_{0}^{\mu_f} dq\,\alpha_{V|\text{PT}}(q). \tag{2.5}$$

The right-hand side of Eq. (2.5) is presumably ill-defined, since it involves an integration over the IR Landau singularity thought to be present in $\alpha_{V|\text{PT}}(q)$, and represents (taking $\mu_f \sim \Lambda$) the $\mathcal{O}(\Lambda)$ ambiguity in the pole mass. To solve this problem, one would be tempted, in analogy with the IR finite coupling approach to power corrections [7], to replace the perturbative effective coupling $\alpha_{V|\text{PT}}(q)$ inside the integral in Eq. (2.5) by the corresponding *non-perturbative* coupling $\alpha_V(q)$ defined by

$$\tilde{V}(q) \equiv -4\pi C_F \frac{\alpha_V(q)}{q^2}, \tag{2.6}$$

where this time $\tilde{V}(q)$ is the Fourier transform of the full non-perturbative potential $V(r)$:

$$V(r) = \int \frac{d^3\vec{q}}{(2\pi)^3} \exp(i\vec{q} \cdot \vec{r}) \tilde{V}(q). \tag{2.7}$$

However, in a confining theory, $\tilde{V}(q)$ either does not exist (e.g., if $V(r) \sim B \log r + C$ for $r \to \infty$), or is anyway too singular at small q (reflecting the singular large distance behavior of $V(r)$), making the integral in Eq. (2.5) (with the non-perturbative $\alpha_V(q)$) divergent at $q = 0$. For instance, in the case of a linearly raising potential $V(r) = \mathcal{O}(r)$ for $r \to \infty$, one gets $\alpha_V(q) = \mathcal{O}(1/q^2)$ for $q \to 0$. This observation suggests one should first subtract out the confining long-distance part of the potential to define a suitable non-perturbative coupling $\alpha_V(q)$. To this end, the following procedure appears the most natural one: expand the potential around $r = \infty$, and subtract from $V(r)$ the first few leading terms in this expansion (including an eventual constant term) which do not vanish for $r \to \infty$. There is by construction only a *finite* number of such terms. Let us call their sum $V_{\text{conf}}(r)$. Then we have

$$V(r) = V_{\text{conf}}(r) + \delta V(r), \tag{2.8}$$

which, assuming the large r expansion can actually be performed, *uniquely* defines $\delta V(r)$, such that $\delta V(r) \to 0$ both for $r \to 0$ (from asymptotic freedom) *and* for $r \to \infty$. It is clear that $\delta V(r)$ now admits a standard Fourier representation

$$\delta V(r) = \int \frac{d^3\vec{q}}{(2\pi)^3} \exp(i\vec{q} \cdot \vec{r}) \delta\tilde{V}(q), \tag{2.9}$$

and one can define the new *non-perturbative* coupling $\tilde{\alpha}_V(q)$ by

$$\delta\tilde{V}(q) \equiv -4\pi C_F \frac{\tilde{\alpha}_V(q)}{q^2}. \tag{2.10}$$

One should note that the perturbative part of these quantities are preserved, namely $\delta V_{\text{PT}}(r) \equiv V_{\text{PT}}(r)$ and $\delta\tilde{V}_{\text{PT}}(q) \equiv \tilde{V}_{\text{PT}}(q)$, since δV differs from V by the $V_{\text{conf}}(r)$ term, which, viewed from short distances, appears as a finite sum of non-perturbative power-like corrections, invisible order by order in perturbation theory. Indeed, the terms occurring in perturbation theory should scale as $1/r$, hence vanish for $r \to \infty$, which excludes them from $V_{\text{conf}}(r)$. Thus $\tilde{\alpha}_{V|\text{PT}}(q) = \alpha_{V|\text{PT}}(q)$ is the same as in Eq. (2.4), i.e., $\tilde{\alpha}_V$ and α_V have identical perturbative expansions.

