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Mahler measure, links and homology growth ☆

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Abstract

Let l be an oriented link of d components with nonzero Alexander polynomial $\Delta(u_1,\ldots,u_d)$. Let Λ be a finite-index subgroup of $H_1(S^3-l)\cong \mathbf{Z}^d$, and let M_Λ be the corresponding abelian cover of S^3 branched along l. The growth rate of the order of the torsion subgroup of $H_1(M_\Lambda)$, as a suitable measure of Λ approaches infinity, is equal to the Mahler measure of Δ. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Associated to any knot $k\subset S^3$ is a sequence of Alexander polynomials Δ_i, $i\geqslant 1$, in a single variable such that Δ_{i+1} divides Δ_i. Likewise, for any oriented link of d components there is a sequence of Alexander polynomials in d variables. Following the usual custom, we refer to the first Alexander polynomial of a knot or a link as *the* Alexander polynomial, and we denote it simply by Δ.

In [9] Gordon examined the homology groups of r-fold cyclic covers M_r of S^3 branched over a knot k. He proved that when each zero of the Alexander polynomial Δ of k has modulus one (and hence is a root of unity), the finite values of $|H_1(M_r)|$ are periodic in r. Gordon conjectured that when some zero of Δ is not a root of unity, the finite values of $|H_1(M_r)|$ grow exponentially.

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More than 15 years later two independent proofs of Gordon's conjecture, one by Riley [16] and another by Gonzaléz-Acuña and Short [8], appeared. Both employed the Gel'fond–Baker theory of linear forms in the logarithms of algebraic integers [2,7].

We extend the above results for knots, replacing the term "finite values of $|H_1(M_r)|$" with "order of the torsion subgroup of $H_1(M_r)$", while at the same time proving a general result for links in S^3. Our proof, which is motivated by Silver and Williams [22], identifies the torsion subgroup of the homology of a finite abelian branched cover with the connected components of periodic points in an associated algebraic dynamical system. Theorem 21.1 of [19], an enhanced version of a theorem of Lind et al. [13], then completes our argument.

Recognizing that relatively few topologists are familiar with algebraic dynamical systems, we have endeavored to make this paper self-contained. The reader who desires to know more about such dynamical systems is encouraged to consult the extraordinary monograph [19].

2. Statement of results

Let $l = l_1 \cup \cdots \cup l_d$ be an oriented link of d components with exterior $E = S^3 - \operatorname{int} N(l)$, where $N(l)$ is a regular neighborhood of l. The meridianal generators of the link group $G_l = \pi_1(S^3 - l)$ represent distinguished generators $u_1, \ldots, u_d$ for the abelianization $G_l/G_l' \cong \mathbf{Z}^d$. We identify these generators with the standard basis of $\mathbf{Z}^d$.

Given a finite-index subgroup $\Lambda \subset \mathbf{Z}^d$ there exists a covering E_Λ of the link exterior corresponding to the epimorphism $G_l \to \mathbf{Z}^d/\Lambda$, the abelianization map composed with the canonical quotient map. By attaching solid tori to E_Λ so that meridians of the tori cover meridians of l while the collection of cores map to the link, we obtain a cover M_Λ of S^3 branched over l.

Let $\mathscr{R}_d$ denote the ring $\mathbf{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}] \cong \mathbf{Z}[\mathbf{Z}^d]$ of Laurent polynomials with integer coefficients. The *Mahler measure* of a nonzero polynomial $f \in \mathscr{R}_d$ is defined by

$$\mathbf{M}(f) = \exp\left(\int_{\mathbf{S}^d} \log |f(\mathbf{s})| \, d\mathbf{s} \right),$$

where d$\mathbf{s}$ indicates integration with respect to normalized Haar measure, and $\mathbf{S}^d$ is the multiplicative subgroup of d-dimensional complex space $\mathbf{C}^d$ consisting of all vectors $(s_1, \ldots, s_d)$ with $|s_1| = \cdots = |s_d| = 1$. Clearly, Mahler measure is multiplicative, and the measure of any unit is 1. It is known that $\mathbf{M}(f) = 1$ if and only if f is equal up to a unit factor to the product of cyclotomic polynomials in a single variable evaluated at monomials (see [19, Lemma 19.1]).

The quantity $\mathbf{M}(f)$, which is the geometric mean of $|f|$ over the d-torus $\mathbf{S}^d$, was introduced by Mahler [14,15]. It is a consequence of Jensen's formula [1, p. 205] that if f is a nonzero polynomial $c_n u^n + \cdots + c_1 u + c_0$ $(c_n \neq 0)$ in one variable, then

$$\mathbf{M}(f) = |c_n| \prod_{j=1}^{n} \max(|r_j|, 1),$$

where $r_1, \ldots, r_n$ are the zeros of f. A short proof can be found in either [5] or [19].

For any finite-index subgroup Λ we let

$$\langle \Lambda \rangle = \min\{|v| : v \in \Lambda - 0\},$$

where $|\cdot|$ denotes the Euclidean metric.

Since M_Λ is a compact manifold, the homology group $H_1(M_\Lambda)$ is finitely generated. (All homology groups in this paper have integer coefficients.) We decompose $H_1(M_\Lambda)$ as the direct sum of a free abelian group of some rank β_Λ and a torsion subgroup $TH_1(M_\Lambda)$. We denote the order of $TH_1(M_\Lambda)$ by b_Λ.

Theorem 2.1. *Let $l = l_1 \cup \cdots \cup l_d$ be an oriented link of d components having nonzero Alexander polynomial $\Delta = \Delta(u_1, \ldots, u_d)$. Then*

$$\limsup_{\langle \Lambda \rangle \to \infty} \frac{1}{|\mathbf{Z}^d/\Lambda|} \log b_\Lambda = \log \mathbf{M}(\Delta).$$

When $d = 1$ the lim sup can be replaced by an ordinary limit.

