COMPACT ABELIAN ACTIONS ON THE TWO CELL

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It is shown that a compact abelian group of continuous self maps of the two dimensional Euclidean cell has a common fixed point. The group identity is not assumed to be represented by the identity homeomorphism.

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common fixed point almost periodic transformation group

0. Introduction

The author's understanding of the history of the Dyer-Isbell problem is as follows. In 1950, Jack Schwartz posed to his roommate at Yale, Sol Schwartzman, whether an abelian family of continuous transformations of a compact convex set in a Banach space need have a common fixed point. Schwartzman passed this problem on to Eldon Dyer who then pointed out the special case of dimension one [personal communication from Sol Schwartzman]. Several early papers suggest that the commuting function problem on the interval occurred independently to Lester Dubins and to Alan Shields. This specific case probably had its first published appearance in Isbell's research announcement [6] which also contained a positive result which we will comment on below. The formulation of the problem as the existence of a common fixed point for a pair of commuting continuous self maps of the unit interval was killed by counterexamples given simultaneously by Boyce [3] and Huneke [5]. Several positive results connected with the problem suggested that a compact abelian group of homeomorphisms of a Euclidean cell should have a common fixed point. This conjecture was killed by Oliver [13] who gave a finite abelian group of differentiable maps acting on a cell whose dimension is probably no greater than $2^2 \cdot 3^2 \cdot 5^2$. Positive results in spaces of small dimension may be of interest in applications so we prove:
Theorem. Let $G$ be a compact abelian group of continuous self maps of $\Delta$, the two-dimensional cell. Then $\text{Fix}(G)$ is not empty.

Note that it is not assumed that the group identity acts as the identity transformation of $\Delta$. In any event the action of the group identity is as a continuous idempotent map of $\Delta$. Thus we may restrict attention to a subset $M$ in $\Delta$ which is the range of the group identity. $M$ is a retract of $\Delta$ and every member of $G$ acts as a homeomorphism on $M$.

1. Preliminaries

The result of this paper also holds for the cell of dimension one (even without assuming that the group is abelian). We will have need of it.

Proposition 1.1. Let $G$ be a compact group of continuous self maps of the interval. Then $\text{Fix}(G)$ is nonempty.

Remark. This result follows from [4, 3.24, p. 333]. Mitchell [8] and Lau [7] have given versions for semigroup actions satisfying various algebraic and analytic assumptions. Mitchell shows how little content there is to the Proposition by obtaining it as a corollary to the observation that such a group of homeomorphisms has at most two elements. This depends on two facts. A homeomorphism belonging to a compact group must have equicontinuous iterates and a homeomorphism of an interval is either isotone or antitone. If $T$ is an isotone homeomorphism and $Tx \lt x$ for some $x$ then $T^n x$ is a strictly decreasing sequence and $\{T^n\}$ cannot possibly be equicontinuous. (The same of course for $Tx \gt x$.) Thus an isotone member of $G$ is necessarily the identity. If $R$ is an antitone member of $G$ then $R^2$ is isotone so $R^2 = \text{id}$.

Now if the group identity is not the identity transformation we can move the action to that retract of the unit interval which is the range of the group identity. The previous remarks apply for a retract of the unit interval is another closed interval.

We will also have need of the next result which has almost the same proof.

Proposition 1.2. Let $K$ be a closed subset of the unit interval. Let $G$ be a compact group of monotone homeomorphisms of $K$. Then either $\text{Fix}(G)$ is nonempty or there is a pair of points, $\{p, q\}$, in $K$ so that $\{p, q \cap K = \emptyset$ and $\{p, q\}$ is invariant as a set under every member of $G$. The fixed-point set is all of $K$, or a single point, or is empty.

The next result is less transparent. This is apparently well known in the transformation group community and can be assembled from theorems of Chapter 6 of Montgomery and Zippin [10] although the statement itself is not to be found there. The group, if effective, must then be Lie under the hypotheses. This is in contrast
to the situation in the Main Theorem where clearly the group represented as continuous maps could be a non-Lie 0-dimensional group.

**Proposition 1.3.** Let $\Delta$ be the 2-cell. Let $G$ be a compact abelian group of homeomorphisms of $\Delta$. Then $\text{Fix}(G)$ is not empty.

