A generalized topological view of motion in discrete space

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1. Introduction

In several areas of Computer Science, one is interested in using abstract mathematical structures as a basis for modelling certain phenomena of the real world. This interest is particularly strong in the knowledge representation subfield of artificial intelligence (AI), and amongst the phenomena studied in this field, those concerned with time, space, motion, and change have enjoyed a special prominence over the last 20 years or so [2,4,16,18].

The standard approach to these phenomena in the natural sciences has always been to use the well-developed theory of functions over the real numbers as the most appropriate starting point. Time is regarded as isomorphic, in point of its ordering and metric properties, with the real line $\mathbb{R}$, and $n$-dimensional space is likewise modelled as $\mathbb{R}^n$. In the field of AI this approach has often been regarded as problematic on account of a perceived mismatch between the forms of representation and reasoning naturally arising from it and the forms employed in everyday qualitative thinking, which it is part of the purpose of AI to emulate. As a result of this, AI researchers have tended to espouse theories based on qualitative descriptions of the phenomena. In practice, this often means discrete rather than continuous descriptions, a typical approach being to describe the state of the world in terms of some small finite set of possible situations collectively constituting a discrete space of qualitative possibilities. Motion and change are then described with respect to this discrete space [7].

Discrete phenomena have also been studied by researchers interested in formulating a systematic and rigorous theory of digital images [12,13,14]. This work has proceeded largely in isolation from the AI work mentioned above, but in recent years a certain rapprochement between the two areas has arisen as a result of a number of researchers’ beginning to explore the common ground between them [19,10]. In particular, it has become apparent that by generalizing certain topological notions it may be possible to provide a unifying framework within which all the disparate forms of discrete space that have engaged the attention of workers in these fields may be described [17].
In this paper, I make use of such a framework to investigate the possible forms that continuous motion may take in discrete spaces. This requires, of course, an exact definition of what is meant by continuity in generalized spaces where the standard topological notion is not immediately applicable. It also requires careful consideration of what should be meant by such terms as ‘position’, ‘distance’, ‘path’, and ‘time’ in the context of structures defined purely mathematically.

I begin by establishing the general framework of closure spaces, which were introduced by Čech [3] and subsequently suggested by Smyth [17] as an appropriate vehicle for the unification of disparate approaches to discrete space—notably the graph-theoretical approaches of the ‘digital topologists’ [13] and the topological approach of [12,14]. Many of the results we require here are mathematically rather straightforward and the proofs are accordingly omitted. Amongst the closure spaces I identify two classes as being of particular interest, which I call adjacency spaces and incidence spaces; both these types of structure have a property called quasi-discreteness which captures the discreteness of the intended models. I then investigate functions of closure spaces with a view to modelling motion, the motion of an object being specified by the function giving its position at each time; this requires, of course, a decision as to how time should be modelled, whether as a closure space itself or in the usual way in terms of $\mathbb{R}$. This in turn impacts on the nature of continuity, and I investigate systematically what continuous motion looks like in various combinations of choices for the modelling of space and time, both for the motion of point objects and for the motion of extended objects, whose positions are regions of space rather than points.

2. Closure spaces

**Definition.** A closure space [3] is a pair $(S, cl)$, where $S$ is any set, and $cl: 2^S \rightarrow 2^S$ is a function associating with each subset $X \subseteq S$ a subset $cl(X) \subseteq S$, called the closure of $X$, such that

1. $cl(\emptyset) = \emptyset$,
2. $X \subseteq cl(X)$,
3. $cl(X \cup Y) = cl(X) \cup cl(Y)$.

Note that by (2), $cl(S) = S$. Closure spaces were suggested by Smyth [17] as an appropriate framework for unifying disparate approaches to discrete space. They generalize topological spaces; in fact, a topological space can be defined as a closure space in which the closure operation is idempotent (see below).

A number of ideas familiar in the topological setting can be straightforwardly generalized to closure spaces.

**Definition.** Let $(S, cl)$ be a closure space, and let $X \subseteq S$. Then

1. The interior $int(X)$ of $X$ is the set $cl(X^c)^c$, i.e., the complement of the closure of the complement of $X$.
2. $X$ is a neighbourhood of an element $x \in S$ if $x \in int(X)$. 
(3) $X$ is closed if $X = \text{cl}(X)$.
(4) $X$ is open if $X = \text{int}(X)$.

As with topological spaces, the open sets are precisely the complements of the closed sets.

**Lemma 1.** In a closure space, $X$ is open iff $X^c$ is closed.

**Lemma 2.** In a closure space, if $X \subseteq Y$ then $\text{cl}(X) \subseteq \text{cl}(Y)$.

**Corollary 1.** In a closure space, if $X \subseteq Y$ then $\text{int}(X) \subseteq \text{int}(Y)$.

**Lemma 3.** In a closure space, $\text{cl}(\bigcap_{i \in I} X_i) \subseteq \bigcap_{i \in I} \text{cl}(X_i)$.

**Lemma 4.** The open sets of a closure space $(S, \text{cl})$ satisfy
(1) $\emptyset$ and $S$ are open,
(2) If $X$ and $Y$ are open, so is $X \cap Y$,
(3) If $X_i$ is open for each $i \in I$, then so is $\bigcup_{i \in I} X_i$.

The proofs of the above lemmas are straightforward and have been omitted.

It is usual to present a topological space in the form $(T, \mathcal{O})$, where $\mathcal{O}$ is the collection of all open sets of the space; the axioms to be satisfied are precisely (1)–(3) of Lemma 4. Here, however, we must be extremely careful. In topology, the closure of a set $X$ is defined to be the intersection all closed sets $Y$ such that $X \subseteq Y$. This is not, in general, the same as the closure in terms of which the closure-space structure is defined. I shall use the symbol $\text{cl}_T$ to denote the topological closure; it is defined as follows:

**Definition.** In a closure space $(S, \text{cl})$, the *topological closure* $\text{cl}_T : 2^S \rightarrow 2^S$ is defined by

$$\text{cl}_T(X) = \bigcap \{ Y \subseteq S \mid X \subseteq Y \land \text{cl}(Y) = Y \}.$$  

It follows immediately from this definition that $X \subseteq \text{cl}_T(X)$.

**Lemma 5.** Fundamental properties of topological closure:

1. $\text{cl}(\text{cl}_T(X)) = \text{cl}_T(X)$ (i.e., $\text{cl}_T(X)$ is closed).
2. If $X \subseteq Y$ then $\text{cl}_T(X) \subseteq \text{cl}_T(Y)$.
3. $\text{cl}(X) \subseteq \text{cl}_T(X)$
4. $X = \text{cl}(X)$ iff $X = \text{cl}_T(X)$

**Proof.** (1) Using Lemma 3 we have

$$\text{cl}(\text{cl}_T(X)) = \text{cl}(\bigcap \{ Y \mid X \subseteq Y \land \text{cl}(Y) = Y \})$$

$$\subseteq \bigcap \{ \text{cl}(Y) \mid X \subseteq Y \land \text{cl}(Y) = Y \}$$
\[
= \bigcap \{ Y \mid X \subseteq Y \land \text{cl}(Y) = Y \}
= \text{cl}_T(X).
\]

Since also \( \text{cl}_T(X) \subseteq \text{cl}(\text{cl}_T(X)) \), it follows that \( \text{cl}(\text{cl}_T(X)) = \text{cl}_T(X) \).

(2) If \( X \subseteq Y \) then \( X \subseteq \text{cl}_T(Y) \), so \( \text{cl}_T(Y) \) is a closed set containing \( X \), and therefore also contains \( \text{cl}_T(X) \).

(3) Since \( X \subseteq \text{cl}_T(X) \), applying Lemma 2 gives \( \text{cl}(X) \subseteq \text{cl}(\text{cl}_T(X)) = \text{cl}_T(X) \) by (1).