As a consequence of the proof of Theorem 2.1 we obtain a new proof of a theorem of Gordon. Recall that for any knot, Δ_i denotes the ith Alexander polynomial.

Corollary 2.2 (Gordon [9])**.** *Let k be a knot in S^3. If Δ_1/Δ_2 divides $t^N - 1$ for some N, then $H_1(M_r) \cong H_1(M_{r+N})$ for all r.*

The Mahler measure of $u_1^2 - u_1 + 1$, the Alexander polynomial of the trefoil knot 3_1, is 1 since both zeros of the polynomial have unit modulus. On the other hand, $u_1^2 - 3u_1 + 1$, the Alexander polynomial of the figure eight knot 4_1, has zeros $(1 \pm \sqrt{5})/2$ and hence it has Mahler measure $(1 + \sqrt{5})/2 \approx 1.618$.

Scanning the table of 2-component links in [17] we find that 6_2^2 is the first link with nonzero Alexander polynomial having Mahler measure greater than 1. The polynomial is $u_1 + u_2 - 1 + u_1^{-1} + u_2^{-1}$, which has Mahler measure approximately equal to 1.285.

The next link in the table, 6_2^3, has Alexander polynomial $2 - u_1 - u_2 + 2u_1u_2$, which can be rewritten as $(2 - u_1) + u_1u_2(2u_1 - 1)$. Using Lemma 19.8 of [19] and an easy change of basis (replacing u_1u_2 with a new variable u_2') we see that Mahler measure of this Alexander polynomial is precisely 2.

No polynomial with integer coefficients is known that has Mahler measure greater than 1 but less than that of $\Delta(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$, which has Mahler measure approximately equal to 1.176. (Only one of the nine zeros of Δ lies outside the unit circle.) Deciding whether or not such a numerical gap truly exists is known as Lehmer's problem, and it remains a vexing open question. (see [5,11]). It is a provocative fact that Δ is the Alexander polynomial of a knot. In fact, there are infinitely many, including infinitely many with complements that fiber over the circle.

For more calculations of Mahler measures of Alexander polynomials of links and further discussion of Lehmer's question see [23].

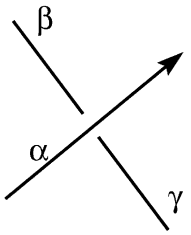

Fig. 1. $\mathbf{T}^{\mathbf{Z}^d}$-Coloring rule.

3. $\mathbf{Z}^d$-shifts and link colorings

A $\mathbf{Z}^d$-*action by automorphisms* on a topological group X is a homomorphism $\sigma : \mathbf{m} \mapsto \sigma_{\mathbf{m}}$ from $\mathbf{Z}^d$ to $\mathrm{Aut}(X)$. Two $\mathbf{Z}^d$-actions σ and σ' on X and X', respectively, are *algebraically conjugate* if there exists a continuous group isomorphism $\phi : X \to X'$ such that $\phi \circ \sigma_{\mathbf{m}} = \sigma'_{\mathbf{m}} \circ \phi$, for every $\mathbf{m} \in \mathbf{Z}^d$.

$\mathscr{R}_d$-modules are an important source of $\mathbf{Z}^d$-shifts via Pontryagin duality. Let $\mathbf{T}$ denote the additive circle group $\mathbf{R}/\mathbf{Z}$. For any $\mathscr{R}_d$-module L, the Pontryagin dual $L^\wedge = \mathrm{Hom}(L, \mathbf{T})$ is a group under pointwise addition. Here L is given the discrete topology, and $L^\wedge$ is endowed with the compact-open topology; $L^\wedge$ is a compact abelian group. For $\mathbf{m} \in \mathbf{Z}^d$, scalar multiplication $a \mapsto \mathbf{m}a$ in L induces $\sigma_{\mathbf{m}} \in \mathrm{Aut}(L^\wedge)$ via its adjoint action. In this way we have a $\mathbf{Z}^d$-action on $L^\wedge$. From a purely algebraic point of view, $L^\wedge$ is a $\mathscr{R}_d$-module. In the case that L is a free $\mathscr{R}_d$-module of rank N, we obtain the compact group $L^\wedge = (\mathbf{T}^N)^{\mathbf{Z}^d}$. The automorphism $\sigma_{\mathbf{m}}$ is the shift map given by $\sigma_{\mathbf{m}}(\alpha_{\mathbf{n}}) = (\alpha_{\mathbf{n}+\mathbf{m}})$ for $\alpha = (\alpha_{\mathbf{n}}) \in (\mathbf{T}^N)^{\mathbf{Z}^d}$. The automorphisms $\sigma_{u_1}, \ldots, \sigma_{u_d}$ will be denoted by $\sigma_1, \ldots, \sigma_d$ for notational ease.

Given a $\mathbf{Z}^d$-action σ on X, we say that a point $x \in X$ is *periodic* under σ if its orbit $\{\sigma_{\mathbf{m}} x \mid \mathbf{m} \in \mathbf{Z}^d\}$ is finite. We will be particularly interested in periodic point sets

$$\mathrm{Fix}_\Lambda(\sigma) = \{x \in X \mid \sigma_{\mathbf{m}} x = x \ \forall \mathbf{m} \in \Lambda\},$$

where Λ is a subgroup of finite index in $\mathbf{Z}^d$. For algebraically conjugate actions such sets clearly correspond under the isomorphism ϕ.