As an example, consider the potential in quenched QCD (this is actually the only case where the analytic form of the $r \to \infty$ expansion is known in low orders). Theoretical expectations give the long distance expansion for $r \to \infty$

$$V(r) \simeq Kr + C - \frac{\pi}{12}\frac{1}{r} + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{2.11}$$

Although the $\mathcal{O}(1/r)$ term is not a rigorous result of QCD, since it has been derived within an effective bosonic string theory [8], it has been numerically confirmed [9] in high precision lattice simulations. We shall therefore assume that Eq. (2.11) gives the correct large distance behavior of the static potential. It follows that

$$V_{\text{conf}}(r) = Kr + C, \tag{2.12}$$

and one defines

$$V(r) \equiv Kr + C + \delta V(r). \tag{2.13}$$

In this case, the couplings $\alpha_V(q)$ (if it can be defined non-perturbatively, i.e., if $C = 0$ as previously noted) and $\tilde{\alpha}_V(q)$ just differ by a $1/q^2$ term, arising from the Fourier transform of the Kr piece.

The prescription for the *non-perturbative* definition of the pole mass now reads as follows. Introduce the 'potential subtracted' mass [5]

$$M_{PS}(\mu_f) = M_{PT} - \delta M_{PT|IR}(\mu_f),\tag{2.14}$$

and define the non-perturbative IR contribution to the pole mass by

$$\delta M_{IR}(\mu_f) = -\frac{1}{2}\delta V_{IR}(\mu_f),\tag{2.15}$$

where

$$\delta V_{IR}(\mu_f) = \int\limits_{|\vec{q}|<\mu_f} \frac{d^3\vec{q}}{(2\pi)^3}\delta\tilde{V}(q),\tag{2.16}$$

which yields

$$\delta M_{IR}(\mu_f) = \frac{C_F}{\pi}\int\limits_0^{\mu_f} dq\,\tilde{\alpha}_V(q),\tag{2.17}$$

in complete analogy with Eqs. (2.1), (2.2) and (2.5). Then the pole mass is given by

$$M = M_{PS}(\mu_f) + \delta M_{IR}(\mu_f) + \cdots,\tag{2.18}$$

where the dots represent non-leading $\mathcal{O}(1/m)$ IR contributions from higher order renormalons, and the μ_f dependence approximatively cancels between the two terms on the right-hand side. The interpretation of the prescription Eq. (2.18) is transparent: it says one should remove from M_{PT} its ambiguous IR part $\delta M_{PT|IR}(\mu_f)$, as suggested in [5], and substitute for it the corresponding non-perturbative (and non-ambiguous) IR contribution $\delta M_{IR}(\mu_f)$. One should note the similarity between Eq. (2.18) and the corresponding expressions in the IR finite coupling approach to power corrections [7]. In the present context, however, the non-perturbative coupling is unambiguously identified. With the pole mass well-defined, the constant term C in the large distance expansion of the potential (Eq. (2.11)) is in turn fixed, since the corresponding constant term in the large distance expansion of $E_{static}(r)$, which should be unambiguous and calculable, is $2M + C$.

3. Constraints on the non-perturbative $\tilde{\alpha}_V(q)$

Eqs. (2.11) and (2.13) yield $\delta V(r) \sim -(\pi/12)(1/r)$ for $r \to \infty$, hence $\delta\tilde{V}(q) \sim -(\pi^2/3)(1/q^2)$ for $q \to 0$, which yields

$$C_F\tilde{\alpha}_V(q=0) = \frac{\pi}{12},\tag{3.1}$$

i.e., $\tilde{\alpha}_V(q=0) \simeq 0.196$, a rather *small* IR fixed point value. Substituting this value as a rough estimate of $\tilde{\alpha}_V(q)$ in the integrand of Eq. (2.17) gives

$$\delta M_{IR}(\mu_f) \simeq \frac{C_F}{\pi}\tilde{\alpha}_V(q=0)\mu_f = \frac{1}{12}\mu_f,\tag{3.2}$$

which represents a correction of about 100 MeV for the range of μ_f quoted in [5] for b-quarks.

A more refined estimate is obtained by inputting the information about the $\mathcal{O}(1/r^2)$ term in Eq. (2.11), which was obtained in [9] from a fit to high precision large r lattice data and yields for $r \to \infty$

$$\delta V(r) \simeq -\frac{\pi}{12}\frac{1}{r} - \frac{\pi}{12}\frac{b}{r^2}\tag{3.3}$$

with $b \simeq 0.04$ fm, hence for $q \to 0$

$$\delta \tilde{V}(q) \simeq -\frac{\pi^2}{3}\frac{1}{q^2} - b\frac{\pi^3}{6}\frac{1}{q} \tag{3.4}$$

and

$$C_F \tilde{\alpha}_V(q) \simeq \frac{\pi}{12}\left(1 + b\frac{\pi}{2}q\right). \tag{3.5}$$