(4) Suppose \( X = \text{cl}(X) \). Then \( X \) is a closed set containing \( X \), and therefore \( \text{cl}_T(X) \) \( \subseteq X \). Since in any case \( X \subseteq \text{cl}_T(X) \), it follows that \( X = \text{cl}_T(X) \). For the converse, it suffices to note (using (3)) that \( X \subseteq \text{cl}(X) \subseteq \text{cl}_T(X) \), so if \( X = \text{cl}_T(X) \) we must have \( X = \text{cl}(X) \). \( \square \)

The next lemma shows that topological closure is a closure operation as defined above, and has the additional property of \textit{idempotence}, i.e., \( \text{cl}_T(\text{cl}_T(X)) = \text{cl}_T(X) \) for all \( X \).

**Lemma 6.** In any closure space \((S, \text{cl})\), the topological closure operation satisfies:

(1) \( \text{cl}_T(\emptyset) = \emptyset \),

(2) \( X \subseteq \text{cl}_T(X) \),

(3) \( \text{cl}_T(X \cup Y) = \text{cl}_T(X) \cup \text{cl}_T(Y) \),

(4) \( \text{cl}_T(\text{cl}_T(X)) = \text{cl}_T(X) \).

**Proof.** (1) Since \( \text{cl}(\emptyset) = \emptyset \), it follows that \( \text{cl}_T(\emptyset) \subseteq \emptyset \) and hence \( \text{cl}_T(\emptyset) = \emptyset \).

(2) Immediate from the definition of \( \text{cl}_T \).

(3) First note that by Lemma 5 (2), \( \text{cl}_T(X) \cup \text{cl}_T(Y) \subseteq \text{cl}_T(X \cup Y) \), so we need only prove the reverse inclusion. By Lemma 5(1), \( \text{cl}_T(X) \) and \( \text{cl}_T(Y) \) are closed, so their union is closed too. Thus \( \text{cl}_T(X) \cup \text{cl}_T(Y) \) is a closed set containing \( X \cup Y \), and hence contains \( \text{cl}_T(X \cup Y) \).

(4) From (2), \( \text{cl}_T(X) \subseteq \text{cl}_T(\text{cl}_T(X)) \). For the reverse inclusion, note that \( \text{cl}_T(X) \) is closed (by Lemma 5(1)) and contains \( \text{cl}_T(X) \), hence also contains \( \text{cl}_T(\text{cl}_T(X)) \). \( \square \)

In an arbitrary closure space, the closure operation \( \text{cl} \) does not necessarily agree with the topological closure \( \text{cl}_T \). The condition that it should do so is that \( \text{cl} \) is idempotent. This is a consequence of the following lemma.

**Lemma 7.** Let \((X, \text{cl})\) be a closure space. Then for every \( X \subseteq S \), \( \text{cl}(X) = \text{cl}_T(X) \) iff \( \text{cl}(\text{cl}(X)) = \text{cl}(X) \).

**Proof.** First, suppose \( \text{cl}(X) = \text{cl}_T(X) \). Then \( \text{cl}(\text{cl}(X)) = \text{cl}(\text{cl}_T(X)) = \text{cl}_T(X) = \text{cl}(X) \) (using Lemma 5(1) for the second step).

Conversely, suppose \( \text{cl}(\text{cl}(X)) = \text{cl}(X) \). Then \( \text{cl}(X) \) is a closed set containing \( X \), and hence contains \( \text{cl}_T(X) \). By Lemma 5(3), \( \text{cl}(X) = \text{cl}_T(X) \). \( \square \)
Starting with an arbitrary closure space \((S, c)\), we can use \(c\) to define the collection \(O\) of open sets, giving us a topological space \((S, O)\). In this space, the topological closure operation \(c_T\) can be defined in the usual way; it is uniquely determined by the original closure operation \(c\), but the latter cannot in general be recovered from the former. This is only the case if the two closure operations coincide, which happens if and only if \(c\) is idempotent. In this case we can say that the closure space \(S\) is a topological space. This is the sense in which the notion of closure space generalizes the notion of topological space.

**Definition.** A closure space \((S, c)\) is topological iff \(c(cl(X)) = cl(X)\) for all \(X \subseteq S\).

In particular, for any closure space \((S, c)\), \((S, c_T)\) is a topological closure space.

The next lemma establishes an important relationship between neighbourhoods and closure.

**Lemma 8.** In any closure space, \(y \in cl(\{x\})\) iff \(x\) is in every neighbourhood of \(y\).

**Proof.** First let \(y \in cl(\{x\})\), and let \(N\) be any neighbourhood of \(y\). Then \(y \in int(N) = cl(N^c)^c\), so \(y \notin cl(N^c)\). Hence \(cl(\{x\}) \notin cl(N^c)\), so by Lemma 2, \(\{x\} \notin N^c\). Hence \(x \in N\). Hence \(x\) is in every neighbourhood of \(y\).

Conversely, suppose \(y \notin cl(\{x\})\). Then \(y \in cl(\{x\})^c = int(\{x\}^c)\), so \(\{x\}^c\) is a neighbourhood of \(y\) not containing \(x\). Therefore, \(x\) is not in every neighbourhood of \(y\). \(\Box\)

**Lemma 9.** Let \((S, c)\) be a closure space. Then the following conditions are equivalent:

1. Each point in \(S\) has a minimal neighbourhood, i.e., for each \(x \in S\) there is a set \(N_x \subseteq S\) such that
   a. \(N_x\) is a neighbourhood of \(x\),
   b. Every neighbourhood of \(x\) contains \(N_x\) as a subset.
2. For each \(X \subseteq S\), \(cl(X) = \bigcup_{x \in X} cl(\{x\})\).

**Proof.** First, assume condition (1) holds (in the terminology of [17], this means that \((S, c)\) is an Alexandroff space). By Lemma 2, for each \(x \in X\) we have \(cl(\{x\}) \subseteq cl(X)\), and therefore \(\bigcup_{x \in X} cl(\{x\}) \subseteq cl(X)\). For the reverse inclusion, suppose \(y \in cl(X)\). We must show that \(y \in cl(\{x\})\) for some \(x \in X\). Suppose not; then for every \(x \in X\) we have \(y \notin cl(\{x\})\). It is immediate from Lemma 8 and the definition of \(N_x\) that \(x \in N_y\) if and only if \(y \in cl(\{x\})\). Hence \(x \notin N_y\). It follows that \(N_y \subseteq X^c\), so by Corollary 1 we have \(int(N_y) \subseteq int(X^c) = cl(X)^c\). Since \(N_y\) is a neighbourhood of \(y\), \(y \in int(N_y)\), so \(y \notin cl(X)\), which contradicts our initial assumption. Therefore, \(y \in cl(\{x\})\) for some \(x \in X\), so condition (2) holds.

Conversely, suppose we have condition (2), and let \(x \in S\). Let

\[
N_x = \bigcap \{N \mid x \in int(N)\}
\]
(so \(N_\varepsilon\) is the intersection of all the neighbourhoods of \(x\)—Čech [3] calls this the ‘star’ of \(\{x\}\)). We wish to show that \(x \in \text{int}(N_\varepsilon)\). Suppose not; then \(x \in \text{cl}(N_\varepsilon) = \bigcup_{y \in N_\varepsilon} \text{cl}(\{y\})\) (by (2)). Hence there is an element \(y\) such that \(y \in N_\varepsilon\) and \(x \in \text{cl}(\{y\})\). From the definition of \(N_\varepsilon\), the former condition means that

\[ y \in \bigcup\{N_\varepsilon \mid x \in \text{int}(N_\varepsilon)\}. \]

Therefore, there is a neighbourhood \(N\) of \(x\) such that \(y \in N\) and \(x \in \text{cl}(\{y\})\). From the definition of \(N_\varepsilon\), the latter condition means that \(y \in \bigcup\{N_\varepsilon \mid x \in \text{int}(N_\varepsilon)\}\). Hence \(x \in \text{cl}(N_\varepsilon)\), which contradicts the assumption that \(N\) is a neighbourhood of \(x\). It follows that \(x \in \text{int}(N_\varepsilon)\), so \(N_\varepsilon\) is the minimal neighbourhood of \(x\) establishing condition (1).