Definition 3.1. Assume that D is a diagram of an oriented link $l = l_1 \cup \cdots \cup l_d$ of d components. A $\mathbf{T}^{\mathbf{Z}^d}$-*coloring* of D is an assignment of elements (*colors*) $\alpha, \beta, \ldots \in \mathbf{T}^{\mathbf{Z}^d}$ to the arcs of D such that the condition

$$\alpha + \sigma_t \beta = \gamma + \sigma_{t'} \alpha \tag{3.1}$$

is satisfied at any crossing. Here α corresponds to an overcrossing arc of the tth component of l, while β and γ correspond to undercrossing arcs of the t'th component. We encounter β as we travel in the preferred direction along the arc labeled by α, turning left at the crossing (see Fig. 1). The terminology is motivated by the concept of Fox coloring for knots [6], which was generalized in [20,22].

If D consists of N arcs, then the set $\mathrm{Col}_{\mathbf{T},\mathbf{Z}^d}(D)$ of all $\mathbf{T}^{\mathbf{Z}^d}$-colorings of D is a closed subgroup of $[\mathbf{T}^{\mathbf{Z}^d}]^N \cong [\mathbf{T}^N]^{\mathbf{Z}^d}$ that is invariant under $\sigma_{\mathbf{m}}$ for each $\mathbf{m} \in \mathbf{Z}^d$. It will follow from observations in Section 4 that if D' is a another diagram for l, then the $\mathbf{Z}^d$-action on $\mathrm{Col}_{\mathbf{T},\mathbf{Z}^d}(D)$ is algebraically conjugate to the action on $\mathrm{Col}_{\mathbf{T},\mathbf{Z}^d}(D')$. In anticipation we make the following definition.

Definition 3.2 (Cf. Silver and Williams [20])**.** Let l be a d-component oriented link with diagram D. The *color* $\mathbf{Z}^d$*-shift* $\mathrm{Col}_{\mathbf{T},\mathbf{Z}^d}(l)$ is the compact abelian group $\mathrm{Col}_{\mathbf{T},\mathbf{Z}^d}(D)$ together with the $\mathbf{Z}^d$-action σ.

4. Alexander module and periodic points

This section contains the proofs of our main results. An example that illustrates the main ideas is given at the end.

A diagram D for l yields a finite Wirtinger presentation $\langle x_1, x_2, \ldots, x_N \,|\, r_1, \ldots, r_N \rangle$ for $G_l = \pi_1(S^3 - l)$. Let P be the canonical 2-complex with $\pi_1 P \cong G_l$, constructed with a single vertex v, directed edges labeled $x_1, \ldots, x_N$, and oriented 2-cells $c_1, \ldots, c_N$ with each boundary ∂c_i attached to 1-cells according to r_i (see Chap. 11 of [12]).

Let $\tilde{P}$ be the maximal abelian cover of P; that is, the cover corresponding to the abelianization map $G_l \to G_l/G_l' \cong \mathbf{Z}^d$. As usual each cell v, x_i, c_j of P lifts to a family $\mathbf{m}\tilde{v}, \mathbf{m}\tilde{x}_i, \mathbf{m}\tilde{c}_j$ of oriented cells indexed by $\mathbf{Z}^d$. By standard construction the chain complex $0 \to \tilde{C}_2 \xrightarrow{\partial_2} \tilde{C}_1 \xrightarrow{\partial_1} \tilde{C}_0 \to 0$ admits a quotient $0 \to \tilde{C}_2 \xrightarrow{\partial_2} \tilde{C}_1 \to 0$ that determines the relative homology group $H_1(\tilde{P}, \tilde{P}^0)$, where $\tilde{P}^0$ is the 0-skeleton of $\tilde{P}$. This $\mathscr{R}_d$-module is the Alexander module of the link, denoted here by A.

By the universal coefficient theorem [24, p. 243] the cohomology group $H^1(\tilde{P}, \tilde{P}^0; \mathbf{T})$ is naturally isomorphic to the dual group of the Alexander module. It is a closed subgroup of $\mathrm{Hom}(\tilde{C}_1, \mathbf{T}) = [\mathbf{T}^N]^{\mathbf{Z}^d}$, the kernel of the coboundary operator

$$\mathrm{Hom}(\tilde{C}_1, \mathbf{T}) \xrightarrow{\mathrm{Hom}(\partial_2, 1)} \mathrm{Hom}(\tilde{C}_2, \mathbf{T})$$

and hence it inherits a $\mathbf{Z}^d$-action from $\mathrm{Hom}(\tilde{C}_1, \mathbf{T})$ (see Section 3).

We observe that $\partial_2 \tilde{c}_1, \ldots, \partial_2 \tilde{c}_N$ closely resemble the relations of the coloring rule (3.1). The Wirtinger relator at the crossing in Fig. 1 has the form $x_i x_{j_1} x_i^{-1} x_{j_2}^{-1}$, and the lifted loop in the cover that begins at $\tilde{v}$ determines the 1-cycle $\tilde{x}_i + u_t \tilde{x}_{j_1} - u_{t'} \tilde{x}_i - \tilde{x}_{j_2}$, which induces the homology relation $\tilde{x}_i + u_t \tilde{x}_{j_1} = \tilde{x}_{j_2} + u_{t'} \tilde{x}_i$. Lifts that begin at other points of the cover are simply translates by elements of $\mathbf{Z}^d$. We can regard the assignment of $\alpha = (\alpha_{\mathbf{m}}) \in \mathbf{T}^{\mathbf{Z}^d}$ to an arc x_i as an assignment of $\alpha_{\mathbf{m}} \in \mathbf{T}$ to the 1-chain $\mathbf{m}\tilde{x}_i$. Then clearly $\mathrm{Col}_{\mathbf{T},\mathbf{Z}^d}(D)$ and $H^1(\tilde{P}, \tilde{P}^0; \mathbf{T})$ are described by identical subsets of $[\mathbf{T}^N]^{\mathbf{Z}^d}$. Since the Alexander module is a link invariant, it follows that the algebraic conjugacy class of $\mathrm{Col}_{\mathbf{T},\mathbf{Z}^d}(D)$ is independent of the diagram for l.

