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Note

About the number of C^∞-words of form $\tilde{w}xw$ ☆

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Abstract

Let $p_i(n)$ denote the number of the C^∞-words of form $\tilde{w}xw$ with length $2n+i$ and gap i, where i is the length of word x. In this paper, we prove the following result: $p_i(n) = 6$ for all $n \geq 1$ and $i = 2, 3, 4$. Moreover, we provide a complete solution of $\tilde{w}xw \in C^{b\omega}$ for $|x| \leq 4$.

Keywords: Kolakoski sequence; Derivative; C^∞-words; Δ-operator; C^ω-words; $C^{b\omega}$-words; Palindrome

1. Introduction

Since the beginning of the last century, combinatorics on words are of an increasing importance in various fields of science like computer science, mathematics, biology, physics or crystallography. In particular, palindromes play an important role among the regular patterns. The palindrome complexity has been studied in [1–7].

Brlek and Ladouceur [3] recently described a general framework for the study of a particular class of infinite words over the 2-letter alphabet $\Sigma = \{1, 2\}$, which is invariant under the action of the run-length encoding operator. This class is related to the curious Kolakoski sequence [8]

$$K = \underbrace{22}_{2}\,\underbrace{11}_{2}\,\underbrace{2}_{1}\,\underbrace{1}_{1}\,\underbrace{22}_{2}\,\underbrace{1}_{1}\,\underbrace{22}_{2}\,\underbrace{11}_{2}\,\underbrace{2}_{1}\,\underbrace{11}_{2}\,\underbrace{22}_{2}\,\underbrace{\cdots}_{\cdots}$$

which received a noticeable attention and shows some intriguing combinatorial properties, constituting mainly a bouquet of conjectures. They proved that the palindromes of this class are characterized by the left palindromic closure of the prefixes of the Kolakoski sequences and revealed an interesting perspective for understanding some of the conjectures. In particular, recurrence, mirror invariance and permutation invariance are all direct consequences of the presence in K of these palindromes.

But in [3], the key step of the proof process of Proposition 7: $\Delta(qx) \notin Pref(K) \Longrightarrow D(qx) \notin Pref(K)$ seems to be false. In fact, we have $\Delta(w) = D(w)u$ for $w = 22\ldots$, where $u = \varepsilon$ or 1. Hence $D(qx) \notin Pref(K) \Longrightarrow \Delta(qx) \notin$

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Pref(K) holds. For avoiding this bug, in [7] we provided a very simple proof of $p(n) = 2$ for all positive integers, by the uniqueness of the palindromic extension of C^∞-palindromes, where $p(n)$ denotes the number of C^∞-palindromes with length n.

Because any palindrome is of form $\tilde{w}w$ or $\tilde{w}\alpha w$, where $\alpha \in \Sigma$, we can consider the palindromes as the C^∞-words of form $\tilde{w}xw$ with gaps 0, 1. So in general we naturally ask what the complexity of C^∞-words of form $\tilde{w}xw$ is. In Section 3, we discuss the number of C^∞-words of form $\tilde{w}xw$ with gaps 2, 3, 4. In Section 4, we explore the infinite C^∞-words w satisfying the condition $\tilde{w}xw \in C^{b\omega}$ and obtain a complete solution.

2. Definitions and notations

This section presents useful notions borrowed from [3]. Let $\Sigma = \{1, 2\}$, Σ^* denotes the free monoid over Σ. A finite word over Σ is an element of Σ^*. If $w = w_1 w_2 \cdots w_n$, $w_i \in \Sigma$ for $i = 1, 2, \ldots, n$ then n is called the length of the word w and denoted by $|w|$. If $|w| = 0$ then w is called the empty word and denoted by ε. The number of occurrences of a letter $\alpha \in \Sigma$ is $|w|_\alpha$. Obviously, the length of a word is given by the number of its letters: $|w| = |w|_1 + |w|_2$.

The set of all right infinite words is denoted by Σ^ω, the set of all left infinite words is denoted by $\Sigma^{l\omega}$, the set of all two-sided infinite words is denoted by $\Sigma^{b\omega}$. Given a word $w \in \Sigma^*$, a factor u of w is a word $u \in \Sigma^*$ satisfying $\exists x, y \in \Sigma^*$ such that $w = xuy$. If $x = \varepsilon$ (*resp.* $y = \varepsilon$) then u is called a prefix (resp. suffix). A block of length k is a factor of the particular form $u = \alpha^k, \alpha \in \Sigma$. *Pref*$(w)$ denotes the set of all prefixes of w. Finally $N^*, N^\omega, N^{l\omega}$ and $N^{b\omega}$ denote the free monoids, the set of all right infinite words, the set of all left infinite words and the set of all two-sided infinite words over N respectively, where N is the set of all positive integers.

