Computing Normal External Descriptions and Feedback Design*

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ABSTRACT

Given a linear system $\dot{x} = Ax + Bu$, we compute a normal external description $(N(s), D(s))$, using the Hessenberg form of the pair (A, B) and embedding techniques. We show how to compute a state feedback K that assigns the closed-loop invariant polynomials using a Diophantine equation. The solution to such an equation corresponds to a back-substitution problem, due to the special structure of the computed normal external description. A procedure to compute an output matrix C that assigns the desired finite zeros of the system is also outlined in terms of a Diophantine equation. The proposed algorithms are easy to implement and computationally efficient and therefore can form a useful toolbox in design problems.

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1. INTRODUCTION

The polynomial-equation approach to the analysis and synthesis in linear multivariable systems has a long history, spanning decades and involving many researchers [7, 9]. A problem that was addressed early in control research is the computation of minimal polynomial bases for matrix pencils [6]. The need for such polynomial bases arises from the fact that they appear in the solution of polynomial equations, thereby unifying the design methodology for a variety of control strategies.

In this paper we show how to compute minimal polynomial bases for the controllability pencil of a linear multivariable system. In particular, we compute a normal external description of the controllability pencil. A normal external description is a minimal polynomial basis for the controllability pencil that also enjoys two more useful properties (cf. Section 2). The computation of such normal descriptions involves the use of staircase forms and embedding techniques. The proposed approach is believed to be simple and computationally attractive.

Due to the structure of the computed normal external description, we can easily solve a Diophantine equation for the problem of state feedback. The solution to such a Diophantine equation is formulated simply as a back-substitution problem for a set of linear equations. We also show how to parametrize the set of all possible state feedbacks that assign the desired closed-loop invariant polynomials.

A second application that we study using the computed normal external description and the solution to a second Diophantine equation is the finite-zero assignment problem for state-accessible systems. Specifically, we reveal the limits of "squaring-down" design technique in assigning invariant factors that reflect the dynamics of the desired finite zeros.

The algorithms proposed in this paper can form a useful toolbox in design problems using the polynomial-equation approach, since they are easy to implement and computationally attractive.

2. PRELIMINARIES AND BACKGROUND

Consider a linear time-invariant system (A, B) governed by

$$\dot{x} = Ax + Bu \tag{2.1}$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. In addition, rank $B = m$, and controllability of (A, B) is further assumed.

Given a polynomial matrix $X(s) \in \mathbb{R}^{r \times p}[s]$, we define the column degree k_i as the highest degree among all the entries of column i. $X(s)$ will be said to be column-degree-ordered if $k_1 \geqslant \cdots \geqslant k_p$. Let $r = p$; then $X(s)$ will be said to be column-reduced if $\deg\det X(s) = \sum_{i=1}^{p} k_i$.

Net $N(s)$, $D(s)$ be $n \times m$ and $m \times m$ polynomial matrices over $\mathbb{R}[s]$. Then $N(s)$, $D(s)$ are said to form a *normal left external description* of (A, B) if [6, 11]

(1) $\begin{bmatrix} N(s) \\ D(s) \end{bmatrix}$ is a minimal polynomial basis of $\ker[sI - A \quad -B]$, i.e.,

$$[sI - A \quad -B]\begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = 0;$$

(2) $D(s)$ is nonincreasingly column-degree-ordered and column-reduced;

(3) $N(s)$ is a minimal polynomial basis of $\ker \Pi(sI - A)$, where Π is a matrix representation of the maximal right annihilator of B, i.e., $\Pi B = 0$.

The transfer function of the system is given by

$$T(s) = (sI - A)^{-1} B = N(s) D(s)^{-1}, \tag{2.2}$$

where $N(s)$, $D(s)$ is a left normal external description of (A, B).

Let $P(s)$ be a $(k_1 + k_2 + \cdots + k_m) \times m$ polynomial and column-reduced matrix with column degrees $k_1 - 1, k_2 - 1, \dots, k_m - 1$ such that

$$P(s) = K \text{ block diagonal}\left\{\begin{bmatrix} 1 \\ s \\ \vdots \\ s^{k_i - 1} \end{bmatrix}\right\},$$

where $K \in \mathbb{R}^{k \times k}$ is nonsingular; $k = \sum_{i=1}^{m} k_i$. Then $P(s)$ is said to be a *polynomial basis* of a k-dimensional and R-linear vector space [16].

Our purpose in this paper is to show how we can compute a normal external description for $[sI - A \quad -B]$ using the block Hessenberg form of the pair (A, B). For that reason we briefly review here the block Hessenberg form. According to [1, 14, 15] there exist orthogonal transformations Q such that

$$
\begin{aligned}
Q^T[sI - A \quad -B]\begin{bmatrix} Q & 0 \\ 0 & I_m \end{bmatrix}
&= [sI - A_H \quad -B_H] \\
&= \begin{bmatrix}
sI_{11} - A_{11} & -A_{12} & \cdots & 0 & 0 \\
-A_{21} & sI_{22} - A_{22} & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
-A_{k-1,1} & \cdots & sI_{k-1,k-1} - A_{k-1,k-1} & -A_{k-1,k} & 0 \\
-A_{k1} & \cdots & -A_{k,k-1} & sI_{k,k} - A_{kk} & -A_{k,k+1}
\end{bmatrix},
\end{aligned}
\tag{2.3}
$$

where the blocks $A_{j,j+1}$, $j \in \{1, 2, \dots, k\}$, have full row rank t_j. Moreover, $A_{k,k+1}$ has full row rank t_k, but from our assumption that B is of full column rank ($t_k = t_{k+1} = m$) it follows that $A_{k,k+1}$ is a square nonsingular matrix of dimensions $m \times m$. The column-minimal indices of the pencil $[sI - A \quad -B]$ are computed as follows [14]:

$$t_{i+1} - t_i \text{ column-minimal indices } k_j \text{ of order } k - i,$$

where $t_0 = 0$. The column-minimal indices of $[sI - A \quad -B]$ are also the controllability indices of the pair (A, B). To simplify the forthcoming notation, we will assume further similarity transformations (T) that bring $A_{i,i+1}$ to the following form:

$$A_{i,i+1} = \begin{bmatrix} I_{t_i} & 0 \end{bmatrix}, \qquad i = 1, 2, \dots, k.$$

In the next section we will show that it suffices to bring $A_{i,i+1}$ to a lower triangular form and therefore use only orthogonal transformations.

