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Journal of Number Theory 104 (2004) 177–192

JOURNAL OF Number Theory

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Kuznietsov trace formula and weighted distribution of Hecke eigenvalues

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Received 20 January 2003; revised 17 April 2003

Communicated by D. Goss

Abstract

Let (X,μ) be a measurable topological space. Let $S_1, S_2, \ldots$ be a family of finite subsets of X. Suppose each $x \in S_i$ has a weight $w_{ix} \in \mathbf{R}^+$ assigned to it. We say $\{S_i\}$ is $\{w_i\}$-distributed with respect to the measure μ if for any continuous function f on X, we have $\lim_{i\to\infty} \frac{\sum_{x\in S_i} w_{ix} f(x)}{\sum_{x\in S_i} w_{ix}} = \int_X f(x)\, d\mu(x)$.

Let $S(N,k)$ be the space of modular cusp forms over $\Gamma_0(N)$ of weight k and let $\mathscr{E}(N,k) \subset S(N,k)$ be a basis which consists of Hecke eigenforms. Let $a_r(h)$ be the rth Fourier coefficient of h. Let x_p^h be the eigenvalue of h relative to the normalized Hecke operator T_p'. Let $\|\cdot\|$ be the Petersson norm on $S(N,k)$. In this paper we will show that for any even integer $k \geqslant 3$, $\{x_p^h : h \in \mathscr{E}(N,k)\}, p \nmid N$ is $\{\frac{|a_r(h)|^2 e^{-4\pi r}}{\|h\|^2}\}$-distributed with respect to a polynomial times the Sato–Tate measure when $N \to \infty$.

1. Introduction

Let $S(N,k)$ be the space of modular cusp forms on $\Gamma_0(N)$ of weight k and for $p \nmid N$, let T_p be the Hecke operator on $S(N,k)$ as defined in [Se2]. We shall consider $T_p' = p^{-(k-1)/2} T_p$. Denote the eigenvalue of a Hecke eigenform h relative to T_p' by x_p^h. The Ramanujan–Petersson conjecture (Deligne's Theorem) asserts that $|x_p^h| \leqslant 2$ for $(p,N) = 1$. Furthermore, it is conjectured that the set $\{x_p^h : (p,N) = 1\}$ is

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0022-314X/$ - see front matter
doi:10.1016/S0022-314X(03)00149-5

equidistributed with respect to the Sato–Tate measure (refer to [Se1, Chapter 1])

$$d\mu_\infty(x) = \begin{cases} \frac{1}{\pi}\sqrt{1-\frac{x^2}{4}}dx & \text{when } x \in [-2,2], \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathscr{E}(N,k)$ be a basis of $S(N,k)$ consisting of Hecke eigenforms. Serre [Se2] considered the distribution of $\{x_p^h : h \in \mathscr{E}(N,k)\}$ for fixed p. He used the Selberg trace formula for the Hecke operators and showed that when $N \to \infty$, the set $\{x_p^h\}$ is equidistributed as

$$d\mu_p(x) = \frac{p+1}{(p^{1/2}+p^{-1/2})^2 - x^2}\, d\mu_\infty(x). \tag{1}$$

In this paper, we established a weighted distribution for $\{x_p^h : h \in \mathscr{E}(N,k)\}$ for fixed p.

Let (X,μ) be a measurable topological space. Let $S_1, S_2, \ldots, S_i, \ldots$ be a family of finite subsets of X. Suppose each $x \in S_i$ has a weight $w_{ix} \in \mathbf{R}^+$ assigned to it. Let δ_x be the Dirac measure at x. Define

$$d\mu_i = \frac{\sum_{x\in S_i} w_{ix}\delta_x}{\sum_{x\in S_i} w_{ix}}.$$

We say $\{S_i\}$ is $\{w_i\}$-*distributed* with respect to measure $d\mu$ if

$$\lim_{i\to\infty} d\mu_i = \lim_{i\to\infty} \frac{\sum_{x\in S_i} w_{ix}\delta_x}{\sum_{x\in S_i} w_{ix}} = d\mu.$$

This means for any continuous function f on X, we have

$$\lim_{i\to\infty} \int_X f(x)\, d\mu_i(x) = \lim_{i\to\infty} \frac{\sum_{x\in S_i} w_{ix} f(x)}{\sum_{x\in S_i} w_{ix}} = \int_X f(x)\, d\mu(x).$$

When $w_{i,x} = 1$, the definition is the same as the definition of equidistribution given in [Se2, Section 1].

In this paper, we will use *Kuznietsov trace formula* to obtain a certain weighted distributions.

Suppose $h \in S(N,k)$ is a Hecke eigenform with Fourier expansion

$$h(z) = \sum_{r=1}^{\infty} a_r e^{2r\pi i z}, \operatorname{Re} z > 0.$$

Write $a_r(h) = a_r$. Let $||\cdot||$ be the Petersson norm on $S(N,k)$ [Ge, p. 24 (2.6)]. We can assume k is even because $S(N,k)$ is empty when k is odd. Define polynomials X_n by

$$X_n(2\cos\phi) = \frac{\sin(n+1)\phi}{\sin\phi}.$$

Let r be a positive integer. Let p be a fixed prime. Let $r_p = \mathrm{ord}_p\, r$. Then we have

Theorem 1.1. *Let k be an even number $\geqslant 3$. Consider the family of sets $S_N = \{(x_p^h) : h \in \mathscr{E}(N,k)\}$, $p \nmid N$ with weight $w_h^r = \frac{|a_r(h)|^2 e^{-4\pi r}}{||h||^2}$ assigned to x_p^h. Then the family of sets $\{S_N : p \nmid N\}$ is $\{w_h^r\}$-distributed with respect to*

$$\sum_{0 \leqslant i \leqslant r_p} X_{2i}(x) d\mu_\infty(x)$$

when $N \to \infty$.

The proof will be given at the end of this paper. A more general result is given in Theorem 5.7.

