New Perturbation Results for Solving the Linear Complementarity Problem With $P_0$-Matrices

A. A. Ebiefung
Department of Mathematics
University of Tennessee at Chattanooga
Chattanooga, TN 37403, U.S.A.

(Received and accepted July 1997)

Abstract—We provide new conditions under which the linear complementarity problem (LCP) can be solved by a perturbation method when the associated matrix is a $P_0$-matrix. The new conditions apply to both degenerate and nondegenerate LCPs. Moreover, these conditions do not require that the $P_0$-matrix belong to another matrix class such as, for example, the $R_0$-matrix class.

Keywords—Complementarity problem, Algorithm, $P$-matrices, $P_0$-matrices.

1. INTRODUCTION

We consider the linear complementarity problem LCP($q, M$): given $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, find $w \in \mathbb{R}^n$, $z \in \mathbb{R}^n$ such that $(I, -M)(w, z)^t = q$, $w^T z = 0$, $w \geq 0$, $z \geq 0$. This is a well-known problem with many applications. See [1,2] for references.

If $M$ is a $P_0$-matrix, the LCP($q, M$) may be solved under suitable conditions by solving the perturbed LCP($q, M(\epsilon)$), where $M(\epsilon) = M + \epsilon I$ and $I$ is an $n \times n$ identity matrix (see [1,3–5]). In this paper, we provide new conditions under which the solution of the LCP($q, M(\epsilon)$) converges to that of the LCP($q, M$).

2. NOTATION AND DEFINITIONS

Let $M$ be an $n \times n$ matrix. For each $\beta \subseteq \{1, 2, \ldots, n\}$, define a matrix $C = C(\beta)$ by $C_{j,j} = -M_{j,j}$ if $j \in \beta$ and $C_{j,j} = I_{j,j}$ if $j \notin \beta$. $C$ is called a complementarity submatrix of $M$ or of $(I, -M)$. A vector $y^* \geq 0$ is a basic feasible solution of the LCP($q, M$) if $Cy^* = q$.

Let $\{\epsilon_k\}$ be a decreasing sequence of positive numbers converging to zero. Let $C(\epsilon_k) = C^k$ be a complementarity matrix solving the LCP($q, M(\epsilon_k)$) for each $k$. Denote by $C$ the limit of $C^k$ as $k \to \infty$ (and $\epsilon_k \to 0$). Let $y(\epsilon_k) = y^k$ be the solution of the LCP($q, M(\epsilon_k)$) associated with $C^k$. Note that $C$ is also a complementarity matrix. Throughout the paper, $C^k$, $y^k$, and $C$ shall be as defined above. Moreover, $M$ shall denote an $n \times n$ $P_0$-matrix.

Project partially supported by the University of Tennessee at Chattanooga Center of Excellence for Computer Applications' Scholars Program.
3. PERTURBATION RESULTS

In what follows, we give new conditions under which the LCP(q, M) can be solved by perturbation methods. Our proof borrows from [1,3,6].

**THEOREM 1.** If the homogeneous system Cy = 0, y ≥ 0 has only the trivial solution, then we have the following.

(a) The sequence \{y^k\} is bounded.
(b) A subsequence of \{y^k\} converges to the solution of the LCP(q, M).
(c) Each accumulation point of \{y^k\} solves the LCP(q, M). (In particular, (a) → (b) → (c)).

**PROOF.**

(a) Suppose \{y^k\} is not bounded. Without loss of generality, assume that ||y^k|| → ∞. Then the bounded infinite sequence \{\tilde{z}^k\} = \{y^k/||y^k|| : y^k ≠ 0\} of nonnegative vectors is well defined. Moreover, it has a converging subsequence, say \{\tilde{z}^k\} = \{\tilde{y}^k/||\tilde{y}^k|| : \tilde{y}^k ≠ 0\}. Assume that \{\tilde{z}^k\} converges to \tilde{z}^* ≥ 0. Observe that \tilde{z}^* ≠ 0.

Let \{\tilde{C}^k\} be a sequence of complementary matrices associated with \{\tilde{z}^k\}. Observe that \tilde{C}^k converges to C since C^k by our assumption, and that \tilde{C}^k\tilde{z}^k = q/||\tilde{y}^k||. By taking limits of both sides as k → ∞, we obtain C\tilde{z}^* = 0, \tilde{z}^* ≥ 0. This contradicts the assumption that Cy = 0, y ≥ 0 has only the trivial solution.

(b) By (a), \{y(ε_k)\} is bounded. So there exists a convergent subsequence, say \{\tilde{y}^k\}. Assume \{\tilde{y}^k\} converges to \tilde{y}^*. Let \{\tilde{C}^k\} be a subsequence of \{C^k\} corresponding to \{\tilde{y}^k\}. Then, \tilde{C}^k\tilde{y}^k → Cy^*. But C\tilde{y}^k = q implies \tilde{C}^k\tilde{y}^k = q. Therefore, \lim_{k→∞} \tilde{C}^k\tilde{y}^k = Cy^* = q.

Thus, \tilde{y}^* is a solution of the LCP(q, N).

(c) This follows from part (b).