3. COMPUTING A NORMAL EXTERNAL DESCRIPTION

In order to find a normal external description $N(s)$, $D(s)$ we will first embed the controllability pencil in a unimodular pencil as follows [3]. Assume a matrix $C_H \in \mathbb{R}^{m \times n}$ of the form

$$C_H = \begin{bmatrix} C_1 & & & 0 \\ X & C_2 & & \\ X & X & \ddots & \\ X & X & \cdots & C_k \end{bmatrix}, \tag{3.1}$$

where C_1 is a square nonsingular matrix of dimensions $t_1 \times t_1$ and $C_i \in \mathbb{R}^{(t_i - t_{i-1}) \times t_i}$ for $i = 2, \dots, k$. Moreover, C_i for $i = 2, \dots, k$ are such that

$$\begin{bmatrix} A_{i-1,i} \\ C_i \end{bmatrix}, \qquad i = 2, \dots, k, \tag{3.2}$$

are nonsingular matrices. In particular, we can select the C_i's as follows:

$$C_i = \begin{bmatrix} 0 & I_{t_i - t_{i-1}} \end{bmatrix}, \qquad i = 1, 2, \dots, k, \tag{3.3}$$

where $t_0 = 0$. It is pointed out that the selection of the C_i's is far from unique [3].

Due to the selection of the matrix C, the pencil

$$U(s) = \begin{bmatrix} sI - A_H & -B_H \\ -C_H & 0 \end{bmatrix} \tag{3.4}$$

is unimodular. That is,

$$\det U(s) = \text{const} \neq 0$$

Therefore, the inverse of $U(s)$ exists and is a polynomial matrix. By denoting the inverse of $U(s)$ as $V(s)$ and partitioning it as

$$V(s) = \begin{bmatrix} F(s) & N(s) \\ G(s) & D(s) \end{bmatrix}, \tag{3.5}$$

it is clear that the pair $N(s)$, $D(s)$ is a basis for the kernel of $[sI - A \quad -B]$. Hence the problem of determining a normal external description has been reduced to the inversion of the unimodular pencil $sL - M$, where

$$L = \begin{bmatrix} I_n & 0 \\ O & 0 \end{bmatrix}, \qquad M = \begin{bmatrix} A_H & B_H \\ C_H & 0 \end{bmatrix}.$$

In the sequel we summarize some properties that the pair $N(s)$, $D(s)$ enjoys.

THEOREM 3.1. *Let the pair $N(s)$, $D(s)$ described in* (3.5). *Then*:

(1) *$N(s)$, $D(s)$ are right coprime.*
(2) *$D(s)$ and $sI - A$ have the same nonunity invariant polynomials.*

Proof. Since the pair $N(s)$, $D(s)$ satisfies

$$U(s)\begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = \begin{bmatrix} 0 \\ I_m \end{bmatrix},$$

it follows that the polynomial matrices $N(s)$, $D(s)$ are right coprime.

Furthermore, $D(s)$ has the same nonunity invariant polynomials as $sI - A$. To see that, let

$$\begin{bmatrix} sI - A_H & -B_H \\ 0 & I \end{bmatrix} V(s) = \begin{bmatrix} I & 0 \\ G(s) & D(s) \end{bmatrix},$$

which shows that the two matrices are equivalent ($\sim$). Furthermore,

$$\begin{bmatrix} sI - A_H & -B_H \\ 0 & I \end{bmatrix} \sim \begin{bmatrix} sI - A_H & 0 \\ 0 & I \end{bmatrix}$$

and

$$\begin{bmatrix} I & 0 \\ G(s) & D(s) \end{bmatrix} \sim \begin{bmatrix} I & 0 \\ 0 & D(s) \end{bmatrix},$$

which shows that $D(s)$ and $sI - A_H$ have the same nonunity invariant polynomials. ∎

The next step in our discussion will be the computationally efficient inversion of the unimodular pencil $sL - M$. It is clear that such a method will provide computationally efficient techniques for the computation of the pair $N(s)$, $D(s)$. For this purpose we will use the block Hessenberg form of the pair (A, B). By defining P to be a permutation matrix we can bring the unimodular pencil $U(s)$ to the following form:

$$P^T U(s) = \begin{bmatrix} -C_1 & 0 & \cdots & 0 & 0 \\ sI_{11} - A_{11} & -A_{12} & \cdots & 0 & 0 \\ X & -C_2 & \cdots & 0 & 0 \\ -A_{21} & sI_{22} - A_{22} & \cdots & 0 & 0 \\ X & X & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ -A_{k-1,1} & \cdots & sI_{k-1,k-1} - A_{k-1,k-1} & -A_{k-1,k} & 0 \\ X & \cdots & X & -C_k & 0 \\ -A_{k1} & \cdots & -A_{k,k-1} & sI_{k,k} - A_{kk} & -A_{k,k+1} \end{bmatrix},$$

which can be written as

$$\begin{bmatrix} -F_1 & 0 & \cdots & 0 & 0 \\ sJ_1 - N_1 & -F_2 & \cdots & 0 & 0 \\ X & sJ_2 - N_2 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ X & \cdots & sJ_{k-1} - N_{k-1} & -F_k & 0 \\ X & \cdots & X & sJ_k - N_k & -F_{k+1} \end{bmatrix} = sJ - F, \tag{3.6}$$

where $F_1 = C_1 = I_{t_1}$, $F_{k+1} = A_{k,k+1} = I_{t_k}$,

$$F_i = \begin{bmatrix} A_{i-1,i} \\ C_i \end{bmatrix} = I_{t_i} \qquad \text{for} \quad i = 2, \dots, k,$$

$$J_k = I_{t_k}, \qquad J_i = \begin{bmatrix} I_{t_i} \\ 0 \end{bmatrix} \quad \text{for} \quad i = 1, \dots, k-1.$$

The infinite elementary divisors of the unimodular pencil $U(s)$ are computed as follows [14]:

$t_{i+1} - t_i$ infinite elementary divisors d_j of order $k - i + 1$.