Corollary 1.2. *Let k, S_N, w_h^r be as above. If $p \nmid r$, then the family of sets $\{S_N : p \nmid N\}$, $N \to \infty$ is $\{w_h^r\}$-distributed with respect to the Sato–Tate measure.*

Proof. If $p \nmid r$, $r_p = \mathrm{ord}_p\, r = 0$ and $X_0(x) = 1$. The corollary follows easily. $\square$

The technique used here can be generalized to other groups (example $G = \mathrm{GSp}(2k)$), refer to [Li] for the generalizations.

2. Construction of test functions

Let $G = \mathrm{GL}_2$. The unipotent group is $N = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \subset G$. Write $e = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$. Let Z be the center of G, let M be the diagonal subgroup of G. Denote $\bar{U} = U/Z$ for any subset U of GL_2. Define $K_\infty = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right\}$. When $p < \infty$, define $K_p = \mathrm{GL}_2(\mathbf{Z}_p)$. Define $K_0(N)_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_p : c \equiv 0 (\mathrm{mod}\, N) \right\}$, $K_0(N) = \prod_{p<\infty} K_0(N)_p$.

Let $\mathbf{A}$ be the adeles of $\mathbf{Q}$. Let $\mathbf{A}_{\mathrm{fin}}$ be the finite component of $\mathbf{A}$.

Let L^2 be the Hilbert space of continuous functions φ on $Z(\mathbf{A})G(\mathbf{Q})\backslash G(\mathbf{A})$ such that $\int_{Z(\mathbf{A})G(\mathbf{Q})\backslash G(\mathbf{A})} |\varphi(g)|^2\, dg < \infty$. The subset of cuspidal functions in L^2 is denoted by L_0^2. Let R be the right regular representation of $G(\mathbf{A})$ on L^2.

There is an embedding $S(N,k) \to L_0^2$ [Ge, p. 42]. The map is denoted by $f \mapsto \varphi_f$. Denote the image of the map by $A(N,k)$.

Suppose (π, V) is a representation of a measurable topological group G and f is a continuous function on G. Define $\pi(f)v = \int_G f(g)\pi(g)v\, dg$.

We are going to construct a function $f = f_\infty f_{\text{fin}}$ on $\overline{G(\mathbf{A})} = \overline{G(\mathbf{R})} \times \overline{G(\mathbf{A}_{\text{fin}})}$. The main property of this function is given in Proposition 2.1. The results quoted below are well known.

The function $f_\infty = f_k$ is a function defined on $\overline{\mathrm{GL}_2(\mathbf{R})}$. It is the conjugate of a normalized matrix coefficient. Explicitly it is defined by $f_k(g) = d_{\pi_k}\overline{\langle \pi(g)v_0, v_0 \rangle}$ where π_k is the discrete series representation of lowest weight k, v_0 is the lowest weight unit vector and d_{π_k} is the formal degree. Explicitly we can take

$$f_k(g) = \begin{cases} \frac{k-1}{4\pi} \frac{2^k (\det g)^{k/2}}{((a+d)+i(b-c))^k} & \text{if } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \det g > 0, \\ 0 & \text{if } \det g < 0. \end{cases} \tag{2}$$

Refer to [Va, p. 192] or [KL] for the details.

Define $\psi(N) = [\Gamma_0(1) : \Gamma_0(N)]$. We take $\text{meas}(K_p) = 1$. One can easily show that $\text{meas}(\overline{K_0(N)}) = \frac{1}{[K_0(1):K_0(N)]} = \frac{1}{\psi(N)}$.

If R be a ring, let $M_2(R)$ be all the 2×2 matrix over R. Now we define

$$M(n, N) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2\left(\prod \mathbf{Z}_p\right) : \ \det(g) \in n \prod \mathbf{Z}_p^* \quad \text{and } c \equiv 0 \ (\text{mod } N) \right\}.$$

Define

$$f^n(g) = \begin{cases} \frac{1}{\text{meas } (\overline{K_0(N)})} = \psi(N) & \text{if } g = zm, z \in Z(\mathbf{A}_{\text{fin}}), m \in M(n, N), \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.1. *Suppose* $f = f_k f^n$, $R(f)$ *vanishes on* $A(N,k)^\perp$. *On* $A(N,k)$ *it acts by*

$$R(f)\varphi_h = \varphi_{n^{-\left(\frac{k}{2}-1\right)} T_n h}.$$

Proof. The idea of the proof can be found in [Ge, Lemma 3.7] and also [Ro, Lemma 2.12]. A complete proof can be found in [KL]. □

Corollary 2.2. *Let* $h \in \mathscr{E}(N,k)$, $f = f_k f^n$ *then*

$$R(f)\varphi_h = n^{1/2} \left(\prod_{p|n} X_{n_p}(x_p^h) \right) \varphi_h.$$

Proof. Recall that $n = \prod p^{n_p}$. We have $T'_n = \prod_p T'_{p^{n_p}}$. By Serre [Se2, Sections 2 and 3, Lemma 1], $T'_{p^{n_p}} = X_{n_p}(T'_p)$. The corollary follows easily from the previous proposition. □

3. Kuznietsov trace formula

Let f be a continuous function on $\overline{G(\mathbf{A})}$. The *kernel* of $R(f)$ is defined as

$$K(x,y) = \sum_{\gamma \in \overline{G(\mathbf{Q})}} f(x^{-1}\gamma y). \tag{3}$$

Another way to express the kernel is

$$K(x,y) = \sum_{\phi} R(f)\phi(x)\overline{\phi(y)}.$$

Here ϕ runs through an orthonormal basis of L^2. When $f = f_k f^n$, $R(f)$ annihilates $A(N,k)^{\perp}$. We can sum over an orthonormal basis of $A(N,k)$. An orthonormal basis can be taken as $\{\frac{\varphi_h(x)}{||\varphi_h||} : h \in \mathscr{E}(N,k)\}$. It is easy to show that

$$K(x,y) = \sum_{h \in \mathscr{E}(N,k)} \frac{R(f)\varphi_h(x)}{||\varphi_h||}\frac{\overline{\varphi_h(y)}}{||\varphi_h||}. \tag{4}$$

Use θ to denote a character on $\mathbf{Q}\backslash\mathbf{A}$. We can decompose θ into $\theta_\infty\theta_{\text{fin}}$. Here θ_∞ (resp. θ_{fin}) is the infinite (resp. finite) component of θ. There exists $r \in \mathbf{Q}$, such that $\theta_\infty(x) = e^{2\pi i r x}$. We assume $r \in \mathbf{Z}^+$ throughout the whole paper. Under this assumption θ_{fin} is trivial on $\prod_p \mathbf{Z}_p$. The character θ can also be regarded as a character on $N(\mathbf{Q})\backslash N(\mathbf{A})$.