We point out that if M is an R_o-matrix, then for each α = 1, 2n, the homogeneous system Cαy = 0, y ≥ 0, where Cα is a complementary matrix of M, has only the trivial solution. As a consequence of Theorem 1, we have the following corollary.

**COROLLARY 2.** If M ∈ P_o ∩ R_o, then we have the following.

(a) The sequence \{y^k\} is bounded.
(b) A subsequence of \{y^k\} converges to the solution of the LCP(q, M).
(c) Each accumulation point of \{y^k\} solves the LCP(q, M).

We note that Corollary 2 is given as Theorem 5.6.2 (a) in [1].

**THEOREM 3.** If C is nonsingular, the sequence \{y^k\} converges to the solution of the LCP(q, M), if one exists.

**PROOF.** For each ε_k > 0, the matrix C^k is invertible. Since C^k converges to C by assumption, it follows that (C^k)^{-1} → C^{-1} by [3]. Hence, \tilde{y}^* = C^{-1}(ε_k)q. Let z^* = lim_{k→∞} y^k = C^{-1}q. Since C is a complementarity matrix and z^* ≥ 0, we have that z^* is a solution of the LCP(q, M).

**EXAMPLE 4.** Let

\[
M = \begin{bmatrix}
0 & 1 & -3 \\
-2 & 1 & -3 \\
4 & 0 & 0
\end{bmatrix}, \quad q = \begin{bmatrix}
-1 \\
1 \\
1
\end{bmatrix}, \quad \tilde{q} = \begin{bmatrix}
-1 \\
-1 \\
-1
\end{bmatrix}.
\]

The LCPs LCP(q, M) and LCP(q, M(ε)), LCP(\tilde{q}, M) LCP(\tilde{q}, M(ε)) can be used to illustrate Theorems 1 and 3.

We point out that M ∉ R_o, M is not positive semidefinite, and that the LCP(q, M) is nondegenerate. In fact, if the LCP(q, M) is nondegenerate, then it has a unique basic solution (see [7]). Consequently, the complementarity matrix solving the problem is nonsingular. By Theorem 3, y^k converges to the solution of the LCP(q, M), if one exists. We summarize this important observation in Corollary 5.
Corollary 5. If the LCP(q, M) is nondegenerate, \( y^k \) converges to the solution of the LCP (q, M), if one exists.

This corollary is also stated in [5, Theorem 3]. The next corollary is an obvious consequence of Theorems 1 and 3.

Corollary 6. If the sequence \( \{y^k\} \) is bounded, then the LCP(q, M) has a solution.

In the above discussions, we assumed that the complementarity matrix \( C^k \) solves the LCP (q, \( M(\varepsilon_k) \)) and converges to \( C \). We now consider the case when \( C^k \) does not solve the LCP (q, \( M(\varepsilon_k) \)), but converges to \( C \), where \( C \) solves the LCP(q, M).

Theorem 7. Suppose \( C \) is nonsingular. Let \( y^* \) be a basic solution of the LCP(q, M) such that \( Cy^* = q, y^* \geq 0 \). Then, the solution of \( C^k y^k = q \) converges to \( y^* \).

Proof. From \( C^k \to C \), we get \( (C^k)^{-1} \to C^{-1} \) by [3]. The unique solutions of the two systems \( C^k y^k = q \) and \( Cy^* = q, y^* \geq 0 \) are \( y^k = (C^k)^{-1}q \) and \( y^* = C^{-1}q \geq 0 \), respectively. Therefore, \( \lim_{\varepsilon_k \to 0} y^k = \lim_{\varepsilon_k \to 0} (C^k)^{-1}q = C^{-1}q = y^* \geq 0 \) by uniqueness.

It is interesting to observe that there is no requirement that \( y^k \) be nonnegative or solves the LCP(q, \( M(\varepsilon_k) \)) in Theorem 7, or that the LCP be nondegenerate.

For the sake of completeness, we state the following theorem proved in [3].

Theorem 8. Suppose the LCP(q, M) has no solution. Then, \( ||y^k|| \to \infty \).

4. CONCLUSION

We have shown that if \( C^k \) solves the LCP(q, \( M(\varepsilon_k) \)) and converges to the nonsingular \( C \) as \( \varepsilon_k \to 0 \), then the solution of the LCP(q, \( M(\varepsilon_k) \)) converges to that of the LCP(q, M). Moreover, if no condition is placed on \( C \) and the homogeneous system \( Cy = 0, y \geq 0 \) has only the trivial solution, then the sequence \( \{y^k\} \) of solutions of the LCP(q, \( M(\varepsilon_k) \)) is bounded and its subsequence converges to the solution of the LCP(q, M).

Some important perturbation results in [1,3,5] are consequences of our results. As examples show, our results apply to both degenerate and nondegenerate LCPs, and to matrices that may or may not belong to the Ro-class, or to the positive semidefinite class.

There are many well-known algorithms for solving the LCP(q, M) when M is a P-matrix (see [1,8]). Since the perturbed problem has a P-matrix, these algorithms can be used to solve LCPs associated with P_\( \varepsilon \)-matrices when above conditions are satisfied.

REFERENCES