Let us denote by $l+1$ the largest infinite elementary divisor. We have

$$U(s)^{-1} = V(s) = \begin{bmatrix} F(s) & N(s) \\ G(s) & D(s) \end{bmatrix}$$

$$= -F^{-1}\left(I + s\overline{N} + s^2\overline{N}^2 + \cdots + s^l\overline{N}^l\right)P^T, \tag{3.7}$$

where

$$\overline{N} = JF^{-1} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \overline{N}_1 & 0 & \cdots & 0 & 0 \\ X & \overline{N}_2 & \ddots & 0 & 0 \\ X & X & \ddots & \vdots & \vdots \\ X & X & \cdots & \overline{N}_k & 0 \end{bmatrix},$$

where

$$\overline{N}_i = \begin{bmatrix} I_{t_i} \\ 0_{t_{i+1}-t_i} \end{bmatrix}.$$

Since k is the maximal controllability index [14] and $d_j = c_j - 1$, it follows that

$$V(s) = -F^{-1}\left(I + s\overline{N} + s^2\overline{N}^2 + \cdots + s^k\overline{N}^k\right)P^T$$

$$= V_0 + sV_1 + s^2V_2 + \cdots + s^kV_k. \tag{3.8}$$

By writing $N(s)$ and $D(S)$ as

$$N(s) = N_0 + sN_1 + s^2N_2 + \cdots + s^kN_k, \tag{3.9}$$

$$D(s) = D_0 + sD_1 + s^2D_2 + \cdots + s^kD_k, \tag{3.10}$$

we can compute

$$N_i = -[I_n \quad 0]V_i \begin{bmatrix} 0 \\ I_m \end{bmatrix} \tag{3.11}$$

$$D_i = -[0 \quad I_m]V_i \begin{bmatrix} 0 \\ I_m \end{bmatrix}, \qquad i = 1,2,\dots,k. \tag{3.12}$$

In the sequel, we will show that the pair $N(s)$, $D(s)$ is column-ordered and column-reduced. For this purpose we will use the special form of the matrices $\overline{N}$ and F in order to compute the leading column coefficients of $N(s)$ and $D(s)$. It can be shown that the powers of $\overline{N}$ have the following form:

$$\overline{N}^i = \begin{bmatrix} 0 & 0 \\ \hat{N}_i & 0 \end{bmatrix}, \qquad i = 1,2,\dots,k, \tag{3.13}$$

where $\hat{N}_i$ are lower-block-triangular matrices and have dimensions $n_i \times n_i$, where $n_i = n + m - \sum_{j=1}^{i} t_j$. The block-diagonal elements of $\hat{N}_i$ are computed as follows:

$$\hat{N}_{i,j} = \prod_{l=i+j-1}^{j} \overline{N}_l, \qquad j + i = 2,3,\dots,k+1. \tag{3.14}$$

Using the special form of the powers of $\overline{N}$, we see that

$$F^{-1}(s\overline{N} - I)^{-1}$$

$$= \begin{bmatrix} \hat{N}_{0,1} & 0 & \cdots & 0 & 0 \\ \hat{N}_{1,1}s + O(s^0) & \hat{N}_{0,2} & \cdots & 0 & 0 \\ \hat{N}_{2,1}s^2 + O(s) & \hat{N}_{1,2}s + O(s^0) & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hat{N}_{k-1,1}s^{k-1} + O(s^{k-2}) & \hat{N}_{k-2,2}s^{k-2} + O(s^{k-3}) & \cdots & \hat{N}_{0,k} & 0 \\ \hat{N}_{k,1}s^k + O(s^{k-1}) & \hat{N}_{k-1,2}s^{k-1} + O(s^{k-2}) & \cdots & \hat{N}_{1,k}s + O(s^0) & \hat{N}_{0,k+1} \end{bmatrix}, \tag{3.15}$$

where $O(s^i)$ denotes polynomial terms of power less or equal to i. Since P^T is just a column permutation matrix, it follows from (3.15) that $V(s)$ is a column-degree-ordered polynomial matrix. Since P^T is a known permutation matrix [see (3.6)], we can easily compute the pair $N(s)$, $D(s)$ by carefully applying the appropriate column permutations in (3.15). In particular, the pair $N(s)$, $D(s)$ has the following form:

$$F^{-1}(s\bar{N} - I)^{-1} P^T \begin{bmatrix} 0 \\ I_m \end{bmatrix}$$

$$= \begin{bmatrix} N(s) \\ D(s) \end{bmatrix}$$

$$= \begin{bmatrix} H_{0,1} & 0 & \cdots & 0 & 0 \\ H_{1,1}s + O(s^0) & H_{0,2} & \cdots & 0 & 0 \\ H_{2,1}s^2 + O(s) & H_{1,2}s + O(s^0) & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ H_{k-1,1}s^{k-1} + O(s^{k-2}) & H_{k-2,2}s^{k-2} + O(s^{k-3}) & \cdots & H_{0,k} & 0 \\ H_{k,1}s^k + O(s^{k-1}) & H_{k-1,2}s^{k-1} + O(s^{k-2}) & \cdots & H_{1,k}s + O(s^0) & H_{0,k+1} \end{bmatrix}, \tag{3.16}$$

where

$$H_{0,i} = \begin{bmatrix} 0_{t_{i-1}} \\ I_{t_i - t_{i-1}} \end{bmatrix}, \qquad i = 1, 2, \ldots, k+1, \qquad t_0 = 0, \tag{3.17}$$

and

$$H_{i,j} = \hat{N}_{i,j} H_{0,i}, \qquad i, j \in Z^+, \qquad i + j = 2, \ldots, k+1. \tag{3.18}$$

Equation (3.16) shows that $D(s)$ is column-degree-ordered.

The next step is to find the high column coefficient of $D(s)$, D_{hc}. From Equation (3.16) we know that

$$D_{hc} = [\, H_{k,1} \quad H_{k-1,2} \quad \cdots \quad H_{1,k} \quad H_{0,k+1} \,],$$

and therefore, if we determine the structure of $H_{k,i}$ for $i = 1, 2, \ldots, k+1$, we actually have computed D_{hc}. We notice here that $H_{0,k+1}$ does not occur in our case, since we assumed that B has full column rank.