We factorize r into $\prod p^{r_p}$.

Normalize measure on $\mathbf{Q}_p$ by taking $\text{meas}(\mathbf{Z}_p) = 1$. Define measure on $\mathbf{A}$ by using the product measure. We can show that $\text{meas}(\mathbf{Q}\backslash\mathbf{A}) = 1$. Measure on $N(\mathbf{A})$ is defined by identifying $\mathbf{A}$ with $N(\mathbf{A})$.

From (3), the kernel $K(x,y)$ is invariant under left multiplication of elements in $N(\mathbf{Q}) \times N(\mathbf{Q})$.

Kuznietsov trace formula is the equality obtained by expanding the following integral using the two formulas (3) and (4)

$$\text{KTF}(f) = \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} K(n_1,n_2)\theta(n_1^{-1}n_2)\, dn_1\, dn_2. \tag{5}$$

The integral is convergent because $\mathbf{Q}\backslash\mathbf{A}$ is compact. The expression obtained using formula (3) is called the *geometric side*. Using formula (4) we obtain the *spectral side*.

Proposition 3.1. *When* $f = f_k f^n$, *then* $\text{KTF}(f)$ *is equal to*

$$n^{1/2} \sum_{h \in \mathscr{E}(N,k)} \left(\prod_{p|n} X_{n_p}(x_p^h) \right) w_h^r,$$

recall that

$$w_h^r = \frac{|a_r(h)|^2 e^{-4\pi r}}{||h||^2}.$$

Proof. Using (4) and Corollary 2.2

$$K(x,y) = n^{1/2} \sum_{h \in \mathscr{E}(N,k)} \left(\prod_{p|n} X_{n_p}(x_p^h) \right) \frac{\varphi_h(x)\overline{\varphi_h(y)}}{||h||^2}.$$

Thus the spectral side of (5) is

$$n^{1/2} \sum_{h \in \mathscr{E}(N,k)} \left(\prod_{p|n} X_{n_p}(x_p^h) \right) \frac{1}{||h||^2} \left| \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \varphi_h(n)\theta(n^{-1}) dn \right|^2.$$

From [Ge, Chapter 3, Lemma 3.6],

$$\int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \varphi_h(n)\theta(n^{-1})\, dn = \begin{cases} a_r e^{-2\pi r} & \text{if } r \in \mathbf{Z}^+, \\ 0 & \text{otherwise.} \end{cases}$$

The proposition follows easily. □

Let $\delta \in G$. We define

$$N_\delta = \{(n_1, n_2) \in N \times N : n_1^{-1} \delta n_2 \sim \delta\};$$

here $g_1 \sim g_2$ if $g_1 = z g_2$ for some z in the center. Denote the image of δ in $N(\mathbf{Q})\backslash\overline{G(\mathbf{Q})}/N(\mathbf{Q})$ by $[\delta]$.

Proposition 3.2.

$$\mathrm{KTF}(f) = \sum_{[\delta] \in N(\mathbf{Q})\backslash\overline{G(\mathbf{Q})}/N(\mathbf{Q})} \int_{N_\delta(\mathbf{Q})\backslash N(\mathbf{A}) \times N(\mathbf{A})} f(n_1^{-1}\delta n_2)\theta(n_1^{-1} n_2)\, dn_1\, dn_2.$$

Proof. The geometric side of KTF(f)

$$= \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \sum_{\gamma \in \overline{G(\mathbf{Q})}} f(n_1^{-1}\gamma n_2)\theta(n_1^{-1} n_2)\, dn_1\, dn_2$$

$$= \sum_{[\delta]\in N(\mathbf{Q})\backslash\overline{G(\mathbf{Q})}/N(\mathbf{Q})} \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \int_{N(\mathbf{Q})\backslash N(\mathbf{A})} \sum_{\gamma\in \overline{N(\mathbf{Q})\delta N(\mathbf{Q})}} f(n_1^{-1}\gamma n_2)\theta(n_1^{-1}n_2)\, dn_1\, dn_2.$$

$$= \sum_{[\delta]} \int \int \sum_{(m_1,m_2)\in N_\delta(\mathbf{Q})\backslash N(\mathbf{Q})\times N(\mathbf{Q})} f(n_1^{-1}m_1^{-1}\delta m_2 n_2)\theta(n_1^{-1}n_2)\, dn_1\, dn_2.$$

Replace n_1 by $m_1^{-1}n_1$, n_2 by $m_2^{-1}n_2$. Since θ is trivial on $m_1, m_2\in N(\mathbf{Q})$, $\mathrm{KTF}(f)$ is equal to

$$\sum_{[\delta]\in N(\mathbf{Q})\backslash\overline{G(\mathbf{Q})}/N(\mathbf{Q})} \int_{N_\delta(\mathbf{Q})\backslash N(\mathbf{A})\times N(\mathbf{A})} f(n_1^{-1}\delta n_2)\theta(n_1^{-1}n_2)\, dn_1\, dn_2. \qquad \square$$

Denote

$$I_\delta(f) = \int_{N_\delta(\mathbf{Q})\backslash N(\mathbf{A})\times N(\mathbf{A})} f(n_1^{-1}\delta n_2)\theta(n_1^{-1}n_2)\, dn_1\, dn_2.$$

An element $\delta\in\overline{G(\mathbf{Q})}$ is said to be *admissible* if the map

$$N_\delta(\mathbf{A})\to\mathbf{C}\ : (n_1, n_2)\mapsto\theta(n_1^{-1}n_2)$$

is trivial.