Note that, since $b > 0$, $\tilde{\alpha}_V(q)$ *increases* from its IR value as q increases, hence must be non-monotonous in the IR region, since asymptotic freedom implies it should ultimately decrease to 0 at large q. At $\mu_f = 1.2$ GeV, the second term in the parenthesis in Eq. (3.5) represents a correction of about 40% to the IR value. Substituting Eq. (3.5) in the integrand of Eq. (2.17) gives

$$\delta M_{\mathrm{IR}}(\mu_f) \simeq \frac{1}{12}\mu_f\left(1 + b\frac{\pi}{4}\mu_f\right), \tag{3.6}$$

which yields $\delta M_{\mathrm{IR}}(\mu_f) \simeq 120$ MeV for $\mu_f = 1.2$ GeV.

4. Short distance expansion of the heavy quark potential

In this section I show that, barring constant terms, the short distance expansion of the heavy quark potential can be obtained directly[1] from Eq. (2.7), despite the singular behavior of $\tilde{V}(q)$ at small q. Introducing again the factorization scale μ_f, Eq. (2.7) can be written as

$$V(r) = -\frac{2\,C_F}{\pi}\left[\int_0^{\mu_f} dq\left(\frac{\sin qr}{qr}\right)\alpha_V(q) + \int_{\mu_f}^{\infty} dq\left(\frac{\sin qr}{qr}\right)\alpha_V(q)\right]. \tag{4.1}$$

At short distances, we can expand the $\sin qr$ factor in the low momentum integral, which gives the IR power corrections. Making the further assumption that $\alpha_V(q)$ has no large power corrections at large q and may be well approximated by its perturbative part $\alpha_{V|\mathrm{PT}}(q)$ above μ_f

$$\alpha_V(q) \simeq \alpha_{V|\mathrm{PT}}(q) \tag{4.2}$$

(this assumption will be modified below, Eq. (4.7)), one ends up with the $r \to 0$ expansion

$$V(r) \simeq V_{\mathrm{PT}}(r, \mu_f) - \frac{2C_F}{\pi}\left[\int_0^{\mu_f} dq\,\alpha_V(q) - \frac{r^2}{6}\int_0^{\mu_f} dq\,q^2\alpha_V(q) + \mathcal{O}(r^4)\right], \tag{4.3}$$

where

$$V_{\mathrm{PT}}(r, \mu_f) = -\frac{2\,C_F}{\pi}\left[\int_{\mu_f}^{\infty} dq\left(\frac{\sin qr}{qr}\right)\alpha_{V|\mathrm{PT}}(q)\right] \tag{4.4}$$

is the IR subtracted perturbative potential [5]. The normalization of the standard $\mathcal{O}(r^0)$ and $\mathcal{O}(r^2)$ renormalon-related power corrections in Eq. (4.3) is thus given[2] by low-energy moments of $\alpha_V(q)$. Note that the $\mathcal{O}(r^0)$ term

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[1] I assume the non-perturbative Fourier transform Eq. (2.7) exists at least in a formal sense, in particular that a long distance $B \log r + C$ contribution is not present, as previously observed.

[2] The corresponding expressions in term of non-local operators can be found in the effective field theory framework of [10].

is actually infinite, as expected from the divergent IR behavior of $\alpha_V(q)$. In particular in quenched QCD Eq. (3.5) implies for $q^2 \to 0$

$$C_F \alpha_V(q) \sim \frac{2K}{q^2} + \frac{\pi}{12}\left(1 + b\frac{\pi}{2}q\right). \tag{4.5}$$

But since the $\mathcal{O}(r^0)$ term contributes only an overall normalization constant to the potential, which in this section is left arbitrary, one can drop it out. On the other hand, the $\mathcal{O}(r^2)$ and higher order r-dependent contributions are *finite*. In particular, using Eq. (4.5) as a rough approximation to $\alpha_V(q)$ in the range $0 < q < \mu_f$ one obtains in quenched QCD for $r \to 0$ (ignoring any constant term)

$$V(r) \simeq V_{\mathrm{PT}}(r, \mu_f) + \left(\frac{2K}{3\pi}\mu_f + \frac{1}{108}\mu_f^3 + \frac{b\pi}{288}\mu_f^4\right)r^2 + \mathcal{O}(r^4). \tag{4.6}$$