The mirror image of $u = u_1 u_2 \cdots u_n \in \Sigma^*$ is the word $\tilde{u} = u_n u_{n-1} \cdots u_2 u_1$. It is obvious that $u \in \Sigma^{l\omega} \Longleftrightarrow \tilde{u} \in \Sigma^\omega$. A palindrome is a word P such that $P = \tilde{P}$. The complement or permutation of the letters, defined by $\bar{1} = 2, \bar{2} = 1$, which is extended to words as well. The permutation of $u = u_1 u_2 \cdots u_n \in \Sigma^*$ is the word $\bar{u} = \bar{u}_1 \bar{u}_2 \cdots \bar{u}_n$.

We see that every word $w \in \Sigma^*$ can be uniquely written as a product of factors as follows:

$$w = \alpha^{i_1} \bar{\alpha}^{i_2} \alpha^{i_3} \bar{\alpha}^{i_4} \ldots, \quad \text{where } i_j > 0.$$

The operator giving the size of the blocks appearing in the coding is a function

$$\Delta : \Sigma^* \to N^*, \quad \text{defined by}$$
$$\Delta(w) = i_1 i_2 i_3 \cdots = \prod_{k \geq 1} i_k$$

which is easily extended to infinite words and two-sided infinite words respectively.

For any $w \in \Sigma^*$ (or Σ^ω), $s(w)$ denotes the first letter of the word w. For each $w \in \Sigma^*$ (or $\Sigma^{l\omega}$), $e(w)$ denotes the last letter of the word w. It is clear that the operator Δ satisfies the property: $\Delta(uv) = \Delta(u)\Delta(v)$ if and only if $e(u) \neq s(v)$.

The function Δ is not bijective because $\Delta(w) = \Delta(\bar{w})$ for every word w. However, pseudo-inverse functions

$$\Delta_1^{-1}, \Delta_2^{-1} : \Sigma^* \to \Sigma^*$$

can be defined by

$$\Delta_1^{-1}(u) = 1^{u_1} 2^{u_2} 1^{u_3} 2^{u_4} \cdots$$
$$\Delta_2^{-1}(u) = 2^{u_1} 1^{u_2} 2^{u_3} 1^{u_4} \cdots$$

which is easily extended to Σ^ω and $\Sigma^{l\omega}$.

But if Δ_i^{-1} is extended from Σ^* to $\Sigma^{b\omega}$ in the similar way as follows:

$$\Delta_1^{-1}, \Delta_2^{-1} : \Sigma^{b\omega} \to \Sigma^{b\omega}, \quad u = \cdots u_{-3} u_{-2} u_{-1} u_0 u_1 u_2 u_3 \cdots$$
$$\Delta_1^{-1}(u) = \cdots 2^{u_{-3}} 1^{u_{-2}} 2^{u_{-1}} 1^{u_0} 2^{u_1} 1^{u_2} 2^{u_3} \cdots$$
$$\Delta_2^{-1}(u) = \cdots 1^{u_{-3}} 2^{u_{-2}} 1^{u_{-1}} 2^{u_0} 1^{u_1} 2^{u_2} 1^{u_3} \cdots.$$

If $w = \cdots w_{-3}w_{-2}w_{-1}w_0w_1w_2w_3\ldots$, and $w = u$, then there exists $k \in \mathcal{Z}$ such that $w_i = u_{i+k}$ for all $i \in \mathcal{Z}$. Clearly, if k is odd, then $\Delta_i^{-1}(w) = \overline{\Delta_i^{-1}(u)}$; if k is even, then $\Delta_i^{-1}(w) = \Delta_i^{-1}(u)$ for $i = 1, 2$. Hence Δ_i^{-1} $(i = 1, 2)$ is not a function from $\Sigma^{b\omega}$ to $\Sigma^{b\omega}$. But $\Delta_i^{-1}(w)$ is unambiguous for a fixed $w \in \Sigma^{b\omega}$. The following property is immediate:

$$\forall u \in \Sigma^*(\Sigma^\omega, \Sigma^{l\omega}, \Sigma^{b\omega}): \qquad \Delta_\alpha^{-1}(u) = \overline{\Delta_{\bar\alpha}^{-1}(u)}.$$

The operator Δ over Σ^ω has two fixpoints, that is $\Delta(K) = K$, $\Delta(1K) = 1K$. Since $\Delta(\tilde{K}1K) = \tilde{K}1K$, $\tilde{K}1K$ is a fixpoint of Δ over $\Sigma^{b\omega}$. But we do not know whether it is the only fixpoint.

We say that a finite word $w \in \Sigma^*$ in which neither 111 or 222 occurs is differentiable, and its derivative, denoted by $D(w)$, is the word whose jth symbol equals the length of the jth run of w, discarding the first and/or the last run if it has length one. It is clear that D is an operator from Σ^* to Σ^* and

$$D(w) = \begin{cases} \varepsilon, & \Delta(w) = 1 \text{or } w = \varepsilon \\ \Delta(w), & \Delta(w) = 2x2 \text{ or } \Delta(w) = 2 \\ x2, & \Delta(w) = 1x2 \\ 2x, & \Delta(w) = 2x1 \\ x, & \Delta(w) = 1x1. \end{cases}$$

Obviously, if $w \in C^\infty$ and $|w| > 0$, then $|D(w)| < |w|$. Moreover D is an operator from Σ^* to Σ^*, and D and Δ can be all iterated.