Lemma 3.3. *If δ is not admissible, then $I_\delta(f) = 0$.*

Proof. Assume that δ is not admissible. Let $(v_1, v_2)\in N_\delta(\mathbf{A})$ such that $\theta(v_1^{-1}v_2)\neq 1$. Replace n_1, n_2 by v_1n_1, v_2n_2, respectively.

$$I_\delta(f) = \int_{N_\delta(\mathbf{Q})\backslash N(\mathbf{A})\times N(\mathbf{A})} f(n_1^{-1}v_1^{-1}\delta v_2 n_2)\theta(n_1^{-1}v_1^{-1}v_2n_2)\, dn_1\, dn_2.$$

Thus $I_\delta(f) = \theta(v_1^{-1}v_2)I_\delta \Rightarrow I_\delta = 0$. $\square$

Theorem 3.4.

$$\mathrm{KTF}(f) = \int_{N(\mathbf{A})} f(n)\theta(n)\, dn + \sum\nolimits_{\mu\in\mathbf{Q}^*} \int_{N(\mathbf{A})} \int_{N(\mathbf{A})} f\left(n_1^{-1}\left(\begin{smallmatrix}0 & \mu\\ 1 & 0\end{smallmatrix}\right)n_2\right)\theta(n_1^{-1}n_2)\, dn_1\, dn_2.$$

Proof. By the Bruhat decomposition $G = NM\cup NM\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)N$. Thus a representative set of $N(\mathbf{Q})\backslash\overline{G(\mathbf{Q})}/N(\mathbf{Q})$ is $\left\{\left(\begin{smallmatrix}\gamma & 0\\ 0 & 1\end{smallmatrix}\right) : \gamma\in\mathbf{Q}^*\right\}\cup\left\{\left(\begin{smallmatrix}0 & \mu\\ 1 & 0\end{smallmatrix}\right) : \mu\in\mathbf{Q}^*\right\}$. From Theorem 3.2, the geometric side $= \sum_{[\delta]} I_\delta$. By Lemma 3.3, $I_\delta(f) = 0$ unless δ is admissible.

When $\delta = \left(\begin{smallmatrix}\gamma & 0\\ 0 & 1\end{smallmatrix}\right)$, simple calculation shows that $N_\delta = \left\{\left(\begin{pmatrix}1 & \gamma t\\ & 1\end{pmatrix}, \begin{pmatrix}1 & t\\ & 1\end{pmatrix}\right)\right\}$. If δ is admissible, we have $\theta((\gamma-1)t) = 1$ for any $t\in\mathbf{A}$. Because θ is non-trivial, this cannot happen unless $\gamma = 1$. Thus

$$I_e(f) = \int_{N_\delta(\mathbf{Q})\backslash N(\mathbf{A})\times N(\mathbf{A})} f\left(n_1^{-1}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}n_2\right)\theta(n_1^{-1}n_2)\,dn_1\,dn_2$$
$$= \int_{\text{diagonal}\backslash N(\mathbf{A})\times N(\mathbf{A})} f(n_1^{-1}n_2)\theta(n_1^{-1}n_2)\,dn_2\,dn_1.$$

Letting $m_1 = n_1, m_2 = n_1^{-1}n_2$, the diagonal becomes (m_1, e). The integral becomes

$$\int_{N(\mathbf{Q})\times e\backslash N(\mathbf{A})\times N(\mathbf{A})} f(m_2)\theta(m_2)\,dm_2\,dm_1 = \int_{N(\mathbf{A})} f(m_2)\theta(m_2)dm_2.$$

When $\delta = \left(\begin{smallmatrix}0 & \mu\\ 1 & 0\end{smallmatrix}\right)$, simple calculation shows that $N_\delta(\mathbf{Q}) = \{(e,e)\}$. Thus δ is admissible. We have

$$I_\delta(f) = \int_{N(\mathbf{A})\times N(\mathbf{A})} f\left(n_1^{-1}\begin{pmatrix}0 & \mu\\ 1 & 0\end{pmatrix}n_2\right)\theta(n_1^{-1}n_2)\,dn_1\,dn_2.$$

We can prove the theorem by summing up all the terms. □

4. Evaluation of Integrals

Lemma 4.1. *For* $u\in\mathbf{Q}$, $\theta_{\text{fin}}(u) = \theta_\infty(-u) = e^{-2\pi ru}$

Proof. We have $1 = \theta(u) = \theta_\infty(u)\theta_{\text{fin}}(u)$. The lemma follows easily. □

The following lemmas show us how to evaluate f^n. Suppose R is a ring, we denote $R^{*2} = \{x^2 : x\in R^*\}$.

Lemma 4.2. *Suppose* $g = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in G(\mathbf{A}_{\text{fin}})$ *and* $\det(g)\in n\prod \mathbf{Z}_p^*$. *Then* $g\in \operatorname{supp} f^n$ *if and only if* $g\in M_2(\prod \mathbf{Z}_p)$ *and* $c\equiv 0 \bmod N$.

Proof. Write $g = zm$, $z = \begin{pmatrix}\zeta & \\ & \zeta\end{pmatrix}\in Z(\mathbf{A}_{\text{fin}})$, $m\in M(n,N)$. Taking the determinant on both sides, we see that ζ is in $\prod \mathbf{Z}_p^*$. Thus z can be absorbed into m, so $g\in M(n,N)$. It is easy to see that $g\in M(n,N)$ if and only if g satisfies the conditions in the lemma. □

Lemma 4.3. *Suppose* $g = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in G(\mathbf{A}_{\text{fin}})$, *then* g *is in* $\operatorname{supp} f^n$ *only if* $\det g\in n\mathbf{A}_{\text{fin}}^{*2}\prod \mathbf{Z}_p^*$.

Under this assumption, say $\det g = n\zeta^2 u$ *for* $\zeta \in \mathbf{A}_{\text{fin}}^*$, $u \in \prod \mathbf{Z}_p^*$. *Let* $z = \begin{pmatrix} \zeta & \\ & \zeta \end{pmatrix}$. Let $m = z^{-1}g = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$. Then $g \in \operatorname{supp} f^n$ if and only if $m \in M_2(\prod \mathbf{Z}_p)$ and $m_{21} \equiv 0 \bmod N$.