Let us now modify the previously mentioned assumption, in order to deal with the possibility that a $\mathcal{O}(1/q^2)$ power correction is actually present in $\alpha_V(q)$. Such a correction has been first suggested in [11] as a consequence of new physics related to confinement, leading to a $\mathcal{O}(r)$ linear correction to the potential at short distances, of the same size (and sign) as the standard long distance correction related to the string tension. It should be noted, however, that a short distance linear piece may have a more conventional (although still non-perturbative) *infrared* origin, as indicated by the position of the leading IR renormalon present in $\tilde{V}(q)$, which also suggests [5] the presence of a $\mathcal{O}(1/q^2)$ correction. Let us thus assume that for $q^2 \to \infty$

$$\alpha_V(q) \simeq \alpha_{V|\mathrm{PT}}(q) + \frac{2K_0}{C_F}\frac{1}{q^2} \tag{4.7}$$

with $K_0 \neq K$ in general. To deal with this correction, one can use the general method of [12], or more conveniently, introduce a new coupling $\tilde{\alpha}_V(q)$ (different in general from the one in Section 2, see below), related to the original $\alpha_V(q)$ by

$$C_F \alpha_V(q) \equiv C_F \tilde{\alpha}_V(q) + \frac{2K_0}{q^2}, \tag{4.8}$$

such that the *redefined* coupling $\tilde{\alpha}_V(q)$ is essentially given by its perturbative part (which coincides with that of $\alpha_V(q)$) at large q^2

$$\tilde{\alpha}_V(q) \simeq \alpha_{V|\mathrm{PT}}(q) \tag{4.9}$$

with no substantial power corrections. Thus from Eq. (4.8)

$$\tilde{V}(q) = -\frac{8\pi K_0}{q^4} - 4\pi C_F \frac{\tilde{\alpha}_V(q)}{q^2}, \tag{4.10}$$

and, upon taking the Fourier transform

$$V(r) = K_0 r + \delta V(r), \tag{4.11}$$

where $\delta V(r)$ is given by Eqs. (2.9) and (2.10), but with $\tilde{\alpha}_V(q)$ now defined by Eq. (4.8). Note that for $K_0 = K$ this definition coincides with that of Section 2 (assuming $C = 0$, see the comment after Eq. (2.13)). Thus, introducing a factorization scale μ_f as in Eq. (4.1) we have

$$\delta V(r) = -\frac{2C_F}{\pi}\left[\int_0^{\mu_f} dq\left(\frac{\sin qr}{qr}\right)\tilde{\alpha}_V(q) + \int_{\mu_f}^{\infty} dq\left(\frac{\sin qr}{qr}\right)\tilde{\alpha}_V(q)\right]. \tag{4.12}$$

Since $\tilde{\alpha}_V(q)$ has no large power corrections, it can be approximated by its perturbative part $\alpha_{V|PT}(q)$ above some scale μ_f, and one deduces the short distance expansion

$$\delta V(r) \simeq V_{PT}(r, \mu_f) - \frac{2C_F}{\pi}\left[\int_0^{\mu_f} dq\,\tilde{\alpha}_V(q) - \frac{r^2}{6}\int_0^{\mu_f} dq\,q^2\tilde{\alpha}_V(q) + \mathcal{O}(r^4)\right]. \tag{4.13}$$

From Eqs. (4.5) and (4.8) we get for $q^2 \to 0$

$$C_F\tilde{\alpha}_V(q) \sim 2(K - K_0)\frac{1}{q^2} + \frac{\pi}{12}\left(1 + b\frac{\pi}{2}q\right). \tag{4.14}$$