**Definition.** A closure space \((S,\text{cl})\) is quasi-discrete [3] if it satisfies either (and therefore both) of the equivalent conditions in Lemma 9.

3. **Closure space derived from relations**

Let \(S\) be any set, and let \(R \subseteq S \times S\) be any binary relation on \(S\). The relation \(R\) gives rise to a closure operation \(\text{cl}_R\) on \(S\) as follows:

\[ \text{cl}_R(X) = X \cup \{y \in S \mid \exists x \in X: yRx\}. \]

That is, the \(R\)-closure of \(X\) contains \(X\) together with all elements of \(S\) that are \(R\)-related to \(X\).

**Lemma 10.** For any relation \(R \subseteq S \times S\), \((S,\text{cl}_R)\) is a closure space.

**Proof.** Straightforward. \(\square\)

We write \(R^=\) to denote the reflexive closure of \(R\), i.e., the relation defined by \(xR^= y \iff xRy \lor x = y\).

**Lemma 11.** The closure space \((S,\text{cl}_R)\) is topological iff \(R^=\) is transitive.

**Proof.** First, suppose \(R^=\) is transitive, and let \(x \in \text{cl}_R(\text{cl}_R(X))\). Then \(xR^= y\) for some \(y \in \text{cl}_R(X)\). Further, \(yR^= z\) for some \(z \in X\). By transitivity, \(xR^= z\), so \(x \in \text{cl}_R(X)\). Hence \(\text{cl}_R(\text{cl}_R(X)) \subseteq \text{cl}_K(X)\), so these two sets are equal, and \((S,\text{cl}_R)\) is topological.

Conversely, suppose \((S,\text{cl}_R)\) is topological. Let \(xR^= y\) and \(yR^= z\). Then \(x \in \text{cl}_R(\{y\})\) and \(y \in \text{cl}_R(\{z\})\), so we have \(x \in \text{cl}_R(\{y\}) \subseteq \text{cl}_R(\text{cl}_R(\{y\})) = \text{cl}_R(\{z\})\), using the fact that \((S,\text{cl}_R)\) is topological. Hence \(xR^= z\), so \(R^=\) is transitive. \(\square\)

Note that if \(R\) is transitive, so is \(R^=\), giving the corollary

**Corollary 2.** If \(R \subseteq S \times S\) is transitive, then \((S,\text{cl}_R)\) is topological.
The interior operation corresponding to $cl_R$ is given by

$$
\text{int}_R(X) = cl_R(X^c)^c = (X^c \cup \{x \in S | \exists y \in X^c: xRy\})^c
$$

$$
= X \cap \{x \in S | \exists y \in X^c: xRy\}^c
$$

$$
= X \cap \{x \in S | \forall y \in X^c: \neg xRy\}
$$

$$
= \{x \in X | \forall y (xRy \rightarrow y \in X)\}.
$$

Thus the $R$-interior of $X$ consists of those elements of $X$ which are not $R$-related to any elements outside $X$.

**Theorem 1.** The closure space $(S, cl)$ is quasi-discrete iff there is a relation $R \subseteq S \times S$ such that $cl = cl_R$.

**Proof.** For the ‘if’ direction we must show that $(S, cl_R)$ is quasi-discrete for any relation $R \subseteq S \times S$. For $x \in S$ define $N_x = \{y \in S | xR^c y\}$. First, it is immediate from the definition of $N_x$ that $x \in N_x$ and $\forall y(xRy \rightarrow y \in N_x)$. Hence $N_x$ is a neighbourhood of $x$. Now let $X$ be any neighbourhood of $x$. This means that $x \in \text{int}_R(X)$, so $x$ can only be $R$-related to elements of $X$. Let $y \in N_x$. If $y = x$ then $y \in X$; and if not, then $xRy$, so again $y \in X$. It follows that $N_x \subseteq X$. Hence $N_x$ is the minimal neighbourhood of $x$, and since $x$ was an arbitrary element of $S$, it follows that $(S, cl_R)$ is quasi-discrete.

For the ‘only if’ part, assume that $(S, cl)$ is quasi-discrete, and define a relation $R \subseteq S \times S$ by the rule

$$
xRy \iff x \in cl(\{y\}).
$$

(By quasi-discreteness, this is equivalent to $y \in N_x$.) Then

$$
y \in cl_R(X) \iff \exists x \in X (y = x \lor yRx)
$$

$$
\iff \exists x \in X (y = x \lor y \in cl(\{x\}))
$$

$$
\iff \exists x \in X (y \in cl(\{x\})) \quad \text{(since $x \in cl(\{x\})$)}
$$

$$
\iff y \in \bigcup_{x \in X} cl(\{x\})
$$

Hence $cl = cl_R$ as desired. \qed

From the proof, it is clear that $R$ can always be taken to be a reflexive relation, and the correspondence between quasi-discrete spaces and reflexive relations was already pointed out by Čech [3]. Equally, it can always be taken to be irreflexive, there being a one-to-one correspondence between irreflexive relations and their reflexive closures;
and indeed, the specific varieties of quasi-discrete space to be introduced in Section 4 will be constructed using irreflexive relations, corresponding better as they do to the spatial relations being modelled.

Theorem 2. \( \text{cl}_R(\{x\}) \) is the minimal neighbourhood of \( x \) if \( R \) is symmetric.

Proof. Suppose \( R \) is symmetric. Then \( N_x = \{x\} \cup \{y \in S \mid yRx\} = \text{cl}_R(\{x\}) \). Conversely, suppose \( N_x \) is \( \text{cl}_R(\{x\}) \), and let \( yRx \). Then \( y \in \text{cl}_R(\{x\}) \), so \( y \in N_x \). Hence either \( y = x \) or \( xRy \). Given that \( yRx \), we must have \( xRy \) in either case. Hence \( R \) is symmetric. \( \Box \)

Lemma 12. If \((S,\text{cl}_R)\) is topological, then the open sets are precisely the unions \( \bigcup_{x \in X} N_x \) for \( X \subseteq S \).

Proof. First, let \( O \) be any open set in \((S,\text{cl}_R)\), so \( O = \text{int}(O) \). Hence \( O \) is a neighbourhood of each of its members, so for each \( x \in O \), \( N_x \subseteq O \). Hence \( \bigcup_{x \in O} N_x \subseteq O \). But since \( x \in N_x \) for each \( x \), we have also \( O \subseteq \bigcup_{x \in O} N_x \). These sets are therefore equal, so \( O \) is a union of minimal neighbourhoods as required.

Conversely, consider any subset \( X \subseteq S \). We must show that \( \bigcup_{x \in X} N_x \) is open. By Lemma 4(3) it suffices to show that \( N_x \) is open. Let \( y \in N_x \), so \( xRy \). Suppose \( yRz \). By transitivity (Lemma 11), \( xRz \), so \( z \in N_x \). Hence \( y \) only bears relation \( R \) to elements of \( N_x \), which means that \( y \in \text{int}(N_x) \). Hence \( N_x \subseteq \text{int}(N_x) \), so \( N_x \) is open. \( \Box \)

To prepare for the next two theorems, recall that a topological space is \( T_0 \) if, for any two distinct points \( x \) and \( y \) of the space, either \( x \) is contained in an open set which does not contain \( y \), or \( y \) is contained in an open set which does not contain \( x \). Put differently, each point of a \( T_0 \) space is uniquely characterized by the collection of open sets containing it. A topological space is \( T_1 \) if, for any two distinct points of the space, each is contained in an open set not containing the other. This is obviously a stronger condition than \( T_0 \).

Theorem 3. If \( R \) is transitive and asymmetric, then \((S,\text{cl}_R)\) is a \( T_0 \) topological space.