Proof. If g is in $\operatorname{supp} f^n$, then $g = zm$ with $z = \begin{pmatrix} \zeta & \\ & \zeta \end{pmatrix} \in Z(\mathbf{A}_{\text{fin}})$ and $m \in M(n, N)$. Thus $\det g = \zeta^2 \det m \in n\mathbf{A}_{\text{fin}}^{*2} \prod \mathbf{Z}_p^*$. This proves the first part.

Suppose $\det g = n\zeta^2 u$ as stated in the lemma. Let $z = \begin{pmatrix} \zeta & \\ & \zeta \end{pmatrix}$. Obviously $g \in \operatorname{supp} f^n$ if and only if $z^{-1}g \in \operatorname{supp} f^n$. One can easily show that $\det z^{-1}g = nu \in n \prod \mathbf{Z}_p^*$. The lemma follows easily from the above lemma. □

From now on we call the $z = \begin{pmatrix} \zeta & \\ & \zeta \end{pmatrix}$ appearing in the previous lemma a *z-part of g*.

Define

$$\text{True}(\textit{statement}) = \begin{cases} 1 & \text{if } \textit{statement} \text{ is true,} \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.4.

$$\int_{N(\mathbf{A}_{\text{fin}})} f^n(n_{\text{fin}})\theta_{\text{fin}}(n_{\text{fin}})\, dn_{\text{fin}} = \psi(N)n^{1/2}\,\text{True}(n^{1/2} \in \mathbf{Z} \quad \text{and } n^{1/2}|r). \tag{6}$$

Proof. Write $n_{\text{fin}} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. Its determinant is 1. From Lemma 4.3, $n_{\text{fin}} \in \operatorname{supp} f^n$ only if $1 \in n\mathbf{Q}_p^{*2}\mathbf{Z}_p^*$ for all primes p. Thus n_p is even for all p. As a result $n \in \pm\mathbf{Q}^{*2}$. We can assume n is positive. Write $n = n'^2$, $n' \in \mathbf{Z}^+$.

Now a z-part of n_{fin} can be $z = \begin{pmatrix} n' & 0 \\ 0 & n' \end{pmatrix}^{-1}$. Let $m = z^{-1}n_{\text{fin}} = \begin{pmatrix} n' & n't \\ 0 & n' \end{pmatrix}$. The lower left entry of m is 0, which is divisible by N. By Lemma 4.3, n_{fin} is in $\operatorname{supp} f^n$ if and only if $\begin{pmatrix} n' & n't \\ 0 & n' \end{pmatrix} \in M_2(\prod \mathbf{Z}_p)$. Or equivalently

$$t' = n't \in \prod \mathbf{Z}_p.$$

Thus (6) is equal to

$$\psi(N) \int_{t \in n'^{-1} \prod \mathbf{Z}_p} \theta_{\text{fin}}(t)\, dt = \psi(N) \int_{\prod \mathbf{Z}_p} \theta_{\text{fin}}\left(\frac{t'}{n'}\right) d\,\frac{t'}{n'}.$$

Because θ_{fin} is trivial on $\prod \mathbf{Z}_p$, integral (6) is equal to

$$\psi(N)n' \sum_{s\in\mathbf{Z}/n'\mathbf{Z}} \int_{s+n'\prod\mathbf{Z}_p} \theta_{\text{fin}}\left(\frac{t'}{n'}\right)dt'$$

$$= \psi(N)n' \sum_{s\in\mathbf{Z}/n'\mathbf{Z}} e\left(-\frac{sr}{n'}\right)\text{meas}\left(s+n'\prod\mathbf{Z}_p\right) = \psi(N) \sum_{s\in\mathbf{Z}/n'\mathbf{Z}} e\left(\frac{sr}{n'}\right).$$

The result follows easily. □

Next, we evaluate

$$\int_{\mathbf{A}_{\text{fin}}\times\mathbf{A}_{\text{fin}}} f^n\left(n_1^{-1}\begin{pmatrix}0 & \mu\\ 1 & 0\end{pmatrix}n_2\right)\theta_{\text{fin}}(n_1^{-1})\theta_{\text{fin}}(n_2)\,dn_1\,dn_2. \tag{7}$$

Define

$$\text{Kl}_u(n,\theta_\infty) = \sum_{s_1,s_2\in\mathbf{Z}/u\mathbf{Z}, s_1s_2\equiv n \,(\text{mod}\, u)} \theta_\infty\left(\frac{s_1}{u}\right)\theta_\infty\left(\frac{s_2}{u}\right).$$

Proposition 4.5. *Integral* (7) $\neq 0$ *only if* $\mu = \pm\frac{n}{u^2}$ *for some integer* $u \equiv 0\,(\text{mod}\,N)$. *Under this assumption, integral* (7) *is equal to* $\psi(N)\,\text{Kl}_u(\mp n, \theta_\infty)$.

Proof. Let $n_i = \begin{pmatrix}1 & t_i\\ & 1\end{pmatrix}$, $i = 1, 2$.

$$n_1^{-1}\begin{pmatrix}0 & \mu\\ 1 & 0\end{pmatrix}n_2 = \begin{pmatrix}-t_1 & \mu - t_1t_2\\ 1 & t_2\end{pmatrix}. \tag{8}$$

Notice $\det\left(n_1^{-1}\left(\begin{smallmatrix}0 & \mu\\ 1 & 0\end{smallmatrix}\right)n_2\right) = -\mu$. From Lemma 4.3, (8) $\in \text{supp}\, f^n$ only if $\mu\in n\mathbf{Q}_p^{*2}\mathbf{Z}_p^*$ for all p. Thus we have $\text{ord}_p(\mu) \equiv n_p\,(\text{mod}\,2)$ for all p. As a result $\mu \in \pm n\mathbf{Q}^2$. Let $\mu = \pm n\zeta^2$ for some $\zeta\in\mathbf{Q}^*$. We can take $z = \begin{pmatrix}\zeta & \\ & \zeta\end{pmatrix}$ as the z-part. Write $m = z^{-1}n_1^{-1}\left(\begin{smallmatrix}0 & \mu\\ 1 & 0\end{smallmatrix}\right)n_2 = \left(\begin{smallmatrix}* & *\\ \zeta^{-1} & *\end{smallmatrix}\right)$. By Lemma 4.3, (8) is in $\text{supp}\, f^n$ only if $\zeta^{-1}\in\mathbf{Z}_p$ for all p. Hence $\zeta = 1/u$ for some $u\in\mathbf{Z}^+$. Let $\mu = \pm\frac{n}{u^2}$, $u\in\mathbf{Z}$.