Consider a finite-index subgroup Λ of $\mathbf{Z}^d$. The unbranched cover E_Λ has the same homology as the quotient complex $\tilde{P}/\Lambda$. A 2-complex Q with the same first homology group as the branched cover M_Λ is obtained from $\tilde{P}/\Lambda$ by attaching additional 2-cells as follows. Each

Wirtinger generator x_i, $1 \leqslant i \leqslant N$, maps to some $u_{t(i)}$ under abelianization. Assume that $u_{t(i)}$ represents an element of order $n(i)$ in $\mathbf{Z}^d/\Lambda$. Then $\tilde{x}_i + u_{t(i)}\tilde{x}_i + \cdots + u_{t(i)}^{n(i)-1}\tilde{x}_i$ is a 1-cycle in $\tilde{P}/\Lambda$, and we attach a 2-cell along it and each of its translates. The additional cells added to the complex in this way correspond to the meridinal disks of tori that we attach to E_Λ when constructing M_Λ. Consequently, Q has the same fundamental group and hence the same first homology group as M_Λ.

Elements of $H^1(Q, Q^0; \mathbf{T})$ correspond to $\mathbf{Z}^d$-colorings in $\mathrm{Fix}_\Lambda(\sigma)$ such that if α is assigned to the ith arc of D, then

$$\alpha + \sigma_{t(i)}\alpha + \cdots + \sigma_{t(i)}^{n(i)-1}\alpha = 0. \tag{4.1}$$

Such $\mathbf{Z}^d$-colorings comprise a subgroup $\mathrm{SFix}_\Lambda(\sigma) \leqslant \mathrm{Fix}_\Lambda(\sigma)$ of *special periodic points*.

The universal coefficient theorem implies that $H^1(Q; \mathbf{T})$ is isomorphic to the dual group of $H_1(Q) \cong H_1(M_\Lambda)$. By decomposing $H_1(M_\Lambda)$ as $TH_1(M_\Lambda) \oplus \mathbf{Z}^{\beta_\Lambda}$ (see Section 2) and recalling that $A^\wedge \cong A$ for any finite abelian group A, we have

$$H^1(Q; \mathbf{T}) \cong [TH_1(M_\Lambda) \oplus \mathbf{Z}^{\beta_\Lambda}]^\wedge \cong TH_1(M_\Lambda) \oplus \mathbf{T}^{\beta_\Lambda}. \tag{4.2}$$

Consider now the portion of the cohomology long exact sequence:

$$H^0(Q^0; \mathbf{T}) \xrightarrow{\delta} H^1(Q, Q^0; \mathbf{T}) \to H^1(Q; \mathbf{T}) \to 0. \tag{4.3}$$

Lemma 4.1. *The image of δ is a direct summand of $H^1(Q, Q^0; \mathbf{T})$ isomorphic to $\mathbf{T}^r$, where $r = |\mathbf{Z}^d/\Lambda| - 1$.*

Proof. The 0-skeleton Q^0 consists of vertices indexed by elements of $\mathbf{Z}^d/\Lambda$. Elements of $H^0(Q^0; \mathbf{T})$ can be regarded as functions $f : \mathbf{Z}^d/\Lambda \to \mathbf{T}$. The image $\delta(f)$ is an edge-labeling of Q, assigning $f(\mathbf{m}') - f(\mathbf{m})$ to an edge from $\mathbf{m}\tilde{v}$ to $\mathbf{m}'\tilde{v}$. Select a maximal tree T in the 1-skeleton of Q. It is clear that $\delta(f)$ is uniquely determined by its values on T, and such values can be prescribed arbitrarily. Since T has r edges, the image of δ is isomorphic to $\mathbf{T}^r$.

The map $H^1(Q, Q^0; \mathbf{T}) \xrightarrow{\varepsilon} \mathbf{T}^r$ given by restricting any cocycle to the edges of the maximal tree T is an epimorphism, by what we have said above. We construct a right inverse η for ε as follows. Given a function $g : T \to \mathbf{T}$, choose an element $f \in H^0(Q^0; \mathbf{T})$ such that $\delta(f)$ agrees with g on T. Define $\eta(g) = \delta(f)$. Hence the image of δ is a direct summand of $H^1(Q, Q^0; \mathbf{T})$. □

Corollary 4.2. *Let $\mathrm{SFix}_\Lambda^0(\sigma)$ be the connected component of the identity in $\mathrm{SFix}_\Lambda(\sigma)$. Then $\mathrm{SFix}_\Lambda(\sigma)/\mathrm{SFix}_\Lambda^0(\sigma) \cong TH_1(M_\Lambda)$.*

Proof. By Eq. (4.2) we have $TH_1(M_\Lambda) \oplus \mathbf{T}^{\beta_\Lambda} \cong H^1(Q; \mathbf{T})$, and by the long exact sequence (4.3) the latter module is isomorphic to $H^1(Q, Q^0; \mathbf{T})/\mathrm{im}(\delta)$. Recall that $H^1(Q, Q^0; \mathbf{T})$ is isomorphic to $\mathrm{SFix}_\Lambda(\sigma)$. By Lemma 4.1 the image of δ is connected and hence contained in $\mathrm{SFix}_\Lambda^0(\sigma)$. Thus $\mathrm{SFix}_\Lambda(\sigma)/\mathrm{SFix}_\Lambda^0(\sigma) \cong TH_1(M_\Lambda)$. □

Corollary 4.3. *The first Betti number β_Λ of M_Λ is equal to* $\dim \mathrm{SFix}_\Lambda(\sigma) - |\mathbf{Z}^d/\Lambda| + 1$.