Definition. (1) $w \in \Sigma^*$ is C^∞ if $\exists k \in N$ such that $D^k(w) = \varepsilon$. The class of C^∞-words is denoted by C^∞.
(2) $w \in \Sigma^\omega$ is C^ω if for all k $\in N$ such that $\Delta^k(w) \in \Sigma^\omega$. The class of C^ω-words is denoted by C^ω.
(3) $w \in \Sigma^{l\omega}$ is $C^{l\omega}$ if for all k $\in N$ such that $\Delta^k(w) \in \Sigma^{l\omega}$. The class of $C^{l\omega}$-words is denoted by $C^{l\omega}$.
(4) $w \in \Sigma^{b\omega}$ is $C^{b\omega}$ if for all k $\in N$ such that $\Delta^k(w) \in \Sigma^{b\omega}$. The class of $C^{b\omega}$-words is denoted by $C^{b\omega}$.

Clearly, $K, \overline{K} \in C^\omega$, $\tilde{K}, \tilde{\overline{K}} \in C^{l\omega}$ and $\tilde{K}1K, \tilde{\overline{K}}2\overline{K} \in C^{b\omega}$.

It is easy to check that Δ and D commute with the mirror image ($\tilde{}$) and are stable for the permutation($\bar{}$):

Lemma 1 (*Proposition 4 in [3]*). (1) *For all* $u \in \Sigma^*$, $D(\tilde{u}) = \widetilde{D(u)}$, $D(\bar{u}) = D(u)$;
(2) *For all* $u \in \Sigma^*(\Sigma^\omega, \Sigma^{l\omega}, \Sigma^{b\omega})$, $\Delta(\tilde{u}) = \widetilde{\Delta(u)}$, $\Delta(\bar{u}) = \Delta(u)$.

These properties indicate that C^∞, C^ω, $C^{l\omega}$ *and* $C^{b\omega}$ *are all closed under these operators:*

$$\begin{aligned} &w \in C^\infty \iff \bar{w}, \tilde{w} \in C^\infty \\ &w \in C^\omega \iff \bar{w} \in C^\omega \\ &w \in C^{l\omega} \iff \bar{w} \in C^{l\omega} \\ &w \in C^{l\omega} \iff \tilde{w} \in C^\omega \\ &w \in C^{b\omega} \iff \bar{w}, \tilde{w} \in C^{b\omega}. \end{aligned}$$

3. The number of C^∞-words of form $\tilde{w}xw$

In this section, we discuss the number of C^∞-words of form $\tilde{w}xw$ with gaps 2, 3, 4. Let $p_i(n)$ denote the number of the C^∞-words of form $\tilde{w}xw$ with length $2n+i$ and gap i, where i is the length of the word x. For $w \in C^\infty$, $\alpha \in \Sigma$, if $\alpha w\alpha \in C^\infty$, then we call $\alpha w\alpha$ a palindromic C^∞-extension of the word w.

Lemma 2. *Let q be a word and $\alpha \in \Sigma$. If each of the words $\tilde{q}\alpha q, \tilde{q}q, \tilde{q}\alpha\alpha\bar{\alpha}q, \tilde{q}\alpha\bar{\alpha}\bar{\alpha}q, \tilde{q}\alpha\bar{\alpha}\alpha q, \tilde{q}\alpha\alpha q, \tilde{q}\alpha\bar{\alpha}\alpha\alpha q, \tilde{q}\alpha\alpha\bar{\alpha}\alpha q$, and $\tilde{q}\alpha\bar{\alpha}\bar{\alpha}\alpha q \in C^\infty$, then any of them has exactly one palindromic C^∞-extension. Moreover if $|q| \geq 1$, then $\tilde{q}\alpha\bar{\alpha}q$ also has exactly one palindromic C^∞-extension.*

Proof. Case 1. $\tilde{q}\alpha q \in C^\infty$.
By induction on $|q|$ (the length of q). It is obvious that q is respectively equal to $\bar{\alpha}, \bar{\alpha}\bar{\alpha}, \bar{\alpha}\bar{\alpha}\alpha, \bar{\alpha}\bar{\alpha}\alpha\alpha, \bar{\alpha}\bar{\alpha}\alpha\alpha\bar{\alpha}, \bar{\alpha}\bar{\alpha}\alpha\alpha\bar{\alpha}\alpha$ for $|q| = 1, 2, 3, 4, 5, 6$. So if $x\tilde{q}\alpha qx \in C^\infty$ for $x \in \Sigma$, then x takes $\bar{\alpha}, \bar{\alpha}, \alpha, \alpha, \bar{\alpha}, \alpha$ for $|q| = 0, 1, 2, 3, 4, 5$ respectively, i.e. the statement holds for $|q| = 0, 1, 2, 3, 4, 5$.