$$m = \begin{pmatrix}-ut_1 & \frac{\pm n-(ut_1)(ut_2)}{u}\\ u & ut_2\end{pmatrix}.$$

Write $t_1' = ut_1$, $t_2' = ut_2$. By Lemma 4.3 again, (8) is in $\text{supp}\, f^n$ if and only if

$$t_1', t_2' \in \prod \mathbf{Z}_p, u \equiv 0\,(\text{mod}\,N), t_1't_2' \equiv \pm n\,(\text{mod}\,u).$$

Integral (7) becomes

$$\begin{aligned}
&\int_{t_1,t_2} f^n\left(\begin{pmatrix} -t_1 & \mu - t_1 t_2 \\ 1 & t_2 \end{pmatrix}\right)\theta_{\text{fin}}(-t_1)\theta_{\text{fin}}(t_2)\,dt_1\,dt_2 \\
&= \int_{t_1',t_2'\in\prod \mathbf{Z}_p,\, t_1't_2'\in \pm n+u\prod \mathbf{Z}_p} f^n\begin{pmatrix} -t_1' & * \\ u & t_2' \end{pmatrix}\theta_{\text{fin}}\left(-\frac{t_1'}{u}\right)\theta_{\text{fin}}\left(\frac{t_2'}{u}\right)d\,\frac{t_1'}{u}\,d\,\frac{t_2'}{u} \\
&= \psi(N) \sum_{s_1,s_2\in \mathbf{Z}/u\mathbf{Z},\, s_1s_2=\pm n} \int_{(s_1+u\prod \mathbf{Z}_p)\times(s_2+u\prod \mathbf{Z}_p)} \theta_{\text{fin}}\left(-\frac{t_1'}{u}\right)\theta_{\text{fin}}\left(\frac{t_2'}{u}\right)d\,\frac{t_1'}{u}\,d\,\frac{t_2'}{u} \\
&= \psi(N) \sum_{s_1,s_2\in \mathbf{Z}/u\mathbf{Z},\, s_1s_2=\pm n} \theta_\infty\left(\frac{s_1}{u}\right)\theta_\infty\left(-\frac{s_2}{u}\right)\text{meas}\left(\prod \mathbf{Z}_p\right)^2 \\
&= \psi(N) \sum_{s_1,s_2\in \mathbf{Z}/u\mathbf{Z},\, s_1s_2=\mp n} \theta_\infty\left(\frac{s_1}{u}\right)\theta_\infty\left(\frac{s_2}{u}\right) = \psi(N)\,\mathrm{Kl}_u(\mp n,\theta_\infty). \qquad \square
\end{aligned}$$

Proposition 4.6. *When $k \geqslant 3$,*

$$\int_{N(\mathbf{R})} f_k(n)\theta_\infty(n)\,dn = \begin{cases} \frac{e^{-4\pi r}(4\pi r)^{k-1}}{(k-2)!} & \text{if } r>0, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Proof. Write $n = \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}$. By (2), the integral in (9) is equal to

$$\frac{k-1}{4\pi}\int_{-\infty}^{\infty} \frac{2^k}{(2+it)^k} e^{2\pi i r t}\,dt.$$

When $r>0$, use the x-axis and the upper semi-circle as the contour. We can get the result easily by evaluating the residue of the integrand at $t = 2i$.

When $r<0$, use the x-axis and the lower semi-circle as the contour. The result follows easily. $\square$

Proposition 4.7. *When $k \geqslant 3$,*

$$\int_{N(\mathbf{R})\times N(\mathbf{R})} f_k\left(n_1^{-1}\begin{pmatrix} 0 & \mu \\ 1 & 0 \end{pmatrix} n_2\right)\theta_\infty(n_1^{-1}n_2)\,dn_1\,dn_2 \tag{10}$$

is non-zero only if $r, -\mu$ are all positive. Under this condition, the integral is equal to

$$\frac{e^{-4\pi r}(4\pi i)^k r^{k-1}}{2(k-2)!}(-\mu)^{\frac{1}{2}} J_{k-1}(4\pi r\sqrt{-\mu});$$

here J_k is the Bessel J function.

Proof. When $\mu > 0$, $\det\left(n_1^{-1}\begin{pmatrix} & \mu \\ 1 & \end{pmatrix} n_2\right) = -\mu < 0$. The value of f_k at it is 0. So we can assume $\mu < 0$. Write $n_i = \begin{pmatrix} 1 & t_i \\ & 1 \end{pmatrix}$, $i = 1, 2$. By (2), (10)=

$$\frac{k-1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{2^k(-\mu)^{k/2}e^{2\pi i r(t_2-t_1)}}{(-t_1+t_2+i(-1+\mu-t_1t_2))^k}\,dt_1dt_2.$$

First assume $r > 0$, evaluate the residue of the integrand at $t_2 = i + \frac{\mu}{t_1+i}$, (10) becomes

$$\frac{k-1}{4\pi}\int_{-\infty}^{\infty}\frac{2\pi i}{(1-it_1)^k}\frac{2^k(-\mu)^{k/2}(2\pi i r)^{k-1}e^{2\pi i r(i+\frac{\mu}{t_1+i}-t_1)}}{(k-1)!}\,dt_1.$$

Use x-axis and lower semi-circle as the contour, the integral can be calculated by evaluating the residue of the integrand at $t_1 = -i$. Notice that the path is counterclockwise.