Proof. By Corollary 2, \((S,\text{cl}_R)\) is topological. Now let \( x, y \in S \), with \( x \neq y \). Since \( R \) is asymmetric, we cannot have both \( xRy \) and \( yRx \). Suppose \( \neg xRy \). Since \( x \neq y \), we have \( \neg xR^=y \), and hence \( y \notin N_x \). Hence \( N_x \) is an open set containing \( x \) but not \( y \). If on the other hand \( \neg yRx \), then \( N_y \) is an open set containing \( y \) but not \( x \). One of these cases must hold, so the space is \( T_0 \). \( \Box \)

Theorem 4. If \( R \setminus I \) is non-empty (where \( I \) is the identity relation on \( S \)) then \((S,\text{cl}_R)\) is not \( T_1 \).

Proof. Let \( xRy \), where \( x \neq y \). Then \( y \notin N_x \). But \( N_x \) is a subset of every open set containing \( x \). Therefore, there does not exist an open set containing \( x \) but not \( y \), and hence the space cannot be \( T_1 \).
4. Connection and distance in quasi-discrete closure spaces

Theorem 1 tells us that in a quasi-discrete closure space \((S, cl)\) there must be a relation \(R\) such that \(cl = cl_R\). We can use this relation to provide natural characterizations of the notions of connection and distance in such spaces.

We begin by defining connection on general closure spaces in a way that precisely generalizes the standard definition of connection in topological spaces:

**Definition.** A closure space \((S, cl_R)\) is connected if and only if \(S\) is not the union of two disjoint non-empty open subsets.

Note that this appears as a theorem in [3], following as an immediate consequence of a closely related definition. Smyth [17] gives the same definition as ours, but in the more general setting of ‘neighbourhood spaces’. We shall show that in the more restricted setting of quasi-discrete spaces connection can be characterized directly in terms of the relation \(R\).

**Definition.** Let \((S, cl_R)\) be a quasi-discrete closure space, and let \(X \subseteq S\). Then the points \(x, y \in S\) are linked if there exists a sequence \(x = x_0, x_1, \ldots, x_n = y\) (where \(n \geq 0\)) such that for \(i = 1, \ldots, n\), either \(x_{i-1}Rx_i\) or \(x_iRx_{i+1}\). In this case we write \(x \sim_R y\). The sequence \((x_i) = x_0, x_1, \ldots, x_n\) is called a path linking \(x\) to \(y\).

It is obvious that \(\sim_R\) is an equivalence relation on \(S\); in fact it is the reflexive, symmetric and transitive closure of \(R\). The connected components of \((S, cl_R)\) will be precisely the equivalence classes under \(\sim_R\). This comes from the characterization of connectedness which we now give.

**Lemma 13.** A quasi-discrete closure space \((S, cl_R)\) is connected if, for all pairs \(x, y \in S\), \(x \sim_R y\).

**Proof.** Suppose first that all pairs are linked, and suppose \(S = Y \cup Z\) where \(Y \cap Z = \emptyset\). Let \(y \in Y\) and \(z \in Z\). By assumption, \(y \sim_R z\), so there is a path \((y_i)\) linking \(y = y_0\) to \(z = y_n\). Since \(y_0 \in Y\) and \(y_n \notin Y\), there must be a greatest value of \(i\) for which \(y_i \in Y\). Then \(y_{i+1} \in Y^c = Z\). By the definition of a path, either \(y_iRy_{i+1}\) or \(y_{i+1}Ry_i\). In the former case, we have \(y_i \notin \text{int}(Y)\), so \(y \notin Y\setminus \text{int}(Y)\), so \(Y\) is not open. Likewise, in the latter case, \(Z\) is not open. Thus in any case \(Y\) and \(Z\) cannot both be open. Thus \(S\) is not the union of two disjoint open sets, i.e., \(S\) is connected.

Conversely, suppose that there exist points \(y, z \in S\) which are not linked by any path. Let \(Y\) be the set of points linked to \(y\), i.e.,

\[ Y = \{ x \in S \mid y \sim_R x \}. \]

Then \(Y\) is open; for suppose \(u \in Y\), and let \(uRv\). Since \(y \sim_R u\), it follows that \(y \sim_R v\), since the path from \(y\) to \(u\) can be extended to \(v\) by the link \(uRv\). Hence \(v \in Y\). It
follows that \( u \in \text{int}(Y) \). Thus \( Y \subseteq \text{int}(Y) \), so \( Y \) is open. Moreover, \( Y^c \) is open also. For if not, there are points \( w \in Y^c, t \in Y \), such that \( tRw \). In that case, since \( y \not\sim t \), we have \( y \not\sim w \), so \( w \in Y \), a contradiction. Moreover, since \( z \in Y^c \), it follows that \( S \) is the union of disjoint non-empty open sets, and hence not connected. \( \square \)

We now define the distance between two points in a quasi-discrete closure space to be the length of the shortest path linking them, if such a path exists.

**Definition.** (1) The *length* of a path \((x_i)\) linking \( x = x_0 \) to \( y = x_n \) in \((S, cl_R)\) is the integer \( n \).

(2) The *distance* between \( x \) and \( y \) in \((S, cl_R)\), written \( d_R(x, y) \), is the least \( n \) such that \( x \) and \( y \) are linked by a path of length \( n \), if such a path exists, otherwise it is undefined.

The proof of the following lemma is straightforward:

**Lemma 14.** The distance function is a metric on each connected component of \( S \).

4.1. Adjacency spaces

**Definition.** An *adjacency space* is a pair \((S, A)\), where \( S \) is any set, and \( A \) is an irreflexive, symmetric relation on \( S \). We read \( xAy \) as ‘\( x \) is adjacent to \( y \)’.

An adjacency space \((S, A)\) can be viewed as a closure space \((S, cl_A)\) as described above. By Theorem 1, this space is quasi-discrete, and by Theorem 2, the minimal neighbourhood of any point \( x \) is the closure of \( \{x\} \): i.e., \( x \) itself together with all points adjacent to it.

We shall refer to the connected components of an adjacency space as the *adjacency components*.

**Lemma 15.** In an adjacency space, the following are equivalent:

1. \( X = \text{int}_A(X) \)
2. \( X = \text{cl}_A(X) \)
3. \( X \) is a union of adjacency components

**Proof.** (1 \( \rightarrow \) 2) Suppose \( X = \text{int}_A(X) \), and let \( y \in \text{cl}_A(X) \). Then there is an \( x \in X \) such that \( yAx \). Since \( X = \text{int}_A(X) \), this means that \( x \in \text{int}_A(X) \), so \( y \in X \). Hence \( \text{cl}_A(X) \subseteq X \), and since the reverse inclusion always holds, we have \( X = \text{cl}_A(X) \).

(2 \( \rightarrow \) 3) Suppose \( X = \text{cl}_A(X) \), and let \( x \in X \). Let \( C_x \) be the adjacency component containing \( x \), and let \( y \in C_x \). Then there is a chain of adjacencies \( x = x_0Ax_1A \cdots Ax_n = y \). We have \( x_0 = x \in X \); moreover, for \( i = 0 \) to \( n - 1 \), if \( x_i \in X \) then since \( x_iAx_{i+1}, x_{i+1} \in \text{cl}_A(X) \) \( = X \). Hence, by induction on \( i \), \( y = x_n \in X \). It follows that \( C_x \subseteq X \). Thus \( X \) includes the whole adjacency component of each of its members, and is therefore itself a union of adjacency components.
(3 → 1) Let $X = \bigcup_{i \in I} C_i$, where each $C_i$ is an adjacency component. Let $x \in X$, so for some $u \in I$, $x \in C_u$. Suppose $yAx$. Then $y \in C_u$, and hence $y \in X$. Thus $\forall y \in S(yAx \rightarrow y \in X)$, which means that $x \in \text{int}_{A}(X)$. Hence $X \subseteq \text{int}_{A}(X)$, and since the reverse inclusion follows immediately from the definition of $\text{int}_{A}$, we have $X = \text{int}_{A}(X)$. □

The upshot of this lemma is that the notions of ‘open set’ and ‘closed set’ are essentially redundant in adjacency spaces, since they coincide with the notion of ‘union of adjacency components’. By Lemma 11, an adjacency space is topological if and only if $A^=$, the reflexive closure of the adjacency relation, is transitive. In that case, being already symmetric, it is an equivalence relation. In each adjacency component, everything is adjacent to everything else, forming a clique. This is not the kind of adjacency space we are primarily interested in. A more typical example would be the two-dimensional rectangular lattice $\mathbb{Z} \times \mathbb{Z}$, with the point $(m,n)$ adjacent either to the four points $(m \pm 1,n),(m,n \pm 1)$ (‘four-adjacency’), or to these together with the further four points $(m \pm 1,n \pm 1)$ (‘eight-adjacency’). These spaces have been investigated in the context of Digital Topology [13], where it is found that for some purposes it is more useful to define a composite adjacency relation which uses eight-adjacency for points within some distinguished ‘figure’ and four-adjacency for points in the remaining ‘ground’ (or vice versa). See also [6].