**Proof.** We will draw freely from the results and methods of [11] and [12] but will give some indication of where those results came from. The argument will depend on detailed knowledge of the structure of a single homeomorphism which generates a compact group. For such a map $T$, the $\omega$-limit set for any point $x$ is a minimal set, contains $x$, and can be given the structure of a compact monothetic group. The action of $T$ on the minimal set is given by a group translation in the induced monothetic group structure. Now not too many compact monothetic groups are simple enough to be embeddable in $\Delta$. A theorem of Abe and Kodaira [11] can be used to show that the minimal sets of $(\Delta, T)$ can only be topological circles or totally disconnected sets. (See also the generalization of this result in [2] which is a more accessible reference.) If $(\Delta, T)$ admits one circle as a minimal set, it is shown in [11] that nearby points must also have circles for their minimal sets. In the case of the existence of one circle for a minimal set the equicontinuity of $\{T^n\}$ shows then that $T$ 'is' an irrational rotation of all of $\Delta$.

The case of no minimal circles is handled by appealing to theorems of Whyburn [17, p. 191] and Montgomery [9] to claim that $T^N = \text{id}$ on $\Delta$ for some integer $N$. Further details in obtaining global periodicity can be found in [12] where an over enthusiastic application of the Whyburn and Montgomery theorems claims global periodicity on the entire interior when it should be only for each connected component of the interior. If $T$ on $\partial \Delta$ is orientation preserving then $T$ on all of $\Delta$ is a (rational) rotation. Unless $T = \text{id}$ on $\Delta$ then $\text{Fix}(T)$ is exactly one fixed point (just as in the case of the irrational rotation).

Finally if $T$ on $\partial \Delta$ is orientation reversing then $T$ has two fixed points on $\partial \Delta$ and interchanges the open boundary arcs between them. Let $p_1$ and $p_2$ be the fixed points on $\partial \Delta$. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be the open arcs between them. For each $q$ in $\mathcal{I}_1$, let $\gamma(q)$ be the straight line in $\Delta$ from $q$ to $T(q)$. Let $\gamma_1 = T \gamma$. Let $E(q) = \gamma \cup \gamma_1$. This set is invariant under $T$ (as $T$ is globally period two in this case). From $E(q)$ we can select an invariant arc in $\Delta$ from $q$ to $T(q)$. (For look at the outer boundary in $\Delta$ of the component of the complement of $E(q)$ which contains $p_1$.). Now $T$ on $E(q)$ is just the situation in Proposition 1.1. There is exactly one point $w(q)$ in $E(q)$ which is fixed under $T$. Using the fact that $T^2 = \text{id}$ on $\Delta$ it is not difficult to show $q \mapsto w(q)$ is a retract of $\Delta$ and that $\text{Fix}(T)$ is a topological diameter of $\Delta$.

Thus we have the necessary ingredients for a downward induction argument. For any finite set $\{T_1, \ldots, T_n\}$ in $G$ the set $\text{Fix}(T_1, \ldots, T_n)$ is nonempty and, as it is always a retract of $\Delta$, has the fixed-point property for a single homeomorphism. The finite covering then implies $\text{Fix}(G)$ is nonempty. Moreover $\text{Fix}(A)$ is either all of $\Delta$, a topological diameter of $\Delta$, or a single point interior to $\Delta$. \(\Box\)
We have one more preliminary result. It is the discrete version of the Theorem. Recall that a finite graph is a finite set of points called nodes together with a collection of unordered pairs of distinct nodes called edges. What it means for a graph to be connected is the obvious thing. If the graph is connected, but removal of any edge disconnects the graph, the graph is called a tree.

**Proposition 1.4.** Let $E$ be a finite tree. Then there is a universal fixed element (either a node or an edge) of $E$ in the sense that the element is fixed for every tree automorphism.

This result can be found in [14, p. 201]. The proof is elementary. If the length of a longest line in $E$ is odd, the middle node of that line is an invariant node. If the length is even, then any automorphism leaves the middle pair of nodes invariant as a set and the fixed element in this case is the edge connecting the middle pair of nodes of this line.

2. Main result

In this section we prove the theorem in two parts. Let $M$ be the retract of $\Delta$ which is the range of the group identity. The two cases are according to whether int($M$) is empty or not.