Proof. By Eq. (4.2) the Betti number β_Λ is equal to the dimension of $H^1(Q;\mathbf{T})$, and by the long exact sequence (4.3) the latter is $\dim \mathrm{SFix}_\Lambda(\sigma) - \dim \mathrm{im}(\delta)$. Lemma 4.1 completes the argument. □

The quantity $\dim \mathrm{SFix}_\Lambda(\sigma)$ can be generally computed as the nullity of a certain matrix. The computation is similar to that which results from the formula of Sakuma [18, Theorem 1.1(2)]. However, the dynamical systems perspective here is new.

Proof of Theorem 2.1. It follows from Corollary 4.2 that $b_\Lambda = |TH_1(M_\Lambda)|$ is equal to the number of connected components of $\mathrm{SFix}_\Lambda(\sigma)$. Now, we apply techniques of symbolic dynamics to count the number of components and determine their exponential growth rate.

We denote the connected component of the identity in $\mathrm{Fix}_\Lambda(\tau)$ by the symbol $\mathrm{Fix}^0_\Lambda(\tau)$. For any $\mathbf{Z}^d$-action τ associated to the dual group of a $\mathscr{R}_d$-module, Theorem 21.1 of [19] implies that the exponential growth rate of $|\mathrm{Fix}_\Lambda(\tau)/\mathrm{Fix}^0_\Lambda(\tau)|$ as $\langle\Lambda\rangle$ approaches infinity is equal to the topological entropy of τ, *provided that the topological entropy of* τ *is finite*. The topological entropy of the $\mathbf{Z}^d$-action σ above is always infinite. However, we will show that there is a related $\mathbf{Z}^d$-shift σ' such that (1) σ' has finite topological entropy equal to $\log \mathbf{M}(\Delta)$; and (2) $|\mathrm{Fix}_\Lambda(\sigma')/\mathrm{Fix}^0_\Lambda(\sigma')| = |\mathrm{SFix}_\Lambda(\sigma)/\mathrm{SFix}^0_\Lambda(\sigma)|$. The main conclusion of Theorem 2.1 follows from these assertions.

A $\mathbf{T}^{\mathbf{Z}^d}$-coloring of D that assigns $0 \in \mathbf{T}^{\mathbf{Z}^d}$ to some arc, say the arc corresponding to Wirtinger generator x_1, will be called a *based* $\mathbf{T}^{\mathbf{Z}^d}$-coloring. The collection of based $\mathbf{T}^{\mathbf{Z}^d}$-colorings is a closed shift-invariant subgroup of $\mathrm{Col}_{\mathbf{T},\mathbf{Z}^d}(D)$, independent of the choice of arc. It is the dual of $B = A/\langle \tilde{x}_1\rangle$, the quotient of the Alexander module by the submodule generated by $\tilde{x}_1$, which one might call the *based Alexander module* of the link. We denote the $\mathbf{Z}^d$-action on B by σ^B.

It is clear from the discussion above that for any knot the based Alexander module is isomorphic to the first homology group of the infinite cyclic cover of the knot exterior. Since all generators of a Wirtinger presentation for a knot group are conjugate, it follows that periodic points of σ^B are always special periodic points. (See the paragraph preceding Lemma 2.7 [21] for details.) Hence in this case $\mathrm{Fix}((\sigma^B)^r)$ is isomorphic to the dual of $H_1(M_r)$, for any r.

In the general case we define special periodic points of σ^B just as we defined them for σ. We claim that

$$\mathrm{SFix}_\Lambda(\sigma^B)/\mathrm{SFix}^0_\Lambda(\sigma^B) \cong \mathrm{SFix}_\Lambda(\sigma)/\mathrm{SFix}^0_\Lambda(\sigma)$$

and hence the number of connected components of $\mathrm{SFix}_\Lambda(\sigma^B)$ is equal to that of $\mathrm{SFix}_\Lambda(\sigma)$. One way to see this is by constructing a maximal tree T for the 1-skeleton of Q, selecting first a maximal number of edges of the form $\mathbf{m}\tilde{x}_1$. Recall that $\mathrm{SFix}_\Lambda(\sigma)$ can be identified with $H^1(Q,Q^0;\mathbf{T})$, and by the proof of Lemma 4.1 this group has a direct summand $\mathbf{T}^r$, the image of the map η defined above. We pass from $\mathrm{SFix}_\Lambda(\sigma)$ to $\mathrm{SFix}_\Lambda(\sigma)/\mathrm{SFix}^0_\Lambda(\sigma)$ in two stages. First, we crush $\eta(\mathbf{T}^r)$; the quotient group is isomorphic to the subgroup of $H^1(Q,Q^0;\mathbf{T})$ consisting of cocyles that vanish on the edges of the maximal tree T. However, in view of Eq. (4.1), any cocycle that vanishes on those edges vanishes on all of the edges $\mathbf{m}\tilde{x}_1$. Thus $\mathrm{SFix}_\Lambda(\sigma)/\eta(T^r) \cong \mathrm{SFix}_\Lambda(\sigma^B)$, the subgroup of $H^1(Q,Q^0;\mathbf{T})$ consisting of based $\mathbf{T}^{\mathbf{Z}^d}$-colorings. The connected component of the identity in this group is also a torus. Crushing it we find that $\mathrm{SFix}_\Lambda(\sigma)/\mathrm{SFix}^0_\Lambda(\sigma) \cong \mathrm{SFix}_\Lambda(\sigma^B)/\mathrm{SFix}^0_\Lambda(\sigma^B)$.

The based Alexander module B has an $(N-1)\times(N-1)$ presentation matrix R, which can be obtained from the matrix for the Alexander module A by deleting the first row and column. Then $\Delta(u_1,\dots,u_d)=(u_1-1)\det R$ (see [12, pp. 119–120]). Since the Mahler measure of u_1-1 is equal to 1, the determinant of R has the same Mahler measure as Δ. By Lind et al. [13, p. 611] the topological entropy of σ^B is equal to the log of the Mahler measure of Δ.