Using (3.14), we can compute each $H_{i,j}$ for $i+j=k+1$ as follows:

$$H_{i,j} = F_{k+1}^{-1} \prod_{l=k}^{j} \overline{N}_l \begin{bmatrix} 0_{t_{i-1}} \\ I_{t_j - t_{j-1}} \end{bmatrix}, \qquad i+j=k+1,$$

which results in

$$D_{hc} = \begin{bmatrix} I_{t_1} & 0 & \cdots & 0 & 0 \\ 0 & I_{t_2-t_1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I_{t_{k-1}-t_{k-2}} & 0 \\ 0 & 0 & \cdots & 0 & I_{t_k - t_{k-1}} \end{bmatrix}. \qquad (3.19)$$

The normal external description for the original matrices (A, B) is given as follows:

$$\begin{bmatrix} QT & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = \begin{bmatrix} QTN(s) \\ D(s) \end{bmatrix},$$

which clearly is a column-reduced and column-degree-ordered external description.

The above discussion yields in the following result.

THEOREM 3.2. *The pair $N(s)$, $D(s)$ computed in* (3.7) *is a normal external description of the system* (A, B).

We summarize this procedure by presenting an algorithm.

ALGORITHM 3.1.

Step 1. Compute similarity transformations Q and T such that (A, B) is in block Hessenberg form (A_H, B_H).

Step 2. Select C_H as in (3.1), (3.2), (3.3), and set

$$\overline{N} = \begin{bmatrix} I_n & 0 \\ O & 0 \end{bmatrix} \begin{bmatrix} A_H & B_H \\ C_H & 0 \end{bmatrix}^{-1}, \qquad F = \begin{bmatrix} A_H & B_H \\ C_H & 0 \end{bmatrix}.$$

Step 3. Compute $N(s)$, $D(S)$ as follows:

$$\begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = \begin{bmatrix} QT & 0 \\ 0 & I_m \end{bmatrix} F^{-1}\left(I + \bar{N}s + \bar{N}^2 s^2 + \cdots + \bar{N}^k s^k\right) \begin{bmatrix} 0 \\ I_m \end{bmatrix}$$

REMARK 3.1. In Section 2 we required that $A_{i,i+1} = [I_{t_i} \ 0]$. This is because we want to compute the entries of $D(s)$ as monic polynomials. If we drop this requirement and instead assume that $A_{i,i+1}$ are in lower triangular form, then D_{hc} will be a nonsingular block-diagonal matrix. Moreover, det D_{hc} will be the proportionality factor between $\det(sI - A)$ and det $D(s)$.

REMARK 3.2. The selection of the sizes of C_i is not arbitrary. The lengths of the Jordan chains of the infinite elementary divisors have to be kept minimal, namely equal to the number of stairs in the Hessenberg form. This guarantees the desired relationship between the column-minimal indices of the pencil $[SI - A \ \ -B]$ and the infinite elementary divisors of $\bar{N}$. This fact is crucial in order to obtain a normal external description. A different relationship will not produce a normal external description. Therefore, though the selection of C_i is not unique, the size of the blocks has to comply with the rule used in (3.3).

REMARK 3.3. The computation of F^{-1} is rather simple, since it is triangular, and so is the construction of $\bar{N}$. In the case that we want $D_{hc} = I$, it is reasonable to require from the very beginning of the algorithm (that is, the block Hessenberg form) that $A_{i,i+1} = [I_{t_i} \ 0]$. This assumption does not make the algorithm more numerically stable, since the inversion of the matrix F is inevitable.

REMARK 3.4. As we have already mentioned, the index of nilpotency of $\bar{N}$ is minimal. Although this is true, the algorithm can produce misleading results when $\bar{N}$ is ill conditioned. However, the algorithm provides a direct and reliable solution to the problem in the case of a well-conditioned $\bar{N}$.

An advantage of Algorithm 3.1 is its ability to provide helpful information on the structure of the pair $(N(s), D(s))$ [see (3.16)]. As a matter of fact, the information about the structure of the diagonal elements in (3.16) is essential, since it provides information about the minimal reachability chains of the pair (A, B). Alternative algorithms for the computation of a normal external description can be found in [2, 4].

We conclude this section with a straightforward application of the proposed algorithm to the transmission-zero assignment for state-accessible systems.

THEOREM 3.3. *Assume a state-accessible system* (A, B). *Then an output matrix* C *that assigns all the zeros of* (A, B) *to infinity with minimal chains is described in* (3.1), (3.2), *and* (3.3).

It is important to mention that the selection of an output matrix C that assigns all the zeros to infinity is far from unique.

4. AN APPLICATION TO STATE FEEDBACK

In this section, we will use the ideas developed in Section 3 for the parametrization of feedback controllers. First, we examine the state-feedback problem. Second, we study the parametrization problem. The main vehicle in the proposed approach is the inversion of a unimodular pencil. To our knowledge, this is the first result that brings together the use of staircase forms in designing controllers with the polynomial approach. Finally, the proposed method provides the parametrization of all possible controllers.

Let a state feedback have the form

$$u(t) = Kx(t) + v(t). \tag{4.1}$$

Then we can show that the poles of the closed-loop system are given as the roots of the following polynomial:

$$\det[D(s) - KN(s)]. \tag{4.2}$$

Assume that the pair (A, B) is already in its block Hessenberg form, (A_H, B_H). Then select a C_H as in (3.1), (3.2), (3.3). The selection of C_H, which clearly is not unique, provides a unimodular pencil $U(s)$ as in (3.4). The controllability indices of the system $k_1, k_2, \dots, k_m$ are decreasingly ordered by magnitude. That is,

$$k_1 = \cdots = k_{t_1} \geqslant k_{t_1+1} = \cdots = k_{t_2} \geqslant \cdots \geqslant k_{t_{k-1}+1} = \cdots = k_{t_k}$$

Moreover, let $c_1(s), c_2(s), \dots, c_m(s)$ be the desired closed-loop invariant polynomials [7] decreasingly ordered by degree.