Refer to [Wa, Chapter 2.1], we have

$$e^{\frac{1}{2}\xi(\tau-\frac{1}{\tau})} = \sum_{-\infty}^{\infty}\tau^n J_n(\xi).$$

Let $\xi = 4\pi r\sqrt{-\mu}$, $\tau = -it_1 + i/\sqrt{-\mu}$ be the residue theorem, (10) becomes

$$\frac{k-1}{4\pi}\frac{2\pi i}{(-i)^k}(-2\pi i)\frac{2^k(-\mu)^{k/2}(2\pi i r)^{k-1}}{(k-1)!}e^{-4\pi r}J_{k-1}(4\pi r\sqrt{-\mu})\frac{(-i)^{k-1}}{(-\mu)^{\frac{k-1}{2}}}.$$

We can get the result easily.

When $r < 0$, use the axis and upper semi-circle as the contour. It is easy to show that (10) is 0. $\square$

Theorem 4.8. *Let k be an even number $\geqslant 3$. Let n, N, r be any positive integers. Factorize n into $\prod_p p^{n_p}$. Assume* $\mathrm{GCD}(N, n) = 1$. *Define $\theta_\infty(x) = e^{2\pi i r x}$, then we have*

$$\sum_{h\in\mathscr{E}(N,k)}\left(\prod_{p|n}X_{n_p}(x_p^h)\right)\frac{|a_r(h)|^2e^{-4\pi r}}{||h||^2}$$
$$= \mathrm{True}(n^{1/2}\in\mathbf{Z}, n^{1/2}|r)\frac{e^{-4\pi r}(4\pi r)^{k-1}}{(k-2)!}\psi(N) \tag{11}$$

$$+\frac{e^{-4\pi r}(4\pi i)^k r^{k-1}}{2(k-2)!}\psi(N)\sum_{v=1}^{\infty}\frac{1}{Nv}J_{k-1}\left(\frac{4\pi n^{1/2}r}{Nv}\right)\mathrm{Kl}_{vN}(n, \theta_\infty). \tag{12}$$

Proof. The spectral side is obtained by Proposition 3.1.

The geometric side is obtained by Theorem 3.4. The integral $I_e(f)$ is the product of (6) and (9). Using the results in Proposition 4.6 and Corollary 4.4, we can get $I_e(f)$.

Let $\delta = \begin{pmatrix} 0 & \mu \\ 1 & 0 \end{pmatrix}$. Then $I_\delta(f)$ is product of (7) and (10). By Proposition 4.5 $(7) \neq 0$ only if $\mu = \frac{\pm n}{u^2}$ and $N|u$. Write $u = Nv$. By Proposition 4.7, $(10) \neq 0$ only if $\mu < 0$. Thus $\sum_\delta I_\delta(f)$ is a sum over $\left\{ \delta = \begin{pmatrix} 0 & \frac{-n}{(Nv)^2} \\ 1 & 0 \end{pmatrix} : v \in \mathbf{Z}^+ \right\}$.

Multiply the results we obtained in Propositions 4.5 and 4.7. Summing up all the terms in the geometric side. We can obtain the formula by equating the spectral side and geometric side and then dividing both sides by $n^{1/2}$. □

5. Weighted distribution

Lemma 5.1.

$$|\mathrm{Kl}_u(n, \theta_\infty)| \leqslant un.$$

Proof. Obviously $|\mathrm{Kl}_u(n, \theta_\infty)| \leqslant |\{s_1 s_2 \equiv n \pmod u\}|$. It suffices to prove $|\{s_1 s_2 \equiv n \pmod u\}| \leqslant un$ for $u = p^{u_p}$ and $n = p^{n_p}$.

If $n_p \geqslant u_p$, $|\{s_1 s_2 \equiv n \pmod u\}| \leqslant u^2 \leqslant un$.

Assume $n_p < u_p$.

$$|\{s_1 s_2 \equiv n \pmod u\}| = \sum_{s=1}^{u} |\{t : st \equiv n \pmod u\}|$$

$$= \sum_{s=1, \gcd(s,u)|n}^{u} \gcd(s,u) = \sum_{s_p=0}^{n_p} p^{s_p} \cdot |\{s : \ \mathrm{ord}_p(s) = s_p, 1 \leqslant s \leqslant u\}|$$

$$\leqslant \sum_{s_p=0}^{n_p} p^{s_p} \frac{p^{u_p}}{p^{s_p}} \leqslant (n_p + 1) p^{u_p} \leqslant nu \qquad \square$$

From [Iw, equation (5.16)],

$$J_k(x) \leqslant \min\{x^k, x^{-1/2}\}.$$

Proposition 5.2. *Let k be an even integer* $\geqslant 3$, *then*

$$\frac{1}{\psi(N)} \sum_{h \in \mathscr{E}(N,k)} \prod_{p|n} X_{n_p}(x_p^h) w_h^r$$

$$= \mathrm{True}(n^{1/2} \in \mathbf{Z}, n^{1/2}|r)\, \frac{e^{-4\pi r}(4\pi r)^{k-1}}{(k-2)!} + O\left(\frac{n^{(k+1)/2}}{N^{k-1}}\right).$$

Here the constant in the O-notation depends on θ only.

Proof. Form the inequalities given above, (12) is

$$\ll \frac{e^{-4\pi r}(4\pi)^k r^{k-1}}{2(k-2)!}\, \psi(N) \sum_{v=1}^{\infty} \frac{1}{Nv}\left(\frac{4\pi n^{1/2} r}{Nv}\right)^{k-1} (Nvn)$$

$$\ll_r \frac{\psi(N)}{N^{k-1}} n^{(k+1)/2} \sum_v \frac{1}{v^{k-1}} \ll_r \frac{\psi(N)}{N^{k-1}} n^{(k+1)/2}.$$

The proposition follows easily from Theorem 4.8. □

Corollary 5.3. *The average of weight*

$$\frac{1}{\psi(N)} \sum_{h \in \mathscr{E}(N,k)} w_h^r = \frac{e^{-4\pi r}(4\pi r)^{k-1}}{(k-2)!} + O\left(\frac{1}{N^{k-1}}\right).$$

Proof. Put $n = 1$ in the above proposition. □

Let $p_1, p_2, \ldots p_\ell$ be distinct primes. For any prime p, let I_p be an interval in $\mathbf{R}$ containing all the possible values of x_p^h for any Hecke eigenform h. By Deligne's result, we can take $I_p = [-2, 2]$, but we do not need this strong result. We only need the fact that I_p is a finite interval. Refer to [Ro, Proposition 2.9] for the proof of this fact.