The $\mathbf{Z}^d$-action σ' that we need is a modification of σ^B. Consider based $\mathbf{Z}^d$-colorings of D, but replace any color β by a pair (β,ζ) of colors, and require in addition to the basic coloring rule (3.1) the condition:

$$\sigma_t\zeta=\zeta+\beta, \tag{4.4}$$

where t is the index of the component of l containing the arc colored by (β,ζ). Denote the associated $\mathbf{Z}^d$-action by σ'.

The $\mathbf{Z}^d$-action σ' is on the dual group of a module B' that we obtain from a presentation for B by adding new generators $z_2,\dots,z_N$ and relations $u_{t(i)}z_i=z_i+x_i$ $(2\leqslant i\leqslant N)$, where $t(i)$ is the index of the component of l containing the arc x_i. The determinant of the new relation matrix R' is $\Delta(u_1,\dots,u_d)$ times a product of polynomials of the form u_t-1. As before, since the Mahler measure of each u_t-1 is equal to 1, we have $\mathbf{M}(\det(R'))=\mathbf{M}(\Delta)$, and by Lind et al. [13, p. 611] the topological entropy of σ' is equal to this value. Hence assertion (1) above holds.

By a straightforward recursion argument we find that $\mathrm{Fix}_\Lambda(\sigma')\cong\mathrm{SFix}_\Lambda(\sigma^B)\oplus\mathbf{T}^s$. Here s is the number of second coordinates ζ that can be freely assigned: Assume that u_t represents an element of order n in $\mathbf{Z}^d/\Lambda$. Condition (4.4) implies

$$\begin{aligned}
\zeta_{\mathbf{m}+u_t}&=\zeta_{\mathbf{m}}+\beta_{\mathbf{m}},\\
\zeta_{\mathbf{m}+2u_t}&=\zeta_{\mathbf{m}}+\beta_{\mathbf{m}}+\beta_{\mathbf{m}+u_t},\\
&\vdots\\
\zeta_{\mathbf{m}+nu_t}&=\zeta_{\mathbf{m}}+\beta_{\mathbf{m}}+\beta_{\mathbf{m}+u_t}+\cdots+\beta_{\mathbf{m}+(n-1)u_t}.
\end{aligned}$$

Clearly, $\zeta_{\mathbf{m}+nu_t}=\zeta_{\mathbf{m}}$ if and only if $\beta_{\mathbf{m}+u_t}+\cdots+\beta_{\mathbf{m}+(n-1)u_t}=0$. Moreover, the coordinates $\zeta_{u_t},\dots,\zeta_{(n-1)u_t}$ are uniquely determined from $\zeta_{\mathbf{0}}$ and coordinates of β. When t' is different from t, condition (4.4) imposes no new requirement; in such a case the coordinates $\zeta_{\mathbf{0}},\zeta_{u_{t'}},\dots,\zeta_{(n'-1)u_{t'}}$ can be chosen arbitrarily, where $u_{t'}$ represents an element of order n' in $\mathbf{Z}^d/\Lambda$. Assertion (2) is immediate, and the proof of the theorem is complete. □

Example 4.4. We will illustrate the ideas and terminology used in the proof of Theorem 2.1 with an example.

The diagram D for the link $l=5_1^2$, shown in Fig. 2, yields a Wirtinger presentation for G_l:

$$\langle x_1,x_2,x_3,x_4,x_5\,|\,x_1x_3=x_5x_1,x_3x_2=x_1x_3,x_5x_4=x_3x_5,x_4x_2=x_1x_4,x_2x_4=x_5x_2\rangle.$$

We assume that under abelianization x_1 and x_2 map to u_1 while the remaining generators are sent to u_2. A portion of the maximal abelian cover $\tilde{P}$ is shown in Fig. 3. The 2-cells are not shown.

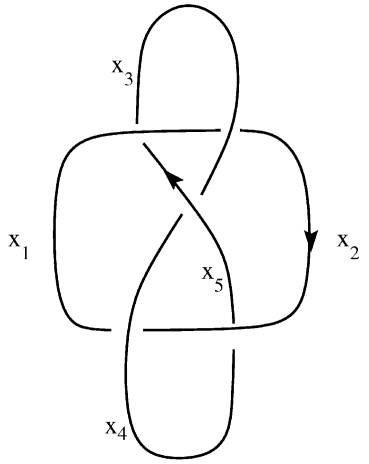

Fig. 2. Diagram for 5_1^2.

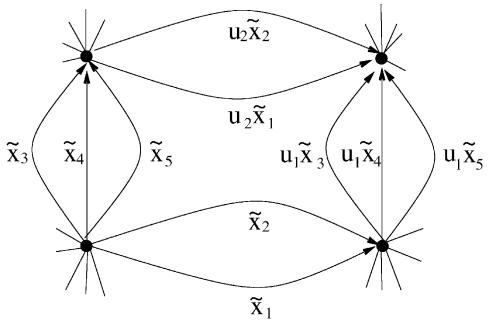

Fig. 3. Portion of $\tilde{P}$.

We consider the subgroup Λ of $\mathbf{Z}^2$ generated by u_1^3 and u_2^2. The 1-skeleton of Q (which is the same as the 1-skeleton of $\tilde{P}/\Lambda$) is shown in Fig. 4. Since Λ has generators parallel to u_1, u_2, it is easy to visualize the additional 2-cells that must be attached to $\tilde{P}/\Lambda$ in order to build Q. In more general examples, the boundaries of the new cells might wind around the graph several times.