Assume that the statement has held for $|q| \le k$ (≥ 5). Let q be in C^∞ such that $x\tilde{q}\alpha q x \in C^\infty$ and $|q| = k+1$. Since $\tilde{q}\alpha q \in C^\infty$ and $|q| \ge 6$, we have $q = \bar{\alpha}\bar{\alpha}u\beta\beta$ or $q = \bar{\alpha}\bar{\alpha}u\beta\bar{\beta}$.

Note that since $\tilde{q}\alpha q \in C^\infty$ and $|\bar{\alpha}\bar{\alpha}u\beta| = |q| - 1 = k$, we see that exactly one of all possible values $\bar{\alpha}\bar{\alpha}u\beta\beta$ and $\bar{\alpha}\bar{\alpha}u\beta\bar{\beta}$ of q such that $x\tilde{q}\alpha q x \in C^\infty$ is in C^∞ by the inductive hypotheses. The similar results will be quoted many times in the following proof process of Lemma 2, but we will not mention it repeatedly.

If $q = \bar{\alpha}\bar{\alpha}u\beta\beta$, $\beta \in \Sigma$, then $x = \bar{\beta}$ and $D(x\tilde{q}\alpha q x) = D(\tilde{q}\alpha q) \in C^\infty$. Hence in this case the statement holds. If $q = \bar{\alpha}\bar{\alpha}u\beta\bar{\beta}$, then $\widetilde{D(q)}1D(q) = D(\tilde{q}\alpha q) \in C^\infty$ and

$$D(x\tilde{q}\alpha q x) = \begin{cases} 1\widetilde{D(q)}1D(q)1, & x = \beta \\ 2\widetilde{D(q)}1D(q)2, & x = \bar{\beta}. \end{cases}$$

Since $|D(q)| < |q| = k+1$, by the inductive hypotheses we see that the statement also holds.

Case 2. $\tilde{q}q \in C^\infty$.

It is obvious that q is respectively equal to $\alpha, \alpha\bar{\alpha}, \alpha\bar{\alpha}\alpha, \alpha\bar{\alpha}\alpha\bar{\alpha}, \alpha\bar{\alpha}\alpha\bar{\alpha}\bar{\alpha}, \alpha\bar{\alpha}\alpha\bar{\alpha}\bar{\alpha}\alpha$ for $|q| = 0, 1, 2, 3, 4, 5, 6$. Hence the statement is true for $|q| = 0, 1, 2, 3, 4, 5$.

If $|q| \ge 6$, then since $\tilde{q}q \in C^\infty$, we get $q = \alpha\bar{\alpha}v$. Hence $q = \alpha\bar{\alpha}u\beta\beta$ or $q = \alpha\bar{\alpha}u\beta\bar{\beta}$, where $\beta \in \Sigma$. In the former case, if $x\tilde{q}qx \in C^\infty$ then we have $x = \bar{\beta}$ and $D(x\tilde{q}qx) = D(\tilde{q})2D(q) = D(\tilde{q}q)$. Therefore the statement is true. In the latter case, we have $\widetilde{D(q)}2D(q) = D(\tilde{q}q) \in C^\infty$ and

$$D(x\tilde{q}qx) = \begin{cases} 2\widetilde{D(q)}2D(q)2, & x = \bar{\beta} \\ 1\widetilde{D(q)}2D(q)1, & x = \beta. \end{cases}$$

By Case 1 we see that the statement also holds.

Case 3. $\tilde{q}\alpha\alpha\bar{\alpha}q \in C^\infty$.

Induction on $|q|$. It is obvious that q is respectively equal to $\bar{\alpha}, \bar{\alpha}\alpha, \bar{\alpha}\alpha\bar{\alpha}, \bar{\alpha}\alpha\bar{\alpha}\bar{\alpha}, \bar{\alpha}\alpha\bar{\alpha}\bar{\alpha}\alpha, \bar{\alpha}\alpha\bar{\alpha}\bar{\alpha}\alpha\bar{\alpha}$ for $|q| = 0, 1, 2, 3, 4, 5, 6$. It follows that the statement is true for $|q| = 0, 1, 2, 3, 4, 5$.

Assume that the statement is true for $|q| \le k$ (≥ 5). Let q be in C^∞ such that $x\tilde{q}\alpha\alpha\bar{\alpha}qx \in C^\infty$ and $|q| = k+1$. Since $|q| \ge 6$ and $\tilde{q}\alpha\alpha\bar{\alpha}q \in C^\infty$, we have $q = \bar{\alpha}\alpha v$ and $\widetilde{D(q)}122D(q) = D(\tilde{q}\alpha\alpha\bar{\alpha}q) \in C^\infty$. Hence $q = \bar{\alpha}\alpha u\beta\beta$ or $q = \bar{\alpha}\alpha u\beta\bar{\beta}$, where $\beta \in \Sigma$.