Our contention for the rest of this section is the development of a computationally efficient algorithm for the assignment of the desired closed-loop polynomials $c_1(s), c_2(s), \dots, c_m(s)$. Our procedure is organized as follows:

(1) Construct a controllable pair $[sI - R \quad -H]$ where R is a matrix that contains the desired closed-loop invariant polynomials $c_1(s), c_2(s), \dots, c_m(s)$ and $[sI - R \quad -B]$ has controllability indices $k_1, k_2, \dots, k_m$.

(2) Compute the normal external description of the pencil $[sI - R \quad -H]$ using Algorithm 3.1 or 3.2, say $P(s), Q(s)$. Then since $N(s), D(s)$ is the normal external description of the original system, we know that $Q(s), D(s)$ are column-reduced with the same column degrees.

(3) Finally, the feedback K can be computed from the Diophantine equation [9]

$$XD(s) + YN(s) = Q(s), \tag{4.3}$$

which always has a pair of constant solutions (X, Y), where X is nonsingular [10]. Furthermore,

$$K = -X^{-1}Y. \tag{4.4}$$

We briefly review here Rosenbrock's theorem [13], in order to draw a connection later with the Diophantine equation (4.3).

THEOREM 4.1. *Given a controllable pair* (A, B) *with controllability indices* $k_1, k_2, \dots, k_m$, *there exists a matrix* $K \in \mathbb{R}^{m \times n}$ *that assigns* $c_1(s), c_2(s), \dots, c_m(s)$ *as closed-loop invariant polynomials if and only if*

$$\sum_{i=1}^{j} \det c_i(s) \geqslant \sum_{i=1}^{j} k_i, \qquad j = 1, 2, \dots, m,$$

are satisfied and equality holds when $j = m$.

The vehicle to our approach will be the Diophantine equation (4.3). A major result concerning the solution of (4.3) was given by Kučera and Zagalak in [9], which we briefly review here.

PROPOSITION 4.2. *The Diophantine equation* (4.3) *has a constant solution* (X, Y), *where* X *is nonsingular if and only if* $Q(s)$ *is column-reduced with the same column indices as* $D(s)$.

It has been shown in [10] that Proposition 4.2 is an equivalent statement to that of Rosenbrock's theorem. The main advantage of Proposition 4.2 over Rosenbrock's theorem is that it provides us with a powerful design Diophantine equation.

Having presented the relation of Rosenbrock's theorem to the existence of solution to (4.3), we are now interested in constructing a pencil $[sI - R \; -H]$ that satisfies the conditions of the closed-loop invariant polynomials and the controllability indices.

Given the desired closed-loop polynomials $c_1(s), c_2(s), \dots, c_m(s)$, of degrees $c_1, c_2, \dots, c_m$ respectively, that satisfy Rosenbrock's theorem, we will use Dickinson's algorithm [5] for selecting a pair (R, H) that is controllable and such that $sI - R$ has invariant polynomials $c_1(s), c_2(s), \dots, c_m(s)$ and the controllability indices of (R, H) are $k_1, k_2, \dots, k_m$. This construction is based on the following lemma [5].

LEMMA 4.3. *Let $[sI - R \; -G]$ have $c_1(s), c_2(s), \dots, c_m(s)$ as its invariant factors, and let $c_1, c_2, \dots, c_m$ (decreasingly ordered) be the controllability indices. Let $c_p < c_q$ for some $p > q$. Then there exists a matrix H such that the controllability indices of $[sI - R \; -H]$ are $k_1, k_2, \dots, k_m$, where*

$$k_i = c_i,$$
$$k_p = c_p + 1,$$
$$k_q = c_q - 1.$$

Given the desired closed-loop polynomials $c_1(s), c_2(s), \dots, c_m(s)$, we can select R as

$$R = \begin{bmatrix} R_{1,1} & 0 & \cdots & 0 & 0 \\ 0 & R_{2,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & R_{m-1,m-1} & 0 \\ 0 & 0 & \cdots & 0 & R_{m,m} \end{bmatrix} \tag{4.5}$$

where $R_{i,i}$ are lower companion matrices of dimensions $c_i \times c_i$, that is,

$$R_{i,i} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ r_{i,1} & r_{i,2} & \cdots & r_{i,c_i-1} & r_{i,c_i} \end{bmatrix}. \tag{4.6}$$

The selection of R guarantees that $sI - R$ has the desired closed-loop polynomials $c_1(s), \dots, c_m(s)$. Selecting G as

$$G = \text{block diag}\{e_i\}, \qquad i = 1, \dots, m, \tag{4.7}$$

where $e_i \in \mathbb{R}^{c_i}$, the controllability of the pair (R, G) is guaranteed, Furthermore, the controllability indices of (R, G) are $c_1, c_2, \dots, c_m$. The construction of H such that (R, H) has controllability indices $k_1, k_2, \dots, k_m$ is summarized in the following algorithm [5].

ALGORITHM 4.4.

Step 1. Initialize (R, G) as in (4.5), (4.6), and (4.7).
Step 2. Let $G = [g_1, \dots, g_m]$. Then compute

$$h_i = g_i,$$

$$h_p = g_q,$$

$$h_q = g_p + Rg_q.$$

REMARK 4.1. Notice that there are in fact no computations, since the product Rg_q is nothing else than the shift up in g_p by one position. This due to the special form of R; namely, it is a block-diagonal matrix having the companion matrices of the $c_i(s)$'s on the main diagonal.

Without loss of generality we assume that the pairs (A, B) and (R, H) are in block Hessenberg form, say (A_H, B_H) and (R_H, H_H) respectively. Performing Algorithm 3.1 on (A_H, B_H) and (R_H, B_H) respectively, we obtain a pair of normal external descriptions $N(s)$, $D(s)$ and $P(s)$, $Q(s)$. It is important to mention the special structure that these two representations enjoy; namely, they are in the form described in (3.16), (3.17), (3.18). Moreover,

$$D_{hc} = Q_{hc} = I_m,$$

and $Q(s)$ contains the desired closed-loop invariant polynomials $c_1(s), \dots, c_m(s)$.