Denote the set of real valued continuous functions on $I = I_{p_1} \times I_{p_2} \times \cdots \times I_{p_\ell}$ by $C(I_{p_1} \times I_{p_2} \times \cdots \times I_{p_\ell})$. Define a topological structure on it by using the L^∞ norm $||f||_\infty = \max\{|f(x)|\}$. Let k be an even integer $\geqslant 3$, define a functional $\mathfrak{F}_N$ on $C(I_{p_1} \times I_{p_2} \times \cdots \times I_{p_\ell})$ by

$$\mathfrak{F}_N f = \frac{\sum_{h \in \mathscr{E}(N,k)} f(x_{p_1}^h, \ldots, x_{p_\ell}^h) w_h^r}{\sum_{h \in \mathscr{E}(N,k)} w_h^r}$$

Proposition 5.4.

$$\lim_{N \to \infty, (N, p_1, \ldots, p_\ell) = 1} \mathfrak{F}_N X_{n_{p_1}} \times \cdots \times X_{n_{p_\ell}} = \prod_{i=1}^{\ell} \mathrm{True}(2|n_{p_i}, n_{p_i}/2 \leqslant r_{p_i})$$

Proof. Let $n = p_1^{n_{p_1}} \cdots p_\ell^{n_{p_\ell}}$. From Corollaries 5.2 and 5.3,

$$\mathfrak{F}_N f = \frac{\prod_{i=1}^{\ell} \mathrm{True}(2|n_{p_i}, n_{p_i}/2 \leqslant r_{p_i}) \frac{e^{-4\pi r}(4\pi r)^{k-1}}{(k-2)!} + O(\frac{n^{(k+1)/2}}{N^{k-1}})}{\frac{e^{-4\pi r}(4\pi r)^{k-1}}{(k-2)!} + O(\frac{1}{N^{k-1}})}.$$

Letting $N \to \infty$, the proposition follows easily. □

Proposition 5.5.

$$\int_{\mathbf{R}} X_n(x) X_m(x)\, d\mu_\infty(x) = \delta_{nm}.$$

Proof. A proof can be found in [Se2, Section 2.2].

$$\frac{\pi}{2}\delta_{nm} = \int_0^\pi \frac{\sin n\theta}{\sin\theta} \sin^2\theta \frac{\sin m\theta}{\sin\theta}\, d\theta.$$

Recall $X_n(x) = \frac{\sin n\theta}{\sin\theta}$, $x = 2\cos\theta$. Make a substitution $x = 2\cos\theta$, we have

$$\frac{\pi}{2}\delta_{nm} = \int_{-2}^{2} X_n(x) X_m(x) \frac{\sin\theta}{2}\, dx.$$

The proposition follows easily. □

Theorem 5.6. *Define measure*

$$d\mu_i(x) = \sum_{0 \leqslant n' \leqslant r_{p_i}} X_{2n'}(x)\, d\mu_\infty(x). \tag{13}$$

Also define

$$\mathfrak{F}(f) = \int_{I_{p_1}} \cdots \int_{I_{p_\ell}} f(x_1, \ldots x_\ell) d\mu_1(x_1) \cdots d\mu_\ell(x_\ell).$$

Then for any $f \in C(I_{p_1} \times I_{p_2} \cdots \times I_{p_\ell})$,

$$\lim_{N\to\infty} \mathfrak{F}_N(f) = \mathfrak{F}(f).$$

Proof. By the previous proposition

$$\int_{I_{p_i}} X_{n_{p_i}}(x)\, d\mu_i(x) = \mathrm{True}(2|n_{p_i}, n_{p_i}/2 \leqslant r_{p_i}).$$

Take the product over $i = 1, \ldots \ell$ and by Proposition 5.4, we have

$$\mathfrak{F}(X_{n_{p_1}} \times \cdots \times X_{n_{p_\ell}}) = \lim_{N \to \infty, (N, p_1, \ldots, p_\ell) = 1} \mathfrak{F}_N(X_{n_{p_1}} \times \cdots \times X_{n_{p_\ell}}).$$

One can easily show that $|\mathfrak{F}_N(f)| \leqslant ||f||_\infty$. Thus $\mathfrak{F}_N$ is a continuous linear functional. Because $\deg X_n = n$, the linear span of $\{X_n\}$ consists of all the one variable polynomials. Thus the linear span of $\{X_{n_{p_1}} \times \cdots \times X_{n_{p_\ell}}\}$ consists of all the possible polynomials. The theorem follows by the fact that polynomials are dense in $C(I_{p_1} \times I_{p_2} \times \cdots \times I_{p_\ell})$. □

Theorem 5.7. *Let k be an even integer $\geqslant 3$. Consider the family of sets $S_N = \{(x_{p_1}^h, \ldots, x_{p_\ell}^h) : h \in \mathscr{E}(N, k)\}$, $(N, p_1 \cdots p_\ell) = 1$ with weight $\{w_h^r\}$ assigned to $(x_{p_1}^h, \ldots, x_{p_\ell}^h)$. Then the family of sets $\{S_N, p \nmid N\}$ is $\{w_h^r\}$-distributed with respect to the measure $d\mu_1 \cdots d\mu_\ell$ when $N \to \infty$. Here μ_i is given by Eq.* (13).

The measure μ_i has the following properties: (a) *it is supported on* $[-2, 2]$, (b) *it is a polynomial times the Sato–Tate measure on* $[-2, 2]$, (c) *it depends only on* $\mathrm{ord}_p(r)$.

Proof. Follows easily from the previous theorem. □

Proof of 1.1. Take $\ell = 1$ and $p_1 = p$. □

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