**Proposition 2.1.** Let $M$ be a retract of $\Delta$ with empty interior. Let $G$ be a compact abelian group of homeomorphisms of $M$. Then Fix($G$) is not empty.

**Proof.** We claim for any finite set $\{T_1, \ldots, T_n\}$ in $G$, that Fix($T_1, \ldots, T_n$) is non-empty, compact, and arcwise connected. It will follow at once from the claim that Fix($G$) is nonempty by the finite-intersection property. For the $n = 1$ case of the claim observe that Fix($T_1$) is certainly compact and is nonempty as any retract of $\Delta$ will also have the fixed-point property for a single continuous self map. Now $M$ itself is arcwise connected, again by the retract property. So if $p$ and $q$ are in Fix($T_1$) let $y$ be any arc from $p$ to $q$ in $M$. Since $M$ is a retract of $\Delta$ with empty interior there are no Jordan curves in $M$. This implies that $y$ and $T_1 y$ have the same carrier set in $M$. Thus $T_1$ on $y$ satisfies the hypothesis of Proposition 1.1 and each point of $y$ is fixed by $T_1$. Then $y$ is actually in Fix($T_1$). This establishes the claims for $n = 1$.

Suppose the assertion is valid for $\{T_1, \ldots, T_n\}$ and that $T_{n+1}$ is another member of $G$. Let $y_1$ be an arc in $M$ from $p$ in Fix($T_1, \ldots, T_n$) to $q$ in Fix($T_{n+1}$). Let $y_2 = T_{n+1} y_1$ and let $y_3$ be an arc from $p = y_1(1)$ to $r = y_2(1)$ which lies in Fix($T_1, \ldots, T_n$). Unless $T_{n+1} p = p$ and Fix($T_1, \ldots, T_{n+1}$) is nonempty. This last set is shown to be arcwise connected by a third application of the same trick. □

**Remark.** This result follows from Isbell's Theorem [6]. In that brief research announcement the case of a single homeomorphism of a tree is not explicitly
established. This case (with a slight variation as to the definition of a tree) is known as the Scherrer fixed-point Theorem [15]. See [16, p. 68] for a discussion of the history of this and related results. The easy argument here is included for completeness and because with our stronger hypotheses we will show that \( \text{Fix}(G) \) is a retract which is not true for homeomorphisms of trees.

For the remaining case, \( \text{int}(M) \) nonempty, we will make use of cyclic element theory. This can be found in [17] and in a more readable form in [18].

**Proposition 2.2.** Let \( M \) be a retract of \( \Delta \) with \( \text{int}(M) \) nonempty. Let \( G \) be a compact abelian group of homeomorphisms of \( M \). Then \( \text{Fix}(G) \) is nonempty.

**Proof.** If \( p \) is an interior point of \( M \), then \( p \) is not a cut point of \( M \) and \( C(p) \) is a topological disk as shown by the Torhorst Theorem [18, p. 126] and the fact that \( C(p) \) is a retract of \( M \) and thus of \( \Delta \) as well. Any member \( T \) of \( G \) must permute the maximal topological disks in \( M \). If \( W \) is such a disk of \( M \) we can measure the size of \( W \) as follows. Let \( R(W) \) be the supremum of the numbers \( r \) such that \( W \) contains the metric disk \( B_r(x) \). We can use the compactness of \( G \) to control the possible distortion by a member \( T \) of \( G \). For a given \( \varepsilon > 0 \) there is a \( \delta > 0 \) so that for \( x \) and \( y \) in \( M \) with \( \text{dist}(x, y) > \varepsilon \) we have

\[
\text{dist}(T^n x, T^n y) > \delta
\]

for all integers \( n \) and all \( T \) in \( G \). This implies that if \( B \) is a metric disk in \( M \) of radius at least \( \varepsilon \) then for any \( n \) and any \( T \) in \( G \) the image \( T^n B \) must contain a metric disk of radius at least \( \delta \). Then for any maximal topological disk \( W \) there is a minimum size for any image of \( W \). We conclude from this that for this disk \( W = C(p) \) the orbit, \( GW \), consists of a finite number of maximal topological disks by simple considerations of area. For \( W \) as above we now give \( GW \) a graph structure. The nodes of \( E \) are the maximal disks of \( GW \). For \( TW \) and \( SW \) in \( GW \) we will say that \( TW \) and \( SW \) are directly connected if there is a curve (perhaps degenerate) from \( TW \) to \( SW \) in \( M \) which meets no other member of \( GW \). The edges of \( E \) are then determined by the direct connections in \( GW \). The maximality of the cyclic elements of \( GW \) implies that \( E \) is a finite tree. So we can appeal to Proposition 1.4. Either \( GW \) contains a topological disk which is fixed under every member of \( G \) or \( GW \) contains a directly connected pair, \( W_1 \) and \( W_2 \), which is invariant as a pair under every member of \( G \). In the first case Proposition 1.3 finishes the proof. We now turn to the second case.