Thus, dropping again the (infinite) $\mathcal{O}(r^0)$ term, and using Eq. (4.14) for $q < \mu_f$, we obtain

$$\delta V(r) \simeq V_{PT}(r, \mu_f) + \left(\frac{2(K - K_0)}{3\pi}\mu_f + \frac{1}{108}\mu_f^3 + \frac{b\pi}{288}\mu_f^4\right)r^2 + \mathcal{O}(r^4), \tag{4.15}$$

hence from Eq. (4.11)

$$V(r) \simeq V_{PT}(r, \mu_f) + K_0 r + \left(\frac{2(K - K_0)}{3\pi}\mu_f + \frac{1}{108}\mu_f^3 + \frac{b\pi}{288}\mu_f^4\right)r^2 + \mathcal{O}(r^4) \tag{4.16}$$

which of course agrees with Eq. (4.6) for $K_0 = 0$. The correlation between the coefficient of the $\mathcal{O}(r)$ correction (which is μ_f independent) and that of the standard OPE $\mathcal{O}(r^2)$ correction should be noted. For $K_0 \neq 0$, we get a neat derivation of the well-known statement [11] that the appearance of a linear short distance term in $V(r)$ is equivalent to the presence of a $\mathcal{O}(1/q^2)$ correction in the standard $\alpha_V(q)$ coupling. Moreover, for $K_0 = K$, one obtains the straightforward, but interesting, result that the appearance of the linear short distance term is equivalent to the statement that the *modified* coupling $\tilde{\alpha}_V(q)$ of Section 2 (rather then $\alpha_V(q)$) has no $\mathcal{O}(1/q^2)$ corrections.

One might attempt an analysis of the lattice short distance data of [13] based on Eqs. (4.11) and (4.13). $V_{PT}(r, \mu_f)$ could be evaluated from Eq. (4.4) by solving the known [14] 3-loop renormalization group equation for $\alpha_{V|PT}(q)$ and performing the integral, similar to the single dressed gluon 'renormalon integral' (with IR cut-off) in [12,15], while the power corrections should be fitted. Unfortunately, one finds that the perturbative expansion of the V-scheme coupling beta function is not reliable at values of μ_f small enough that the low momentum integral in Eq. (4.12) can be meaningfully expanded and parametrized in term of a few power correction terms, even at the shortest values of r presently measured on the lattice. Thus no reliable fit of the power corrections can be performed yet. It should be noted that in the present approach standard IR power corrections appear from an OPE-like separation of long and short distances in the Fourier transform of the momentum space potential, and their presence is mandatory. This is to be contrasted with the result of [13], where no power corrections were needed[3] if the potential is predicted in term of the renormalization group equation of the position space effective charge α_F associated [17] to the force $F(r) = dV/dr = C_F\alpha_F(1/r)/r^2$. However, the implicit definition[4] of the power corrections in the later case is different, and does not make use of a momentum space IR cutoff to separate long from short distances.

5. Conclusion

We have shown that it is possible to fix in a natural way the $\mathcal{O}(\Lambda)$ renormalon ambiguity in the pole mass, thus giving a *non-perturbative* definition of the pole mass in QCD at this level of accuracy, which represents a natural

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[3] The alternative analysis of [16] also finds no room for power corrections.

[4] The convergence of the expansion of the α_F beta function is only slightly better [13] than that of the momentum space α_V beta function, which presumably makes a quantitative analysis of the power corrections difficult also in the scheme of [13].

non-perturbative extension of the 'potential subtracted' mass, in the spirit of the IR finite coupling approach to power corrections. This definition is an optimal one, in the sense the prescription is to remove from the heavy quark potential contribution to the self-energy those terms and *only* those one (the confining ones contained in $V_{\text{conf}}(r)$) which would give a meaningless (infinite) result for the pole mass. For instance, one should *not* remove from $\delta V(r)$ the $\mathcal{O}(1/r)$ 'Lüscher term' to include it in $V_{\text{conf}}(r)$ (see Eq. (2.11)) (which, moreover, would make the modified IR finite V-scheme coupling $\tilde{\alpha}_V(q)$ non-asymptotically free!). The applications of the proposed mass definition are similar to those of the 'potential subtracted' mass, to which it provides the leading power correction, allowing an accurate relation to the standard $\overline{\text{MS}}$ mass, but it can be used consistently with non-perturbative extensions of the Coulomb static potential (such as implied by phenomenological potential models or the potential determined on the lattice). The remaining challenge is to fix the $\mathcal{O}(\Lambda^2/m)$ ambiguities in the pole mass arising from higher order renormalons.

We have also discussed the OPE like analysis of the short distance potential. The magnitude of the standard OPE contributions have been estimated from Eq. (4.5). However, the resulting short distance expansion is unreliable at the lowest values of r measured so far on the lattice, due to the poor convergence of perturbation theory for the momentum space V-scheme coupling beta function.

Acknowledgements

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