An element $x \in \mathrm{Fix}_\Lambda(\sigma^B)$ can be represented by 3×2 matrices $\beta, \gamma, \delta, \varepsilon$ assigned to arcs corresponding to x_2, x_3, x_4, x_5, respectively. These matrices have entries in $\mathbf{T}$, and are the restrictions of the elements of $\mathbf{T}^{\mathbf{Z}^d}$ to a fundamental region of $\mathbf{Z}^d/\Lambda$. The element x is in $\mathrm{SFix}_\Lambda(\sigma^B)$ if the column sums of β and the row sums of $\gamma, \delta, \varepsilon$ are all zero.

An element of $\mathrm{Fix}_\Lambda(\sigma')$ assigns additional 3×2 matrices $\zeta^\beta, \zeta^\gamma, \zeta^d, \zeta^\varepsilon$. We can prescribe the coordinates $\zeta^\beta_{0,0}, \zeta^\beta_{0,1}$ arbitrarily; the other coordinates of ζ^β are uniquely determined by these and β. Similarly, the coordinates $\zeta^\gamma_{0,0}, \zeta^\gamma_{1,0}, \zeta^\gamma_{2,0}, \zeta^\delta_{0,0}, \zeta^\delta_{1,0}, \zeta^\delta_{2,0}, \zeta^\varepsilon_{0,0}, \zeta^\varepsilon_{1,0}, \zeta^\varepsilon_{2,0}$ are arbitrary. We have $\mathrm{Fix}_\Lambda(\sigma') \cong \mathrm{SFix}_\Lambda(\sigma) \oplus \mathbf{T}^{11}$.

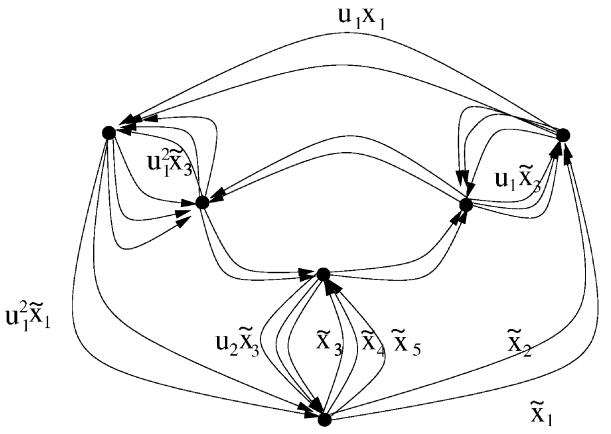

Fig. 4. Skeleton of Q.

We remark that for this link the Alexander polynomial is $(1-u_1)(1-u_2)$. Since the Mahler measure of the polynomial is 1, the orders b_Λ have zero exponential growth rate.

Proof of Corollary 2.2. Since Δ_1/Δ_2 annihilates the Alexander module of any knot [3], it follows that $\sigma^N = \mathrm{id}$. From this we have $\mathrm{Fix}(\sigma^{r+N}) = \mathrm{Fix}(\sigma^r)$. Recall that for any knot, $\mathrm{Fix}(\sigma^r)$ is isomorphic to the dual of $H_1(M_r)$. Hence $H_1(M_{r+N}) \cong H_1(M_r)$, for every $r \geqslant 1$. □

5. Coloring with nonabelian groups

The coloring rule (3.1) generalizes in a natural way, allowing one to replace $\mathbf{T}$ with an arbitrary topological group Σ.

Definition 5.1. Assume that D is a diagram of an oriented link $l = l_1 \cup \cdots \cup l_d$ of d components. A $\Sigma^{\mathbf{Z}^d}$-*coloring* of D is an assignment of elements (*colors*) $\alpha, \beta, \ldots \in \Sigma^{\mathbf{Z}^d}$ to the arcs of D such that the condition

$$\alpha \cdot \sigma_t \beta = \gamma \cdot \sigma_{t'} \alpha \tag{5.1}$$

is satisfied at any crossing. The colors α, β, γ correspond to arcs that are described as in Definition 3.1.

As before, if D consists of N arcs, then the set $\mathrm{Col}_{\Sigma,\mathbf{Z}^d}(D)$ of all $\Sigma^{\mathbf{Z}^d}$-colorings of D is a closed subspace of $[\Sigma^N]^{\mathbf{Z}^d}$ that is invariant under $\sigma_{\mathbf{m}}$ for each $\mathbf{m} \in \mathbf{Z}^d$. That $\mathrm{Col}_{\Sigma,\mathbf{Z}^d}(D)$ does not depend on the choice of diagram for l follows immediately from the following. Let $\tilde{E}$ denote the maximal abelian cover of the link exterior with projection $p : \tilde{E} \to E$, and let $*$ be a point of E. Let $\tilde{*}$ denote a fixed lift of $*$. Any covering automorphism of $\tilde{E}$ induces a homeomorphism of the quotient space $\tilde{E}/p^{-1}(*)$, and hence induces an automorphism of $\pi_1(\tilde{E}/p^{-1}(*), \tilde{*})$. By considering the adjoint action we obtain a homeomorphism of the representation space

$\mathrm{Hom}[\pi_1(E/p^{-1}(*),\tilde{*}),\Sigma]$. In this way we obtain a $\mathbf{Z}^d$-action σ' on $\mathrm{Hom}[\pi_1(E/p^{-1}(*),\tilde{*}),\Sigma]$. A $\mathbf{Z}^d$-action on a topological space is defined just as for $\mathbf{Z}^d$-action on a topological group, eliminating the requirement of a group structure on the space; two $\mathbf{Z}^d$-actions, σ acting on X and σ' acting on X', are *topologically conjugate* if there is a homeomorphism $\phi : X \to X'$ such that $\phi \circ \sigma_{\mathbf{m}} = \sigma'_{\mathbf{m}} \circ \phi$, for each $\mathbf{m} \in \mathbf{Z}^d$.