4.2. Incidence spaces

**Definition.** An **incidence space** is a pair $(S,B)$, where $S$ is any set, and $B$ is an irreflexive, transitive relation on $S$. We read $xBy$ as ‘$x$ bounds $y$’.

Note that the irreflexivity and transitivity of $B$ immediately imply that bounding is an asymmetric relation; a non-empty relation cannot therefore be considered as both an adjacency relation and a bounding relation.

An incidence space $(S,B)$ can be viewed as a closure space $(S,\text{cl}_B)$ as above. The closure of a set consists of the set itself together with any element which bounds an element of the set. By Theorem 1 this space is quasi-discrete, and since $B$ is defined to be transitive, the space is topological by Corollary 2; moreover, by Theorems 3 and 4 it is $T_0$ but not $T_1$ (except in the trivial case where the bounding relation is empty).

**Definition.** Two points $x, y$ in an incidence space are **incident**, written $xIy$, if either of them bounds the other.

Thus incidence is the symmetric closure of bounding, $I = B \cup B^{-1}$. The linkage relation $\sim$ could equally well be written $\sim_{L}$. We shall refer to the connected components not as bounding components but as **incidence components**.

Various forms of incidence spaces have been studied in the literature. They include the ‘cellular complexes’ of [14], the ‘hyper-rasters’ of [19], the ‘star-topology’ of [1], the ‘topological mode spaces’ of [11], and the Khalimsky spaces [12].
5. Mappings between closure spaces

**Definition.** A mapping $f : (S_1, cl_1) \to (S_2, cl_2)$ between closure spaces is **continuous** so long as $f(cl_1(X)) \subseteq cl_2(f(X))$ for every $X \subseteq S_1$.

If the closure spaces are topological, then this gives the usual topological definition of continuity (normally expressed in the equivalent form that if $X$ is open in $S_2$ then $f^{-1}(X)$ is open in $S_1$).

**Lemma 16.** In a $T_0$ space, if $x \in cl\{y\}$ and $y \in cl\{x\}$ then $x = y$.

**Proof.** Suppose $x \neq y$. Then since the space is $T_0$, one of $x, y$ is in an open set not containing the other. Suppose without loss of generality it is $x$, and let $U$ be an open set such that $x \in U$ and $y \in U^c$. Since $U$ is open, $U^c$ is closed, so $cl\{y\} \subseteq U^c$. Since $x \in cl\{y\}$, this implies $x \in U^c$, and we have a contradiction. Hence $x = y$. □

**Theorem 5.** Let $(X,A)$ be an adjacency space, and let $(Y, cl)$ be a $T_0$ space. Let $f : X \to Y$ be continuous. Then $f$ is constant on each adjacency component of $X$.

**Proof.** Suppose $xAy$. Then since $f$ is continuous, $f(cl_A\{x\}) \subseteq cl\{f(x)\}$. Hence, if $y \in cl_A\{x\}$, then $f(y) \in cl(f(x))$. Similarly, since $A$ is symmetric, $f(x) \in cl(f(y))$. By Lemma 16, $f(y) = f(x)$. We can extend this to all points connected to $x$ by a chain of adjacencies, and hence to the whole adjacency component of $X$ in which $x$ lies. □

The following result is well known, so we omit the proof:

**Lemma 17.** In a $T_1$ space, every singleton set is closed.

**Theorem 6.** Let $(X,B)$ be an incidence space, and let $(T, cl)$ be a $T_1$ space. Let $f : X \to T$ be continuous. Then $f$ is constant on each incidence component of $X$.

**Proof.** Suppose $xBy$, so $x \in cl_B\{y\}$. By continuity of $f$, we have $f(x) \in cl\{f(y)\}$. By Lemma 17, $cl\{f(y)\} = \{f(y)\}$, so $f(x) = f(y)$. This can be extended to all points connected to $x$ by a chain of incidences, and hence to the entire incidence component of $X$ in which $x$ lies. □

**Lemma 18.** Let $(X, cl_B)$ and $(Y, cl_{R'})$ be quasi-discrete closure spaces, and let $f : X \to Y$ be continuous. Then for all $x, y \in X$, if $xRy$ then either $f(x) = f(y)$ or $f(x) \not\in cl_{R'}f(y)$.

**Proof.** If $xRy$ then $x \in cl_B\{y\}$. Since $f$ is continuous, $f(cl_B\{y\}) \subseteq cl_{R'}\{f(y)\}$. Hence $f(x) \in cl_{R'}\{f(y)\}$. But this just means that either $f(x)R'f(y)$ or $f(x) = f(y)$. □
Theorem 7. Let \((X,\text{cl}_R)\) and \((Y,\text{cl}_{R'})\) be quasi-discrete closure spaces, and let \(f : X \to Y\) be continuous. Then whenever \(d_R(x,y)\) is defined,

\[
0 \leq d_{R'}(f(x), f(y)) \leq d_R(x, y).
\]

Proof. If \(d_R(x,y)\) is defined, and has the value \(n\), then there is a path \(x = x_0, x_1, \ldots, x_n = y\). For \(i = 1, \ldots, n\) we have either \(x_i \text{R} x_{i+1}\) or \(x_{i+1} \text{R} x_i\). By Lemma 18, this means that we have one of \(f(x_i) = f(x_{i+1})\), \(f(x) \text{R}' f(x_{i+1})\), or \(f(x_{i+1}) \text{R}' f(x_i)\). Consider the sequence \(f(x) = f(x_0), f(x_1), \ldots, f(x_n) = f(y)\). Whenever consecutive terms of the sequence are equal (as they may be from above), delete one of the duplicates. If this is repeated until no two consecutive terms are equal, then any two consecutive terms of the resulting sequence must be linked by either \(R'\) or \((R')^{-1}\), so the sequence is a path linking \(f(x)\) to \(f(y)\) in \((Y,\text{cl}_{R'})\). Moreover, its length is at most \(n\). The result is immediate. \(\square\)

The import of this theorem is that the gradient of a continuous function between quasi-discrete spaces is bounded above. Because of the discrete character of the spaces, it is not possible to define an ‘instantaneous gradient’ in the manner of the differential calculus; instead, we stop short at the precursor notion of ‘average gradient’ over an interval.

Definition. Let \((X,\text{cl}_R)\) and \((Y,\text{cl}_{R'})\) be quasi-discrete closure spaces, and let \(f : X \to Y\) be continuous. Let \(x\) and \(y\) be distinct elements of \(X\) such that \(x \text{R} y\). Then the gradient of \(f\) between \(x\) and \(y\) is the quantity

\[
D_f(x, y) = \frac{d_{R'}(f(x), f(y))}{d_R(x, y)}.
\]

Theorem 7 simply says that for a continuous function between quasi-discrete spaces the gradient between two points cannot exceed 1. This fact has often been observed in relation to the speed of continuous motion in discrete space and time; the general result for incidence spaces (specifically, cellular complexes) is stated in [15].