If $q = \bar{\alpha}\alpha u\beta\beta$ and $x\tilde{q}\alpha\alpha\bar{\alpha}qx \in C^\infty$ then $x = \bar{\beta}$ and $D(x\tilde{q}\alpha\alpha\bar{\alpha}qx) = D(\tilde{q}\alpha\alpha\bar{\alpha}q) \in C^\infty$. Therefore the statement is true.

If $q = \bar{\alpha}\alpha u\beta\bar{\beta}$ and $x\tilde{q}\alpha\alpha\bar{\alpha}qx \in C^\infty$, then

$$D(x\tilde{q}\alpha\alpha\bar{\alpha}qx) = \begin{cases} 1\widetilde{D(q)}122D(q)1, & x = \beta \\ 2\widetilde{D(q)}122D(q)2, & x = \bar{\beta}. \end{cases}$$

Hence

$$D(\widetilde{x\tilde{q}\alpha\alpha\bar{\alpha}qx}) = \begin{cases} 1\widetilde{D(q)}221D(q)1, & x = \beta \\ 2\widetilde{D(q)}221D(q)2, & x = \bar{\beta}. \end{cases}$$

Since $\widetilde{D(q)}221D(q) = \widetilde{\widetilde{D(q)}122D(q)} \in C^\infty$ and $|D(q)| < |q| = k+1$, by the inductive hypothesis we see that exactly one of $1\widetilde{D(q)}221D(q)1$ and $2\widetilde{D(q)}221D(q)2$ is also in C^∞, i.e. exactly one of $1\tilde{q}\alpha\alpha\bar{\alpha}q1$ and $2\tilde{q}\alpha\alpha\bar{\alpha}q2$ is also in C^∞. Thus the statement is also true.

Case 4. $\tilde{q}\alpha\bar{\alpha}\bar{\alpha}q \in C^\infty$.

From $\tilde{q}\alpha\bar{\alpha}\bar{\alpha}q \in C^\infty$ it follows that $\tilde{q}\bar{\alpha}\bar{\alpha}\alpha q = \widetilde{\tilde{q}\alpha\bar{\alpha}\bar{\alpha}q} \in C^\infty$. By Case 3 $\tilde{q}\bar{\alpha}\bar{\alpha}\alpha q$ has exactly one palindromic C^∞-extension. Therefore $\tilde{q}\alpha\bar{\alpha}\bar{\alpha}q$ also has exactly one palindromic C^∞-extension.

Case 5. $\tilde{q}\alpha\bar{\alpha}\alpha q \in C^\infty$.

Since $\widetilde{(\alpha q)}\bar{\alpha}(\alpha q) = \tilde{q}\alpha\bar{\alpha}\alpha q \in C^\infty$, by Case 1 we see that $\tilde{q}\alpha\bar{\alpha}\alpha q$ has exactly one palindromic C^∞-extension.

Case 6. $\tilde{q}\alpha\alpha q \in C^\infty$.

Since $\tilde{q}\alpha\alpha q = (\widetilde{\alpha q})(\alpha q)$, by Case 2, we see that the statement is true.

Case 7. $\tilde{q}\alpha\bar{\alpha}q \in C^\infty$.

Since $\tilde{q}\alpha\bar{\alpha}q \in C^\infty$, it is clear that q is respectively equal to $\bar{\alpha}, \bar{\alpha}\alpha, \bar{\alpha}\alpha\alpha, \bar{\alpha}\alpha\alpha\bar{\alpha}, \bar{\alpha}\alpha\alpha\bar{\alpha}\alpha, \bar{\alpha}\alpha\alpha\bar{\alpha}\alpha\alpha$, or $\alpha, \alpha\bar{\alpha}, \alpha\bar{\alpha}\bar{\alpha}, \alpha\bar{\alpha}\bar{\alpha}\alpha, \alpha\bar{\alpha}\bar{\alpha}\alpha\bar{\alpha}, \alpha\bar{\alpha}\bar{\alpha}\alpha\bar{\alpha}\bar{\alpha}$ for $1 \le |q| \le 6$. Hence the statement is true for $1 \le |q| \le 5$.

Now let q be such that $|q| \ge 6$ and $\tilde{q}\alpha\bar{\alpha}q \in C^\infty$. By $\tilde{q}\alpha\bar{\alpha}q \in C^\infty$, we have $q = \bar{\alpha}\alpha v$ or $q = \alpha\bar{\alpha}v$. If $q = \bar{\alpha}\alpha v$ and $x\tilde{q}\alpha\bar{\alpha}qx \in C^\infty$ for $x \in \Sigma$, then

$$D(x\tilde{q}\alpha\bar{\alpha}qx) = \begin{cases} \widetilde{D(q)}112D(q), & q = \bar{\alpha}\alpha u\beta\beta, x = \bar{\beta} \\ 1\widetilde{D(q)}112D(q)1, & q = \bar{\alpha}\alpha u\beta\bar{\beta}, x = \beta \\ 2\widetilde{D(q)}112D(q)2, & q = \bar{\alpha}\alpha u\beta\bar{\beta}, x = \bar{\beta}. \end{cases}$$

Therefore in view of Case 3, we see that exactly one of $x\tilde{q}\alpha\bar{\alpha}qx$ for $x \in \Sigma$ is also in C^∞.