Since $D(s)$ and $Q(s)$ are column-reduced with the same column degrees by Proposition 4.2, Equation (4.3) has always a solution. Let $S(s)$ be a $(k_1 + k_2 + \cdots + k_m) \times m$ polynomial and column-reduced matrix with column degrees $k_1 - 1, k_2 - 1, \dots, k_m - 1$ such that

$$S(s) = \text{block diag}\left\{ \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{k_i - 1} \end{bmatrix} \right\}.$$

Then Equation (4.3) can be written as

$$\begin{aligned} XD_{hc} \operatorname{diag}[s^{k_1}, \dots, s^{k_m}] + YN_{lc}S(s) \\ = Q_{hc} \operatorname{diag}[s^{k_1}, \dots, s^{k_m}] + (Q_{lc} - D_{lc})S(s), \end{aligned} \tag{4.8}$$

where D_{hc} and Q_{hc} are the leading coefficient matrices of $D(s)$ and $Q(s)$ respectively, and N_{lc}, D_{lc}, and Q_{lc} are constant matrices. In fact, we are to solve the following two systems of equations:

$$XD_{hc} = Q_{hc},$$

$$YN_{lc} = Q_{lc} - D_{lc}.$$

On the other hand, since $D(s)$, $Q(s)$ are in the special form described in (3.16), we know that $D_{hc} = Q_{hc} = I$, which results in $X = I$. Therefore

$$K_H = -(Q_{lc} - D_{lc})N_{lc}^{-1}. \tag{4.9}$$

The computation of K_H follows without the inversion of N_{lc}. Due to the special form of the matrix $N(s)$ and the structure of $S(s)$, N_{lc} has the following form:

$$N_{lc} = \begin{bmatrix} 1 & & \\ & \ddots & \\ X & & 1 \end{bmatrix}. \tag{4.10}$$

Therefore the solution of (4.9) reduces to a simple back-substitution problem.

We can now summarize our proposed technique in the following algorithm.

ALGORITHM 4.5.

Step 1. For the given invariant polynomials $c_1(s), \dots, c_m(s)$, construct (R, H) such that the controllability indices of (R, H) are $k_1, \dots, k_m$, by executing Algorithm 4.5.

Step 2. Compute $(N(s), D(s)), (P(s), Q(s))$ by executing Algorithm 3.1.

Step 3. Compute K_H from (4.9) using back substitution. Then

$$K = K_H T^{-1} Q^T.$$

REMARK 4.2. The computation of $Q(s)$ using Algorithm 3.1 avoids any inversion that is incorporated for general matrices. This is due to the special structure of the matrices R and H.

We conclude this section by presenting a parametrization technique for all state feedbacks that assign the desired closed-loop invariant polynomials $c_1(s), \dots, c_m(s)$ for a given normal external description $(N(s), D(s))$.

Without loss of generality we assume that (R, H) is in block Hessenberg form. Then any matrix C that is constructed according to (3.1), (3.2), and (3.3) will produce a desired $Q(s)$. Clearly, the selection of C is not unique, and as a result the desired $Q(s)$ is not unique. In particular, different selections of C will produce different $Q(s)$. By denoting as $\mathscr{C}$ the collection of all C's that satisfy (3.1), (3.2), and (3.3) we can define as $\mathscr{Q}$ the set of all column-reduced, column-ordered polynomial matrices with degrees $k_1, \dots, k_m$ and invariant factors $c_1(s), \dots, c_m(s)$.

On the other hand, because of the structure of $Q(s)$ and $D(s)$, the solution to (4.3) is unique and is given by (4.9). Therefore, we can define the set of all possible feedbacks that assign the desired closed-loop invariant polynomials as follows:

$$\mathscr{K} = \{K = -X^{-1}Y \mid XD(s) + YN(s) = Q(s)\ \forall Q(s) \in \mathscr{Q}\}.$$

The main motivation for such a parametrization is the issue of robustness in the following sense. Given a normal external description, we wish to find a $Q(s)$ which leads to a K for which the invariant polynomials of $D(s) - KN(s)$ are as insensitive to perturbations as possible.

5. AN APPLICATION TO ZERO ASSIGNMENT

In this section we present a further application that arises from the computation of a normal external description and the solution of a Diophantine equation. In particular we will show that in the case of state-accessible systems the problem of assigning the zeros is equivalent to a state-feedback problem. This approach relates the column-minimal indices of the original system to the problem of zero assignment by squaring. That is, we want to find an output matrix $C \in \mathbb{R}^{m \times n}$ such that

$$y = Cx$$

and such that the invariant factors of $CN(s)$ reflect the dynamics of the desired zeros.

Given a normal external description $(N(s), D(s))$ for the pair (A_H, B_H) described in (3.16), we can define a new normal external description $(\overline{N}(s), \overline{D}(s))$ by simply eliminating the last m rows in (3.16). That is,

$$\begin{bmatrix} \overline{N}(s) \\ \overline{D}(s) \end{bmatrix} = \begin{bmatrix} H_{0,1} & 0 & \cdots & 0 \\ H_{1,1}s + O(s^0) & H_{0,2} & \cdots & 0 \\ H_{2,1}s^2 + O(s) & H_{1,2}s + O(s^0) & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H_{k-1,1}s^{k-1} + O(s^{k-2}) & H_{k-2,2}s^{k-2} + O(s^{k-3}) & \cdots & H_{0,k} \end{bmatrix}, \tag{5.1}$$

where

$$H_{0,i} = \begin{bmatrix} 0_{t_{i-1}} \\ I_{t_i - t_{i-1}} \end{bmatrix}, \qquad i = 1, 2, \ldots, k, \qquad t_0 = 0,$$

and

$$H_{i,j} = \hat{N}_{i,j} H_{0,i}, \qquad i, j \in Z^+, \qquad i + j = 2, \ldots, k$$

The new normal external description also enjoys all the properties discussed in Section 3. Furthermore, $(\overline{N}(s), \overline{D}(s))$ is a normal external description for the pair $(\bar{A}_H, \bar{B}_H)$, where

$$\bar{A}_H = [I_{n-m} \quad 0] A_H \begin{bmatrix} I_{n-m} \\ 0 \end{bmatrix}, \qquad \bar{B}_H = [I_{n-m} \quad 0] A_H \begin{bmatrix} 0 \\ I_m \end{bmatrix}.$$

LEMMA 5.1. *The pair* $(\bar{A}_H, \bar{B}_H)$ *is also in Hessenberg form with controllability indices* $k_1 - 1, k_2 - 1, \dots, k_{t_{k-1}} - 1$.