6. Modelling time, space, and motion

We now descend to a less abstract level and consider how the above ideas can be applied to the issue of modelling motion in various kinds of spatio-temporal framework. Motion occurs whenever an object occupies distinct positions in space at different moments in time. A formal description of motion therefore requires us to set up a framework consisting of representations of space, time, and the positions of objects. We shall be particularly interested in what happens when we choose various types of closure space to be the formal representation of space and time.

In this context, the most important features of time are that it is connected, one-dimensional, and non-circular. For simplicity we shall also assume that it is unbounded.
For the continuous time of classical physics these properties are secured by representing the time line as the real-number continuum \( \mathbb{R} \). When this is topologized in the usual way, we have a closure space which is topological but not quasi-discrete.

If we want to represent the time line by means of a quasi-discrete space, we must first determine how the conditions of connectedness, one-dimensionality, and unboundedness are to be represented. The simplest approach is to demand that the space is ordered like the integers. We shall adopt the following definition

**Definition.** A quasi-discrete closure space \((T, cl_T)\) is timelike if there is a bijection \( f : \mathbb{Z} \to T \) such that for all \( m, n \in \mathbb{Z} \), \( f(m)(R \cup R^{-1})f(n) \) iff \(|m - n| = 1\).

A consequence of the definition is that in a timelike space, the closure of any singleton set has at most three elements:

\[
cl_T(\{f(n)\}) \subseteq \{f(n - 1), f(n), f(n + 1)\}.
\]

This is related to the notion of a COTS (connected ordered topological space) introduced by [12]; a COTS is a connected topological space with the property that out of any three points, one can choose one of them such that the other two fall in separate connected components of its complement.

A timelike adjacency space (‘adjacency time’) can be modelled as \( \mathbb{Z} \) with \( A = \{(n, n+1), (n+1, n) \mid n \in \mathbb{Z}\} \). This is an example of the natural generalization of COTS to closure spaces.

A timelike incidence space (‘incidence time’) can be modelled as \( \mathbb{Z} \) with either \( B = \{(2n, 2n+1), (2n, 2n-1) \mid n \in \mathbb{Z}\} \) or \( B = \{(2n+1, 2n), (2n-1, 2n) \mid n \in \mathbb{Z}\} \). This is because if we have \((n-1)Bn\) and \(nB(n+1)\), then by transitivity we have \((n-1)B(n+1)\), so the space is not timelike. This means that each point either bounds both its immediate neighbours, or is bounded by them. Let us call the points which bound both their neighbours, ‘instants’, and those which are bounded by their neighbours, ‘atomic intervals’. We then have the picture of a timelike incidence space as an alternating sequence of instants and atomic intervals, with the instants bounding the atomic intervals on either side of them. This is a COTS, exactly as defined by [12], the one-dimensional ‘Khalimsky space’.

Rather than using even integers for instants and odd integers for intervals, or vice versa, as indicated above, it is more intuitive to represent incidence time using integers and consecutive integer pairs: each \( n \in \mathbb{Z} \) is an instant, and each pair \((n, n+1)\) is an interval, with \( n \) bounding both \((n-1, n)\) and \((n, n+1)\) but no other intervals.

Of course, exactly the same models can be used for a one-dimensional space. This enables us to construct some particularly simple, but instructive, examples representing the motion of a particle along a line. Here we have two one-dimensional structures, \((T, cl_T)\) representing time and \((S, cl_S)\) representing space. The position of the particle is given by a function \( \text{pos} : T \to S \). We might model each of \( T \) and \( S \) in three different ways: as a timelike adjacency space (type A), as a timelike incidence space (type B—for ‘bounding’), and as a continuous line like \( \mathbb{R} \) (type C). In principle, this gives us nine different cases to consider, which we can label \( AA, AB, \ldots, CC \) (here the first
letter describes the time structure, the second the space structure). We already know that some of these are trivial:

1. By Theorem 5, continuous motion in the cases $\text{AB}$ and $\text{AC}$ means remaining in the same position. This is because both cases $\text{B}$ and $\text{C}$ are $T_0$ topological spaces, so any function $\text{pos}: T \rightarrow S$ must be constant.

2. By Theorem 6, continuous motion in the case $\text{BC}$ means remaining in the same position. This is because case $\text{C}$ is a $T_1$ topological space.

Case $\text{CC}$ is far from trivial: it represents the paradigm of classical physics, in which space and time are both represented by the real numbers. But there is little of further interest to say about this case in the current context.

Cases $\text{BA}$ and $\text{CA}$ are strange for the following reason. It is natural to interpret the points of adjacency space as representing some fixed minimal positive extent. In moving from one point to an adjacent point one is moving a certain distance. If we execute a sequence of moves, say from $n$ to $n+1$ to $n+2$, then we expect to spend some positive length of time at $n+1$. Now compare this with the natural interpretation of incidence time: here we have instants and atomic intervals, and it is natural to interpret the former as durationless, the latter as having some fixed minimal positive duration. The durationless instants occupy the interstices between pairs of atomic intervals. In both the $\text{BA}$ and $\text{CA}$ cases it is possible to have a continuous motion in which, during the passage from position $n$ to $n+2$ via $n+1$, the middle position is occupied only for an instant. As an example, consider the function $\text{pos}: T \rightarrow S$ defined as follows:

\[
\text{pos}(n) = 2n, \\
\text{pos}((n, n+1)) = 2n + 1.
\]

Here, the moving particle is occupying the even-numbered positions in adjacency space at the instants of incidence time, and the odd-numbered positions on the atomic intervals. This function is continuous, since we have

\[
\text{pos}(\text{cl}_R(\{n\})) = \text{pos}(\{n\}) \\
\subseteq \text{cl}_R(\text{pos}(\{n\})),
\]

\[
\text{pos}(\text{cl}_R(\{(n,n+1)\})) = \text{pos}(\{n,(n,n+1),n+1\}) \\
= \{2n, 2n+1, 2n+2\} \\
= \text{cl}_R(\{2n+1\}) \\
= \text{cl}_R(\{\text{pos}((n,n+1))\})
\]

continuity now following from the distributivity of closure over union in quasi-discrete spaces. The motion is illustrated in Fig. 1. Precisely, the same illustration will work for the case $\text{CA}$, the only difference being that we now interpret the definition clause $\text{pos}((n,n+1)) = 2n+1$ to mean that $\text{pos}(x) = 2n+1$ for all $x$ in the range $n < x < n+1$. Here, again the even-numbered positions in the adjacency space are only occupied for fleeting instants.
The problematic character noted for BA and CA does not arise in the remaining cases AA, BB, and CB. We deal with these cases in turn.

In AA we have \( \text{pos}(n+1) = \text{pos}(n) \), which means that in one time step the continuously moving object can change its position by at most one space step. This is exemplified by the motion of the king in chess. Two examples of such motions are illustrated in Fig. 2. These examples show that the notion of gradient in quasi-discrete spaces, as defined above, is a good deal 'messier' than the parallel notion for continuous spaces. In the left hand motion, \( \text{pos}(t) = \lfloor \frac{1}{2} t \rfloor \), where we should intuitively like to say that the 'overall' gradient is essentially a constant \( \frac{1}{2} \), the full picture is in reality rather more complicated:

\[
D_f(x, y) = \begin{cases} 
\frac{1}{2} & \text{(if } x \text{ and } y \text{ are both even or both odd)}, \\
\frac{1}{2} - \frac{1}{2(y-x)} & \text{(if } x \text{ is even and } y \text{ is odd)}, \\
\frac{1}{2} + \frac{1}{2(y-x)} & \text{(if } x \text{ is odd and } y \text{ is even)}. 
\end{cases}
\]

Thus, depending on precisely which points are considered, the gradient may vary between 0 (e.g., between 2 and 3) and 1 (between 3 and 4), with many intermediate values which in a clear sense average to \( \frac{1}{2} \). The gradient here can of course be called 'speed', since the spaces involved are interpreted as time and space.