If $q = \alpha\bar{\alpha}v$ and $x\tilde{q}\alpha\bar{\alpha}qx \in C^\infty$ for $x \in \Sigma$, then

$$D(x\tilde{q}\alpha\bar{\alpha}qx) = \begin{cases} \widetilde{D(q)}211D(q), & q = \alpha\bar{\alpha}u\beta\beta, x = \bar{\beta} \\ 1\widetilde{D(q)}211D(q)1, & q = \alpha\bar{\alpha}u\beta\bar{\beta}, x = \beta \\ 2\widetilde{D(q)}211D(q)2, & q = \alpha\bar{\alpha}u\beta\bar{\beta}, x = \bar{\beta}. \end{cases}$$

From Case 4, it follows that $\tilde{q}\alpha\bar{\alpha}q$ has exactly one palindromic C^∞-extension. From the above discussion we see that the statement also holds for $|q| \ge 6$.

Case 8. $\tilde{q}\alpha\bar{\alpha}\alpha\alpha q \in C^\infty$.

Since $\tilde{q}\alpha\bar{\alpha}\alpha\alpha q = (\widetilde{\alpha q})\bar{\alpha}\alpha(\alpha q)$ and $|\alpha q| = 1 + |q| \ge 1$, from Case 7 it follows that the statement holds.

Case 9. $\tilde{q}\alpha\alpha\bar{\alpha}\alpha q \in C^\infty$.

Since $\tilde{q}\alpha\alpha\bar{\alpha}\alpha q = \widetilde{\tilde{q}\alpha\bar{\alpha}\alpha\alpha q}$, from Case 8 we can get the required result.

Case 10. $\tilde{q}\alpha\bar{\alpha}\bar{\alpha}\alpha q \in C^\infty$.

Since $\tilde{q}\alpha\bar{\alpha}\bar{\alpha}\alpha q = (\widetilde{\bar{\alpha}\alpha q})(\bar{\alpha}\alpha q)$, by Case 2, we see that the statement holds. □

Theorem 1. *Let $p_i(n)$ denote the number of C^∞-words of form $\tilde{q}xq$ with length $2n + i$ and gap $|x| = i$. Then $p_i(n) = 6$ for $i = 2, 3, 4$ and $n \ge 1$.*

Proof. If $i = 2$, then remarking that the words $\alpha\bar{\alpha}$ have exactly two palindromic C^∞-extension, from $x = 12, 21, 11$ or 22, and the cases $\tilde{q}\alpha\alpha q$ and $\tilde{q}\alpha\bar{\alpha}q$ of Lemma 2, it follows that $p_2(n) = 6$.

If $i = 3$, then from $x = 121, 212, 221, 122, 112$ or 211, and the cases $\tilde{q}\alpha\alpha\bar{\alpha}q$, $\tilde{q}\alpha\bar{\alpha}\bar{\alpha}q$, and $\tilde{q}\alpha\bar{\alpha}\alpha q$ of Lemma 2 it follows that $p_3(n) = 6$.

If $i = 4$, then from $x = 2122, 2212, 1211, 1121, 1221$ or 2112, and the cases $\tilde{q}\alpha\bar{\alpha}\alpha\alpha q$, $\tilde{q}\alpha\alpha\bar{\alpha}\alpha q$, and $\tilde{q}\alpha\bar{\alpha}\bar{\alpha}\alpha q$ of Lemma 2 it follows that $p_4(n) = 6$. □

4. The infinite C^∞-words w satisfying the condition $\tilde{w}xw \in C^{b\omega}$

In [7] we discussed the infinite C^∞-words w satisfying the condition $\tilde{w}xw \in C^{b\omega}$ with $|x| = 0, 1$ and proved that

Lemma 3 ([7, Lemma 6–7]). (1) $\tilde{q}1q \in C^{b\omega} \iff q = K$.

(2) $\tilde{q}2q \in C^{b\omega} \iff q = \overline{K}$.

(3) $\tilde{q}q \in C^{b\omega} \iff q = \alpha\Delta_{\bar{\alpha}}^{-1}(\overline{K})$, *where* $\alpha = 1, 2$.

If $u = xy \in \Sigma^\omega$, $x \in \Sigma^*$, for the convenience, in what follows we shall use the following notation: $y = x^{-1}u$.