Having in mind the special structure of B_H, in order to assign $n - m$ finite zeros, we ought to find an output matrix C_H such that

(1) $C_H B_H$ is nonsingular [8];
(2) we have

$$C_H = \begin{bmatrix} \bar{N}(s) \\ \bar{D}(s) \end{bmatrix} = \bar{Q}(s),$$

where $\bar{Q}(s)$ contains the desired invariant polynomials.

The first condition implies that if we partition C_H as $[\bar{Y}\ \bar{X}]$, $\bar{X}$ must be nonsingular. The second condition involves the solution of

$$\bar{X}\bar{D}(s) + \bar{Y}\bar{N}(s) = \bar{Q}(s) \tag{5.2}$$

for some $\bar{X}, \bar{Y}$ such that $\bar{X}$ is nonsingular. This result shows that the finite-zero assignment is a disguised version of a state-feedback problem on a reduced-order system.

Due to the structure of the problem, we can apply Rosenbrock's theorem for state feedback in order to understand the structure of the invariant polynomials that reflect the dynamics of the finite zeros. This is summarized in the following result, the proof of which follows from our discussion.

THEOREM 5.2. *Given a state-accessible system* (A, B), *we can find an output matrix* $C \in \mathbb{R}^{m \times n}$ *that assigns at most* t_{k-1} *invariant polynomials* $c_1(s), \dots, c_{t_{k-1}}(s)$ *that reflect the dynamics of* $n - m$ *finite zeros if and only if the set of inequalities*

$$\sum_{i=1}^{j} \deg c_i(s) \geqslant \sum_{i=1}^{j} (k_i - 1), \qquad j = 1, 2, \dots, t_{k-1}, \tag{5.3}$$

are satisfied and equality holds when $j = t_{k-1}$.

Another advantage of formulating the problem of finite-zero assignment arises from the fact that the solution of (5.2) can be computed easily using the proposed algorithms in Section 4.

6. A NUMERICAL EXAMPLE

Let (A, B) be given as follows:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then the Hessenberg form of (A, B) is

$$A_H = \begin{bmatrix} 0.5000 & 1.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.2500 & 0.5000 & 1.0000 & 0.0000 & 0.0000 \\ -1.0000 & 0.0000 & 1.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.5000 & 0.5000 \\ 0.0000 & 0.0000 & 0.0000 & 0.5000 & -1.5000 \end{bmatrix},$$

$$B_H = \begin{bmatrix} 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 1.0000 & 0.0000 \\ 0.0000 & 1.0000 \end{bmatrix}.$$

Therefore $t_1 = t_2 = t_3 = 1$ and $t_4 = t_5 = 2$. Notice that it is not required that $A_{i,i+1} = [I_{t_i} \ 0]$; consequently only orthogonal transformations are involved. The structure of C_H is of the form

$$C_H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then using (3.16) we obtain

$$\begin{bmatrix} 1.0000 & 0.0000 \\ -0.5000 & 0.0000 \\ 0.0000 & 0.0000 \\ 1.0000 & 0.0000 \\ 0.0000 & 1.0000 \\ -0.5000 & -0.5000 \\ -0.5000 & 1.5000 \end{bmatrix} + \begin{bmatrix} 0.0000 & 0.0000 \\ 1.0000 & 0.0000 \\ -1.0000 & 0.0000 \\ 1.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 0.5000 & 0.0000 \\ -0.5000 & 1.0000 \end{bmatrix} s$$

$$+ \begin{bmatrix} 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 1.0000 & 0.0000 \\ -2.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 2.0000 & 0.0000 \\ 1.0000 & 0.0000 \end{bmatrix} s^2$$

$$+ \begin{bmatrix} 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 1.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ -2.5000 & 0.0000 \\ -0.5000 & 0.0000 \end{bmatrix} s^3 + \begin{bmatrix} 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 1.0000 & 0.0000 \\ 0.0000 & 0.0000 \end{bmatrix} s^4,$$

from which we can extract

$$H_{01} = 1, \qquad H_{03} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad H_{1,1} = 1, \qquad H_{21} = 1$$

and

$$H_{31} = \begin{bmatrix} 1.0000 \\ 0.0000 \end{bmatrix}, \qquad H_{13} = \begin{bmatrix} 0.0000 \\ 1.0000 \end{bmatrix}.$$

Notice that only these $H_{i,j}$'s appear in (3.16), due to the values of t_i's. Moreover, observe that $D_{hc} = I_m$.

The normal external description for the original pair (A, B) is

$$\begin{bmatrix} N(s) \\ D(s) \end{bmatrix} = \begin{bmatrix} 1.4142 & 0.0000 \\ -0.7071 & 0.7071 \\ 0.0000 & 0.0000 \\ -0.7071 & -0.7071 \\ 0.0000 & 0.0000 \\ 0.0000 & 1.4142 \\ 0.7071 & -0.7071 \end{bmatrix} + \begin{bmatrix} -1.4142 & 0.0000 \\ -0.7071 & 0.0000 \\ -1.4142 & 0.0000 \\ -0.7071 & 0.0000 \\ 1.4142 & 0.0000 \\ -0.7071 & 0.7071 \\ 0.0000 & -0.7071 \end{bmatrix}$$

$$+ \begin{bmatrix} 0.0000 & 0.0000 \\ 1.4142 & 0.0000 \\ 0.0000 & 0.0000 \\ 1.4142 & 0.0000 \\ -1.4142 & 0.0000 \\ -0.7071 & 0.0000 \\ -2.1213 & 0.0000 \end{bmatrix} s^2 + \begin{bmatrix} 0.0000 & 0.0000 \\ -0.7071 & 0.0000 \\ 0.0000 & 0.0000 \\ -0.7071 & 0.0000 \\ 0.0000 & 0.0000 \\ 1.4142 & 0.0000 \\ 2.1213 & 0.0000 \end{bmatrix} s^3$$

$$+ \begin{bmatrix} 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ -0.7071 & 0.0000 \\ -0.7071 & 0.0000 \end{bmatrix} s^4.$$