Proposition 5.2. *The $\mathbf{Z}^d$-actions on $\mathrm{Hom}[\pi_1(\tilde{E}/p^{-1}(*),\tilde{*}),\Sigma]$ and $\mathrm{Col}_{\Sigma,\mathbf{Z}^d}(D)$ are topologically conjugate.*

Proof. The quotient space $\tilde{E}/p^{-1}(*)$ has the same fundamental group as the quotient complex $\tilde{P}/\tilde{P}^0$, where $\tilde{P}$ is defined in Section 4 and $\tilde{P}^0$ is the 1-skeleton. There is a group presentation for $\pi_1(\tilde{P}/\tilde{P}^0)$ in which the generators correspond to the edges of $\tilde{P}$; lifts in $\tilde{P}$ of closed paths representing Wirtinger relators are closed paths in $\tilde{P}/\tilde{P}^0$ representing the relators. Assignments of colors to the arcs of D, or equivalently to the edges of P, such that condition (5.1) holds at each crossing then correspond to homomorphisms from $\pi_1(\tilde{P}/\tilde{P}^0)$ to Σ. The correspondance defines a homeomorphism $\phi : \mathrm{Col}_{\Sigma,\mathbf{Z}^d}(D) \to \mathrm{Hom}[\pi_1(\tilde{E}/p^{-1}(*),\tilde{*}),\Sigma]$ such that $\phi \circ \sigma_{\mathbf{m}} = \sigma'_{\mathbf{m}} \circ \phi$, for each $\mathbf{m} \in \mathbf{Z}^d$. □

In view of this proposition we call $\mathrm{Col}_{\Sigma,\mathbf{Z}^d}(D)$ the *color Σ^d-shift* of the link l, and we denote it by $\mathrm{Col}_{\Sigma,\mathbf{Z}^d}(l)$.

The abelianization of $\pi_1(\tilde{E}/p^{-1}(*))$ is $H_1(\tilde{P},\tilde{P}^0)$. It is isomorphic to $H_1(\tilde{E},p^{-1}(*))$, the Alexander module of the link. Hence we propose that $\pi_1(\tilde{E}/p^{-1}(*))$ be called the *Alexander group* of the link; we denote the group by $\mathscr{A}_l$. (Shortly after completing the first draft of this paper, the authors discovered that the Alexander group is a special case of the *derived group of a permutation representation*, introduced by Crowell [4].)

It is much easier to write a presentation for the Alexander group than for the commutator subgroup of $\pi_1(S^3 - l)$. One begins with families of generators $a_{\mathbf{m}}, b_{\mathbf{m}}, c_{\mathbf{m}}, \ldots (\mathbf{m} \in \mathbf{Z}^d)$ corresponding to the arcs of the diagram. Each crossing gives rise to a family of relations: a crossing such as in Fig. 1 imposes the relation $a_{\mathbf{m}} \cdot b_{\mathbf{m}+u_l} = c_{\mathbf{m}} \cdot a_{\mathbf{m}+u'_l}$. In the case of a knot, when $d = 1$, presentations of this sort are well known; Hausmann and Kervaire [10] termed them **Z**-*dynamic*. A presentation for the Alexander group of a link such as the one we have described might be called $\mathbf{Z}^d$-*dynamic*.

The next proposition describes the relationship between the Alexander group $\mathscr{A}_l$ of a link l and the commutator subgroup G'_l. We use the terminology of Section 4. Recall that T is a maximal tree in the 1-skeleton of the cover $\tilde{P}$.

Proposition 5.3. *Let l be an oriented link of d components. The generators of $\mathscr{A}_l$ corresponding to the edges of T freely generate a subgroup $F(E_T)$ of $\mathscr{A}_l$. Moreover, $\mathscr{A}_l$ is isomorphic to the free product $G'_l * F(E_T)$.*

Proof. Let $C\tilde{P}^0$ denote the cone on the 0-skeleton of $\tilde{P}$. The fundamental group of $X = \tilde{P} \cup_{\tilde{P}^0} C\tilde{P}^0$ is isomorphic to $\mathscr{A}_l$. We can regard X as the union of $\tilde{P}$ and $T \cup_{\tilde{P}^0} C\tilde{P}^0$, which have

contractible intersection T. An application of the Seifert van-Kampen theorem completes the argument. □

Example 5.4. We examine two examples. Both are simple, but they highlight some of the advantages of working with the Alexander group rather the commutator subgroup of a link:

(i) Let l be the trivial 2-component link. The group G_l is free on two generators. Choosing a diagram without crossings, we find that the Alexander group $\mathscr{A}_l$ is free on generators $a_{i,j}, b_{i,j}$, where i, j range over $\mathbf{Z}$. The commutator subgroup G' is also free, but it does not admit a natural $\mathbf{Z}^2$-action by automorphisms as does $\mathscr{A}_l$.

(ii) Next consider the link $l = 2_1^2$, a Hopf link. The group G_l is free abelian of rank 2. The Alexander group $\mathscr{A}_l$ has presentation $\langle a_{i,j}, b_{i,j} \mid a_{i,j} b_{i+1,j} = b_{i,j} a_{i,j+1} \rangle$. In this example the commutator subgroup G_l' is trivial.

6. Conclusion

A possible direction for further inquiry involves links with zero Alexander polynomial.

The homology growth rate in Theorem 2.1 was computed as the topological entropy of a $\mathbf{Z}^d$-action σ'. When the Alexander polynomial of the link is zero, the entropy of σ' can be shown to be infinite; in such a case we obtained no information. However, we offer the following.

Conjecture 6.1. *If l is an oriented link of d components, then*

$$\lim_{\langle \Lambda \rangle \to \infty} \frac{1}{|\mathbf{Z}^d/\Lambda|} \log |TH_1(M_\Lambda(l))| = \log \mathbf{M}(\Delta_i),$$

where Δ_i is the first nonzero Alexander polynomial of the link.

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