In this section, we discuss the cases for $|x| = 2, 3, 4$ and give a complete solution. For this, we need the following important infinite sequence h: it is an infinite word of symbols 1 and 2, the first symbol is 1, from the second run starting, the length of the ith run is the $(i - 1)$th symbol, i.e.

$$h = 121121221221121122121 1\cdots\cdots = 1(2^{-1}K)$$

and

$$\Delta(1^{-1}h) = \Delta(2112122122112112112211211\cdots\cdots) = h.$$

It is easy to check that

Lemma 4. (1) $\Delta(\tilde{h}221h) = \tilde{h}122h$, $\Delta(\tilde{h}122h) = \tilde{h}221h$.
(2) $\tilde{h}221h, \tilde{h}122h \in C^{b\omega}$.
(3) $\Delta^2(\tilde{h}221h) = \tilde{h}221h$, $\Delta^2(\tilde{h}122h) = \tilde{h}122h$.

Lemma 5. *Let* $h_{21} = 211\Delta_2^{-1}\Delta_1^{-1}(h)$, $h_{12} = 122\Delta_1^{-1}\Delta_1^{-1}(h)$. *Then* $\widetilde{h_{21}}\alpha\bar{\alpha}h_{21}, \widetilde{h_{12}}\alpha\bar{\alpha}h_{12} \in C^{b\omega}$ *for* $\alpha = 1, 2$.

Proof. Since $\Delta^2(\widetilde{h_{21}}\alpha\bar{\alpha}h_{21}) = \Delta^2(\widetilde{(211)^{-1}h_{21}}112\alpha\bar{\alpha}211(211)^{-1}h_{21}) = \Delta^2(\widetilde{\Delta_2^{-1}\Delta_1^{-1}(h)}112\alpha\bar{\alpha}211\Delta_2^{-1}\Delta_1^{-1}(h))$

$$= \begin{cases} \Delta(\widetilde{\Delta_1^{-1}(h)}22112\Delta_1^{-1}(h)), & \alpha = 2 \\ \Delta(\widetilde{\Delta_1^{-1}(h)}21122\Delta_1^{-1}(h)), & \alpha = 1 \end{cases}$$
$$= \begin{cases} \tilde{h}221h, & \alpha = 2 \\ \tilde{h}122h, & \alpha = 1. \end{cases}$$

In view of Lemma 4(2), we have $\widetilde{h_{21}}\alpha\bar{\alpha}h_{21} \in C^{b\omega}$ for $\alpha = 1, 2$. By a similar discussion, we can get $\widetilde{h_{12}}\alpha\bar{\alpha}h_{12} \in C^{b\omega}$. $\square$

Lemma 6. $\tilde{h}221h, \tilde{\bar{h}}112\bar{h}, \tilde{h}122h, \tilde{\bar{h}}211\bar{h} \in C^{b\omega}$.

Proof. Since $\Delta(\tilde{\bar{h}}112\bar{h}) = \Delta(\overline{\tilde{h}221h}) = \Delta(\tilde{h}221h)$, and $\Delta(\tilde{\bar{h}}211\bar{h}) = \Delta(\overline{\tilde{h}122h}) = \Delta(\tilde{h}122h)$, by Lemma 4(2) we see that $\tilde{h}221h, \tilde{\bar{h}}112\bar{h}, \tilde{h}122h, \tilde{\bar{h}}211\bar{h} \in C^{b\omega}$. $\square$

Lemma 7. (1) *Let* $h_{11} = 11\Delta_2^{-1}\Delta_1^{-1}(h)$, $h_{22} = 22\Delta_1^{-1}\Delta_1^{-1}(h)$. *Then* $\widetilde{h_{11}}2122h_{11}$, $\widetilde{h_{11}}2212h_{11}$, $\widetilde{h_{22}}1211h_{22}$, *and* $\widetilde{h_{22}}1121h_{22} \in C^{b\omega}$.
(2) *Let* $K_{\alpha^{-1}} = \alpha^{-1}\Delta_\alpha^{-1}(\bar{K})$, $\alpha = 1, 2$. *Then* $\widetilde{K_{\alpha^{-1}}}\alpha\bar{\alpha}\bar{\alpha}\alpha K_{\alpha^{-1}} \in C^{b\omega}$ *for* $\alpha = 1, 2$.

Proof. (1) Since $\Delta^2(\widetilde{h_{11}}2122h_{11}) = \Delta^2(\widetilde{\Delta_2^{-1}\Delta_1^{-1}(h)}11212211\Delta_2^{-1}\Delta_1^{-1}(h)) = \Delta(\widetilde{\Delta_1^{-1}(h)}21122\Delta_1^{-1}(h)) = \tilde{h}122h$, and $\widetilde{h_{11}}2212h_{11} = \widetilde{h_{11}2122h_{11}}$. $\Delta^2(\widetilde{h_{22}}1211h_{22}) = \Delta^2(\widetilde{\Delta_1^{-1}\Delta_1^{-1}(h)}22121122\Delta_1^{-1}\Delta_1^{-1}(h)) = \Delta(\widetilde{\Delta_1^{-1}(h)}21122\Delta_1^{-1}(h)) = \tilde{h}122h$, and $\widetilde{h_{22}}1121h_{22} = \widetilde{h_{22}1211h_{22}}$, in view of Lemma 4(2) we obtain $\widetilde{h_{11}}2122h_{11}, \widetilde{h_{11}}2212h_{11}, \widetilde{h_{22}}1211h_{22}$, and $\widetilde{h_{22}}1121h_{22} \in C^{b\omega}$.