In the original coordinates the output matrix C is

$$C = \begin{bmatrix} 0.7071 & 0.0000 & -0.7071 & 0.0000 & 0.0000 \\ 0.0000 & 0.7071 & 0.0000 & -0.7071 & 0.0000 \end{bmatrix},$$

and

$$CN(s) = \begin{bmatrix} 1.0000 & 0.0000 \\ 0.0000 & 1.0000 \end{bmatrix}.$$

The system (A, B) has two controllability indices; namely $k_1 = 4$ and $k_2 = 1$. Therefore, we can assign either two closed-loop invariant polynomials of degrees $\deg c_1 = 4$, $\deg c_2 = 1$ or one closed-loop invariant polynomial of degree $\deg c_1 = 5$. To motivate and also to exhibit the proposed technique we will examine both cases.

Case 1

In this case we will assign the following two invariant polynomials:

$$c_1(s) = s^4 + 4s^3 + 10s^2 + 10s + 5, \qquad c_2(s) = s + 2.$$

The roots of $c_1(s)c_2(s)$ are $\{-1, -2, -2, -1 \pm j\}$.

By executing Algorithm 4.4 we obtain

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -4 & -10 & -10 & -5 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}, \qquad H = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Using Algorithm 4.5, we obtain $Q(s)$, K_H, and K as follows:

$$Q(s) = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix} s + \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix} s^2$$
$$+ \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} s^3 + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} s^4,$$

$$K_H = -\begin{bmatrix} 9.5000 & 25.0000 & 23.0000 & 7.5000 & 0.5000 \\ 0.0000 & 0.0000 & 0.0000 & 0.5000 & 0.5000 \end{bmatrix},$$

$$K = -\begin{bmatrix} 1.5000 & 3.5000 & 11.0000 & 3.5000 & 11.5000 \\ 1.5000 & 3.5000 & 11.0000 & 4.5000 & 11.5000 \end{bmatrix}.$$

Case 2

In this case we assign the following invariant polynomial:

$$c_1(s) = s^5 + 7s^4 + 20s^3 + 30s^2 + 24s + 8.$$

The roots of $c_1(s)$ are $\{-1, -2, -2, -1 \pm j\}$.

As before, by executing Algorithm 4.4 we obtain

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -24 & -20 & -30 & -7 & -8 \end{bmatrix}, \qquad H = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Using Algorithm 4.5, we obtain $Q(s)$, K_H, and K as follows:

$$Q(s) = \begin{bmatrix} 0 & 1 \\ -8 & 7 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -24 & 1 \end{bmatrix} s$$

$$+ \begin{bmatrix} 0 & 0 \\ -30 & 0 \end{bmatrix} s^2 + \begin{bmatrix} 0 & 0 \\ -20 & 0 \end{bmatrix} s^3 + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} s^4,$$

$$K_H = -\begin{bmatrix} -2.0000 & 0.0000 & 3.0000 & 2.5000 & 1.5000 \\ -25.0000 & -74.0000 & -70.0000 & -19.5000 & 5.5000 \end{bmatrix},$$

$$K = -\begin{bmatrix} 7.0000 & 13.0000 & 30.0000 & 9.0000 & 36.5000 \\ -5.0000 & -12.0000 & -32.0000 & -5.0000 & -33.5000 \end{bmatrix}.$$

We will compute an output matrix that assigns the desired invariant polynomials that reflect the dynamics of the finite zeros of the system. Since the original system has two controllability indices $k_1 = 4$ and $k_2 = 1$, the reduced-order system has one controllability index, $k_1 = 3$. This implies that we can only assign one invariant polynomial c_1, with $\deg c_1 = 3$. The pair $(\overline{N}(s), \overline{D}(s))$ is

$$\begin{bmatrix} 1.0000 & 0.0000 \\ -0.5000 & 0.0000 \\ 0.0000 & 0.0000 \\ 1.0000 & 0.0000 \\ 0.0000 & 1.0000 \end{bmatrix} + \begin{bmatrix} 0.0000 & 0.0000 \\ 1.0000 & 0.0000 \\ -1.0000 & 0.0000 \\ 1.0000 & 0.0000 \\ 0.0000 & 0.0000 \end{bmatrix} s$$

$$+ \begin{bmatrix} 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 1.0000 & 0.0000 \\ -2.0000 & 0.0000 \\ 0.0000 & 0.0000 \end{bmatrix} s^2 + \begin{bmatrix} 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \\ 1.0000 & 0.0000 \\ 0.0000 & 0.0000 \end{bmatrix} s^3,$$

which has $\overline{D}_{hc} = I$. By selecting

$$c_1 = s^3 + 3s^2 + 4s + 2,$$

the roots of which are $\{-1, -1 \pm j\}$, and executing Algorithm 4.5, we obtain

$$\overline{Q}(s) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} s + \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} s^2 + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} s^3,$$

$$C_H = \begin{bmatrix} 5.0000 & 8.0000 & 5.0000 & 1.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 \end{bmatrix},$$

$$C = \begin{bmatrix} -0.5000 & 1.0000 & 4.5000 & 0.0000 & 2.5000 \\ -0.5000 & 0.0000 & 4.5000 & 1.0000 & 2.5000 \end{bmatrix}.$$

All the algorithms described in this paper have been implemented in MATLAB [12].

7. CONCLUSIONS

We have presented an algorithm for the computation of normal external descriptions for linear state-variable systems using staircase forms and embedding techniques. We associated the computed normal external description to the state-feedback problem in linear systems through a Diophantine equation. Solutions to such a Diophantine equation were obtained in a computationally efficient way. Furthermore we showed how to solve the parametrization problem effectively. Finally, we applied the results of state feedback to finite-zero assignment, through the solution of a second Diophantine equation. Then we studied the limits of squaring down in state-accessible systems in assigning the finite structure of the zeros.

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