We have \( W_1 = C(p_1) \) and \( W_2 = C(p_2) \). Let \( K \) be the set of cut points of \( M \) which lie between \( W_1 \) and \( W_2 \). Assume for the moment that \( K \) is not empty. Now each arc in \( M \) running from \( W_1 \) to \( W_2 \) must go through the points of \( K \) in the same order. The natural order of \( K \) is exactly that induced in \( K \) by the order of \( [0, 1] \) and any such arc \( \gamma \). It is clear that any \( T \) in \( G \) is a homeomorphism of \( K \) and is either isotone or antitone in the order of \( K \). Thus Proposition 1.2 implies either a common fixed point for \( G \) in \( K \) (and we are done) or a pair of cut points \( x_i \) and
x, which are between W, and W, and have no cut point between them. The set \( \{x_1, x_2\} \) is G invariant. Let \( p_3 \) be a point on \( \gamma \) between \( x_1 \) and \( x_2 \). Then \( C(p_3) \) cannot be a cut point so must be a topological disk (which contains \( x_1 \) and \( x_2 \)). This disk \( C(p_3) \) is G invariant so we are back to Proposition 1.3 for a common fixed point.

Finally we consider the case of \( K \) being empty. If the curve \( \gamma \) was trivial \( W_1 \) and \( W_2 \) meet at a single (invariant) cut point of \( M \). Otherwise pick \( p_3 \) on that arc between \( p_1 \) and \( p_2 \) and consider \( C(p_3) \). This is a topological disk which contains \( p_1 \) and \( p_2 \) so must be invariant. A last appeal to Proposition 1.3 finishes the proof. \( \square \)

3. Retract properties

One standard way of establishing a common fixed-point property is to show that Fix(\( T_1, \ldots, T_n \)) has the fixed-point property for single continuous maps for each finite set \( \{T_1, \ldots, T_n\} \). If it were shown that Fix(\( T_1, \ldots, T_n \)) was a retract of \( \Delta \) that would be the case. But we have not shown this. We will now argue that Fix(G) is a retract and this, in turn, implies that Fix(\( T_1, \ldots, T_n \)) is as well. This last claim is by looking at \( G_1 \), the minimal compact group containing \( \{T_1, \ldots, T_n\} \).

To show Fix(G) is a retract we define first
\[
K = \text{Fix}(G) \cup \{C(p) : p \in \text{Fix}(G)\}.
\]
We can show that \( K \) is compact by considering \( \{q_n\} \) in \( K \) with \( q_n \to q_0 \). Then \( \{q_0\} \) is compact so for any convergent sequence \( \{p_n\} \) in \( M \setminus q_0 \) either the sequence converges to \( \{q_0\} \) or is ultimately in a single component of \( M \setminus q_0 \). In either case then we find that \( K \) is closed under sequential limits. Now \( K \) compact implies that any convergent sequence \( \{p_n\} \) in \( M \setminus K \) either converges to \( K \) or is ultimately in a single component of \( M \setminus K \). Thus each component of \( M \setminus K \) has exactly one limit point in \( K \). This tells us how a retract of \( M \) onto \( K \) must be defined. The same trick shows that the map that sends each entire component of \( M \setminus K \) to that limit point of \( K \) is continuous so is a retract. Now \( K \) differs from Fix(G) by including all of any topological disk \( C(p) \) which has Fix(G) \( \cup C(p) \) as a proper subset. The subset is either a single point or a topological diameter. In each of these cases the disk can be collapsed to the fixed subset of the disk. That this can be done simultaneously over \( K \) to be a continuous retract map follows once again from the twice used property of locally connected spaces.

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