(2) Since $\Delta(\widetilde{K_{\alpha^{-1}}}\alpha\bar{\alpha}\bar{\alpha}\alpha K_{\alpha^{-1}}) = \Delta((\alpha^{-1}\Delta_\alpha^{-1}(\bar{K}))\alpha\bar{\alpha}\bar{\alpha}\alpha(\alpha^{-1}\Delta_\alpha^{-1}(\bar{K}))) = \Delta(\widetilde{\Delta_\alpha^{-1}(\bar{K})}\bar{\alpha}\bar{\alpha}\Delta_\alpha^{-1}(\bar{K})) = \tilde{\bar{K}}2\bar{K} = \overline{\tilde{K}1K} \in C^{b\omega}$, we have $\widetilde{K_{\alpha^{-1}}}\alpha\bar{\alpha}\bar{\alpha}\alpha K_{\alpha^{-1}} \in C^{b\omega}$. $\square$

Now we can state our second main result:

Theorem 2. (1) $\tilde{w}\alpha\alpha w \in C^{b\omega} \iff w = \Delta_{\bar{\alpha}}^{-1}(\bar{K})$ *for* $\alpha = 1, 2$.
(2) $\tilde{w}\alpha\bar{\alpha}w \in C^{b\omega} \iff w = 211\Delta_2^{-1}\Delta_1^{-1}(h)$ *or* $122\Delta_1^{-1}\Delta_1^{-1}(h)$.
(3) $\tilde{w}\alpha\alpha\bar{\alpha}w \in C^{b\omega} \iff w = h(\alpha = 2)$ *or* $w = \bar{h}(\alpha = 1)$.
(4) $\tilde{w}\alpha\bar{\alpha}\bar{\alpha}w \in C^{b\omega} \iff w = h(\alpha = 1)$ *or* $w = \bar{h}(\alpha = 2)$.
(5) $\tilde{w}\alpha\bar{\alpha}\alpha w \in C^{b\omega} \iff w = 1^{-1}\bar{K}(\alpha = 1)$ *or* $w = 2^{-1}K(\alpha = 2)$.
(6) $\tilde{w}\alpha\bar{\alpha}\alpha\alpha w \in C^{b\omega} \iff w = 11\Delta_2^{-1}\Delta_1^{-1}(h)(\alpha = 2)$ *or* $w = 22\Delta_1^{-1}\Delta_1^{-1}(h)(\alpha = 1)$.
(7) $\tilde{w}\alpha\alpha\bar{\alpha}\alpha w \in C^{b\omega} \iff w = 11\Delta_2^{-1}\Delta_1^{-1}(h)(\alpha = 2)$ *or* $w = 22\Delta_1^{-1}\Delta_1^{-1}(h)(\alpha = 1)$.
(8) $\tilde{w}\alpha\bar{\alpha}\bar{\alpha}\alpha w \in C^{b\omega} \iff w = \alpha^{-1}\Delta_\alpha^{-1}(\bar{K})$.

Proof. (1) It follows from Lemmas 2 and 3(3).
(2) It follows from Lemmas 2 and 5.
(3) It follows from Lemmas 2 and 6.

(4) It follows from Lemmas 2 and 6.
(5) It follows from Lemmas 2 and 3(1–2).
(6) It follows from Lemmas 2 and 7(1).
(7) It follows from (6).
(8) It follows from Lemmas 2 and 7(2). □

The significance of Theorem 2 is due to give the construct of C^∞-words of form $\tilde{w}xw$ with gaps 2, 3, 4.

5. Remarks

From the above discussion we can see that the key element of computing of $p_n(i)$ for $n \leq 4$ is the fact that for each $x \in C^\infty$ and $|x| \leq 4$, every palindromic C^∞-extension of x has an unique palindromic C^∞-extension. But if $|x| \geq 5$, then the same result does not hold. For example, taking $x = \bar{\alpha}\alpha\alpha\bar{\alpha}\alpha$, the palindromic C^∞-extension $\bar{\alpha}x\bar{\alpha}$ of x has no palindromic C^∞-extension. Also neither does $\bar{\alpha}\alpha\alpha\bar{\alpha}\alpha\bar{\alpha}\bar{\alpha}\alpha\mathbf{x}\alpha\bar{\alpha}\bar{\alpha}\alpha\bar{\alpha}\alpha\alpha\bar{\alpha}$. Therefore the computing of $p_n(i)$ for $n \geq 5$ would become more complicated than the case for $n \leq 4$.

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