For the case BB, we have \( \text{pos}(t)B^+ \text{pos}((t-1), t) \) and \( \text{pos}(t)B^+ \text{pos}((t, t+1)) \). Two examples, analogous to the ones we showed for AA, are shown in Fig. 3. The left-hand figure shows the function pos defined by

\[
\text{pos}(2n-1, 2n) = (n-1, n), \\
\text{pos}(2n) = n,
\]
Fig. 2. Two examples of continuous motion in adjacency space and adjacency time. Left: $pos(t) = \lfloor \frac{1}{2} t \rfloor$. Right: $pos(t) = t$.

Fig. 3. Two examples of continuous motion in incidence space and incidence time.

We can regard a timelike incidence space as a quotient of the real line under the equivalence relation $\sim$ defined by

$x \sim y \iff \exists n \in \mathbb{Z}[\{x = y = n\} \lor (n < x, y < n + 1)]$.

If both $S$ and $T$ are regarded in this way, then the function $pos$ defined above is obtained as the image of the real function $s = \frac{1}{2} t$ under the quotient mapping.

The right-hand graph in Fig. 3 illustrates the identity function $pos(x) = x$ (where $x$ may be either an instant or an interval), which may be regarded as the image of the real function $s = t$.

Viewing the BB case in terms of a quotient mapping from CC is not necessarily the best way of handling this case. Two considerations arise here; they are illustrated in Fig. 4. In the left-hand illustration is shown another (as nearly as possible uniform)
motion in which the gradient averages out to $\frac{1}{2}$; but this motion is not the image of any real function with constant gradient $\frac{1}{2}$. The right-hand illustration shows what we would obtain if we took the image under the quotient construction of the real function $s = 2t$. This is not a function in B space; rather it is a ‘multi-function’, in which more than one position is assigned to some of the times (cf., [15, Section 5.1]).

The latter problem does not arise in our remaining case, CB. Fig. 5 shows three motions, corresponding to the CC motions given by $s = \frac{1}{2}t$, $s = t$, and $s = 2t$. Note that since time is here represented by a space that is not quasi-discrete, Theorem 7 does not apply; there is no upper limit to the speed of motion.

Case CB is of particular importance in applications to qualitative reasoning in AI. An example is the little eight-element incidence space consisting of the ‘RCC’ relations of [16]. These are a basic set of qualitative spatial relations describing the possible configuration of two regions in space; they are DC (disconnected), EC (externally connected), PO (partially overlapping), TPP (tangential proper part) and its inverse TPPI, NTTP (non-tangential proper part) and its inverse NTTPPI, and EQ (equal). The bounding relations are as shown in Fig. 6. The possible changes in the qualitative spatial relation between two regions, under continuous motion or deformation of those
regions, are exactly determined by continuous motions of a notional point (representing the current spatial configuration) through the RCC incidence space. Since time is here being treated as continuous, we have a clear example of the CB case. For more details, see [10,11].

7. Motion of an extended body

The preceding discussion only considered the motion of a point object, i.e., an object whose position at any one time is given by a single space element. In practice, however, we are often concerned with the motion of extended bodies, whose positions are given as regions in space. The position function for such a body takes the form

\[ \text{pos} : T \rightarrow 2^S, \]

that is, it assigns to each time element a set of space elements which collectively represent the region on which the object is positioned at that time.

The first problem is to define what is meant by continuity for motions of this kind. An obvious ploy would be to define a closure structure on \( 2^S \), and then invoke the usual notion of continuity. The problem is to do this in such a way that the results look ‘reasonable’ from the point of view of the typical concrete interpretations of the mathematical structures involved. An attractive point of view we can take—which, however, is by no means forced on us—is to adopt what may be described as the ‘atomic theory of motion’ [8]. This states that the motion of an extended object is entirely dependent on the motion of its constituent parts. In the discrete case we can read ‘parts’ as ‘atomic parts’. To formalize this notion we use the notion of ‘pointwise continuity’ defined for general closure spaces as follows:

**Definition.** Let \( (X, cl_1) \) and \( (Y, cl_2) \) be any closure spaces. A function \( f : X \rightarrow 2^Y \) is **pointwise continuous** if for each \( x \in X \) and \( y \in f(x) \) there exists a continuous function \( f_{x,y} : X \rightarrow Y \) such that \( f_{x,y}(x) = y \) and for all \( x' \in X \), \( f_{x,y}(x') \in f(x') \).
Thus the sets which are values of $f$ consist of the points which are values of the continuous functions $f_{x,y}$. The continuity of the set-valued function is inherited from the continuity of the point-valued functions.

We shall be particularly interested in what pointwise continuous functions look like when the range and domain are both quasi-discrete. Part of the answer is supplied by the following lemma.

**Lemma 19.** Let $(X,cl_R X)$ and $(Y,cl_R Y)$ be quasi-discrete closure spaces, and let $f : X \to 2^Y$ be pointwise-continuous. Further, let $R$ be the irreflexive binary relation on $2^Y$ such that for all $P, Q \subseteq Y$,

$$PRQ \equiv \forall x \in P \exists y \in Q (xR^+_y y) \land \forall y \in Q \exists x \in P (xR^+_y y) \land P \neq Q.$$  

Then $f$ is continuous with respect to the closure structures $(X,R_X)$ and $(2^Y,R)$.

**Proof.** Suppose $x_1 R_X x_2$, where $x_1, x_2 \in X$, and let $y \in f(x_1)$. By pointwise continuity, there is a continuous function $f_{x_1,y} : X \to Y$ such that $f_{x_1,y}(x_1) = y$ and for all $x' \in X$, $f_{x_1,y}(x') \in f(x')$. By the continuity of $f_{x_1,y}$, since $x_1 R_X x_2$, we have $f_{x_1,y}(x_1) R^+_y f_{x_1,y}(x_2)$, i.e., $y R^+_y f_{x_1,y}(x_2)$. Since $f_{x_1,y}(x_2) \in f(x_2)$, we have

$$\forall y \in f(x_1) \exists z \in f(x_2) : y R^+_z z.$$

Similar reasoning applied to an arbitrary $y \in f(x_2)$ gives us

$$\forall z \in f(x_2) \exists y \in f(x_1) : y R^+_z z.$$

Putting these two results together gives $f(x_1) R^+_y f(x_2)$. Since this holds whenever $x_1 R_X x_2$, the function $f$ is continuous with respect to the stated closure structures. 

Does the converse hold? That is, must a function $f : X \to 2^Y$ that is continuous with respect to the closure structures $(X,R)$ and $(2^Y,R)$ also be pointwise-continuous? In the case where $(X,R)$ is timelike—which is, after all, the case that is of interest to us—we can prove that this is indeed so.

**Theorem 8.** Let $(T,cl_R T)$ be a timelike quasi-discrete closure space, and let $(S,cl_R S)$ be any other quasi-discrete closure space. Then a function $f : T \to 2^S$ is pointwise continuous if and only if it is continuous with respect to the closure structure on $2^S$ defined by

$$PRQ \equiv \forall x \in P \exists y \in Q (xR^+_y y) \land \forall y \in Q \exists x \in P (xR^+_y y) \land P \neq Q.$$  

**Proof.** For simplicity we shall label the elements of $T$ by integers, so that $n(R_T \cup R_T^{-1})m$ iff $|n - m| = 1$.

The ‘only if’ part is a specialization of Lemma 19.

For the ‘if’ part, let $f$ be continuous with respect to $(2^S,cl_R S)$. Let $n \in \mathbb{Z}$. Then either $n R_T (n + 1)$ or $(n + 1) R_T n$. In the former case, we know from continuity of $f$
that \( f(n) \mathcal{R}^= f(n + 1) \), and in the latter case, \( f(n + 1) \mathcal{R}^= f(n) \). If \( f(n) \mathcal{R}^= f(n + 1) \) then let \( S(n) = \{(y, z) \mid y \in f(n) \land z \in f(n + 1) \land y \mathcal{R}^= z \} \). From the definition of \( \mathcal{R} \), for each \( y \in f(n) \), \( S(n) \) contains at least one pair with \( y \) as its first element, and for each \( z \in f(n + 1) \), \( S(n) \) contains at least one pair with \( z \) as its second element. If \( f(n + 1) \mathcal{R}^= f(n) \), we put \( S(n) = \{(y, z) \mid y \in f(n) \land z \in f(n + 1) \land z \mathcal{R}^= y \} \).

Now let \( n \in T \) and \( y \in f(n) \). We define a continuous function \( f_{n,y} : T \to S \) as follows. First let \( f_{n,y}(n) = y \). Moving ‘forward in time’, once we have defined \( f_{n,y}(n + k) \) (where \( k \geq 0 \)), we define \( f_{n,y}(n + k + 1) \) to be any \( z \) such that \( (f_{n,y}(n + k), z) \in S(n + k) \). This can always be done; and inductively we know that this construction allows us to define \( f_{n,y}(i) \) for all integers \( i > n \), and moreover that \( f_{n,y}(i) \in f(i) \). Next, working ‘backward in time’, once we have defined \( f_{n,y}(n − k) \), we define \( f_{n,y}(n − k − 1) \) to be any \( z \) such that \( (z, f_{n,y}(n − k)) \in S(n − k − 1) \). Inductively this gives us \( f_{n,y}(i) \in f(i) \) for all \( i < n \). We now have, for each \( n \in T \) and \( y \in f(n) \) a function \( f_{n,y} : T \to S \) such that \( f_{n,y}(n) = y \) and, for all \( i \in T \), \( f_{n,y}(i) \in f(i) \). It remains to verify that all these functions are continuous.

Suppose \( m \mathcal{R}_T m' \). Then either \( m = m' − 1 \) or \( m = m' + 1 \). In the former case, we have \( m \mathcal{R}_T (m + 1) \). Now \( (f_{n,y}(m), f_{n,y}(m + 1)) \in S(m) \). Since \( m \mathcal{R}_T (m + 1) \) we have \( f(m) \mathcal{R}^= f(m + 1) \) (since \( f \) is continuous), and hence by the construction of \( S(m) \), \( f_{n,y}(m) \mathcal{R}^= f_{n,y}(m + 1) \). Similarly, if \( (m + 1) \mathcal{R}_T m \), we deduce that \( f_{n,y}(m + 1) \mathcal{R}^= f_{n,y}(m) \). It follows that \( f_{n,y} \) is continuous, as required, and hence that \( f \) is pointwise continuous. □

Taking pointwise continuity as the basis for continuous motion of extended bodies, let us now consider what happens for each of the three types of space \((A, B, \text{and } C)\).

**Adjacency space \((S, A)\):** From Theorem 8 it follows that continuity in the motion of an extended body is determined by the closure structure on \( 2^S \) defined by

\[
X \mathcal{A} Y \iff X \subseteq cl_A(Y) \land Y \subseteq cl_A(X) \land X \neq Y.
\]

Note that \( A \) is symmetric and irreflexive, so \((2^S, A)\) is an adjacency space. We can consider paths and distances in \((2^S, A)\) in the usual way. The distance \( d_A(X, Y) \) between two regions is then the least \( n \) such that for each point in \( X \) there is a point in \( Y \) at a distance no greater than \( n \) and for each point in \( Y \) there is a point in \( X \) at a distance no greater than \( n \). It is, in other words, the Hausdorff distance between \( X \) and \( Y \) as normally defined. Successive regions occupied by a continuously moving body are ‘nearest neighbours’ with respect to this measure. Each point in either of two successively occupied positions is adjacent or equal to a point in the other. See [9] for a more detailed treatment of this case.

**Incidence space \((S, B)\):** For incidence space, the closure structure on \( 2^S \) given by Theorem 8 is

\[
X \mathcal{B} Y \iff \forall x \in X \exists y \in Y (x B^= y) \land \forall y \in Y \exists x \in X (xB^= y) \land X \neq Y
\]

which may be expressed more compactly as

\[
X \mathcal{B} Y \iff X \subseteq cl_B(Y) \land Y \subseteq cl_B^{-1}(X) \land X \neq Y.
\]
Fig. 7. Positions of a closed disc in $\mathbb{R}^2$, with their digital images.

Note that the second conjunct uses the closure relation derived from the *inverse* of the bounding relation. It is easy to see that $B$ is irreflexive and transitive, and hence is itself a bounding relation, conferring an incidence-space structure on $2^S$.

An example of continuous motion of an extended object in incidence space, taken from [10], is shown in Fig. 7. The upper row of images shows a sequence of positions of a small closed disc in $\mathbb{R}^2$ (the dotted grid-lines represent points with one or more integer coordinates). The sequence is complete in the sense that as the disc moves continuously along a particular path between the positions in images 1 and 8, all the qualitatively distinct configurations relative to the grid-lines are shown. The continuous plane may be discretized as an incidence space (Khalimsky space) consisting of raster cells, their edges, and vertices, with vertices bounding ($B$) edges and cells, and edges bounding cells. This discretization induces a further discretization of the continuous space of possible positions of the disc, in which those positions having the same qualitative configuration relative to the grid-lines are identified. The bounding relation $B$ for this space is derived from the bounding relation $B$ in the manner described above. The sequence of digital images of the moving disc thereby obtained is shown in the lower row of the figure. Between the two rows of images is a sequence of labels showing a possible timing for the motion. The labels $(t_0, t_1)$, etc., may be interpreted as intervals in the continuous time of the upper sequence of images, or as bounded elements of the incidence space obtained by discretizing the time line in accordance with the qualitative ‘landmarks’ punctuating the motion. The arrows show the bounding relation, which may be construed equally as relation $B$ on $2^S$ or as the bounding relation on $T$, the discretized time line: by continuity, these two bounding relations are necessarily in conformity.

Continuous space: Theorem 8 no longer applies, so we must look elsewhere for an appropriate topological structure on $2^S$. The problem is how to characterize continuous change in subsets of $\mathbb{R}^n$. Since the primary focus of this paper is on discrete space, we shall not discuss this here, but instead refer the reader to [8,11,5].

8. Concluding remarks

In this paper, I have attempted to apply topological ideas to the problem of describing motion in various models of space and time. The descriptive problem is motivated by
the concerns of artificial intelligence researchers who seek appropriate conceptual tools for modelling real-world phenomena in a way that conforms to the normal workings of human intelligence and intuition; the technical tools I have brought to bear on the problem are derived from pure mathematics.

The particular focus of this paper has been on discrete space. A number of mathematical models for discrete space have been proposed in the literature, some of them topological, others in more of a graph-theoretical style. I used the notion of closure spaces to provide a uniform framework in which both styles of model can be accommodated, and in particular, I focussed on ‘quasi-discreteness’ as an appropriate formal analogue, within this framework, of the notion of discreteness. Two particular classes of quasi-discrete spaces were singled out, the adjacency spaces and the incidence spaces. Most of the discrete spaces studied in the literature belong to one or other of these categories.

With continuity for closure spaces defined in the obvious way, I then systematically investigated the forms of continuous motion that are possible when space and time are variously modelled as adjacency spaces, incidence spaces, continuous spaces, or some combination of these. Where possible, the phenomena described were illustrated with examples taken from existing work, thereby helping to clarify the relationships between such work and the framework expounded in this paper.

It should be stressed that this enterprise is not purely mathematical. In using mathematics to model phenomena of the empirical world, it is always necessary to determine how the elements of a mathematical theory are to be interpreted in relation to the world, and this determination cannot, by its nature, form a part of pure mathematics. By the same token, one cannot expect, in the application of mathematical theories to these phenomena, to meet with the same high standard of rigour as is customarily—and rightly—demanded in the mathematics itself.

References


