Population equilibrium with support in evolutionary matrix games

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Abstract

In this paper we consider symmetric bimatrix games \([A, A^T]\). We use a matrix operator \(s(A)\), defined as the sum of the cofactors of the given matrix \(A\), for finding the population equilibrium and its fitness in evolutionarily matrix games with all supported strategies, and to complete Bishop–Cannings Theorem.

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1. Preliminaries

In common opinion, for instance [1], Evolutionary Game Theory (EGT) was born in 1973 when John Maynard Smith (biologist) and George R. Price (mathematician) wrote their famous article in Nature [2] describing the idea of applying game theory to the behavior of animals. So it was quite natural, that at first the EGT was the application of game theory in evolutionary biology and was used to understand situations in which the payoffs of what an individual did depend on what other individuals were doing in the population. A really large increasing of interest in the EGT could be observed since 1994 when The Central Bank of Sweden in Economic Science in Memory of Alfred Nobel was award to John Nash, John Harsanyi and Reinhard Selten for their contribution to game theory. Nowadays the EGT has found applications in many fields like...
economics, for example [3], psychology [4], anthropology [5], ecology [6], language theory [7],
and language learning [8], sociology [9], ethics [10], philosophy [11], and others.

It can be said that the EGT is an extension to traditional game theory as it studies games
played by individuals from a large population and the action they take is determined by natural
selection. A population is in equilibrium (PE) when there is Nash equilibrium for the frequency
of occurrence of competing behavioral strategies in the population, or in other words, when the
population achieves the balance of nature, what is the metaphor taken from population ecology
[12].

In general case if we consider a bimatrix game \([A, B]\), where \(A, B \in \mathbb{R}^{n \times n}\), and \(A\) is the
payoff matrix for first individual, called Row, \(B\) is the payoff matrix for second individual, named
Column, then a pair \((p, q)\) of strategies constitutes a Nash equilibrium (NE) of the game \([A, B]\)
if and only if

\[
\begin{align*}
    p^T A q &\geq x^T A q \quad \text{for all Row’s strategies } x \ (p \text{ is the best response to } q), \\
    p^T B q &\geq p^T B y \quad \text{for all Column’s strategies } y \ (q \text{ is the best response to } p).
\end{align*}
\]

In the EGT, used to model the evolution of species, a strategy is a behavioral phenotype and it
specifies what an individual will do in any situation. The EGT restricts itself mostly to symmetric
bimatrix games \([A, A^T]\). Therefore, in this restricted case, since \(q=p, y=x, B = A^T\), a profile \(p\)
is a population equilibrium (PE) if and only if

\[ p^T A p \geq x^T A p \quad \text{for all strategies } x. \]

In other words the PE is a set of actions \(p\) which must be the best response to all strategies.
Furthermore, the PE is the condition must be met if the proportion of individuals is to be stable.
The concept of the PE is of particular importance in evolutionary matrix games as it is applied in
considering behavior adopted by various organisms. We put standard assumptions:

- the same set of behavioral phenotypes is available to each individual, where the phenotype is
  predetermined by the genotype of the species;
- considered population is a large set of individuals and the action an individual takes in the
  game is determined by natural selection;
- two individuals, Row and Column, are repeatedly and randomly matched to play the game and
every time the same game is played;
- the payoff an individual receives from playing the game against other individuals provides a
  fitness measure used as the basic for selection.

As it is known, in bimatrix games the individuals are not completely antagonistic to one another
(contrary to two person zero-sum games \([A, -A]\), where one individual wins what another indi-
vidual loses); if Row playing a strategy \(x\) meets Column playing a strategy \(y\), then Row’s fitness
is \(E(x, y) = x^T A y\) and Column’s fitness is \(E(y, x) = y^T A x\).

In game theory Nash equilibrium usually is found by Lemke–Howson algorithm and its later
different modifications or by linear programming (on May 13, 2005 died George Dantzig who in
1947 made the contribution to mathematics for which he was most famous, the simplex method
of optimization. The known term linear programming was proposed by Tjalling Koopmans one
year later, and since then Dantzig was nicknamed Father of linear programming). The paper
also develops algebraic approach to game theory, and equilibrium is found by sums of cofactors.
Therefore let introduce now the sums of cofactors of the given matrix.
Definition 1. Denoting by $A_{rs}$ the $rs$-th cofactor of $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ we set:

$$s_k(A) = \sum_{j=1}^{n} A_{kj} \quad (k = 1, 2, \ldots, n),$$

(1)

$$s^k(A) = \sum_{i=1}^{n} A_{ik} \quad (k = 1, 2, \ldots, n),$$

(2)

$$s(A) = \sum_{k=1}^{n} s_k(A) = \sum_{k=1}^{n} s^k(A).$$

(3)

It is easy to observe that

$$s_k(A) = \det A_{(k)}, \quad s^k(A) = \det A^{(k)},$$

(4)

where $A_{(k)}$ ($A^{(k)}$, respectively) is the matrix of $A$ with all entries of the $k$th row ($k$th column, respectively) replaced by ones.

Lemma 1. Some properties of $s(A)$ coincide with corresponding properties of the determinant, however some properties of $s(A)$ differ from those of $\det A$, namely:

1. $s(A^T) = s(A)$.
2. $s(B) = -s(A)$, where $B$ is obtained from $A$ by exchanging two rows (columns).
3. $s(cA) = c^{n-1} \cdot s(A)$, whereas $\det(cA) = c^n \cdot \det A$.
4. For a matrix $A$ with a row (column) of zeros $s(A)$ differs in general from zero, whereas $\det A = 0$.
5. $s(AB)$ differs in general from $s(A)s(B)$, whereas $\det(AB) = \det A \det B$.
6. If $B$ results from $A$ by adding a multiple of one row (column) to another row (column), then $s(A)$ differs in general from $s(B)$, whereas $\det B = \det A$.

Definition 2. As the game with the preference parameter $\lambda \in \langle 0, 1 \rangle$ we define a family of games $[A_{\lambda}, A_{\lambda}^T]$, where

$$A_{\lambda} = (1 - \lambda)A + \lambda A^T. \quad (5)$$

The matrix $A_{\lambda}^T$ is called the altruistic dual matrix of $A_{\lambda}$. It is easy to see, that the matrix $A_{\lambda}$ and its altruistic dual matrix are connected as follows:

$$A_{\lambda}^T = A_{1-\lambda}. \quad (6)$$

$A_{\lambda}$ reduces to the original fitness matrix $A$ for $\lambda = 0$ (egoistic value) and to the opponent’s fitness matrix $A^T$ for $\lambda = 1$ (altruistic value). The parameter $\lambda$ can be also interpreted as a social norm (socially imposed norm by tax enforces an individual to maximize payoff with preference $\lambda$), an imperfect information or as an error. Note that according to Definition 2, altruism and egoism are context dependent. It implies that one individual preferences affect both his own behavior and the behavior of his opponent. Moreover, we can interpret the original game $[A, A^T]$ in an unexpected way: it is the game, where an individual as an egoist plays against himself as an altruistic alter ego. That interpretation follows immediately from (6), which implies—as particular case—the following equalities:
\[ A_0 = A_1^T = A, \quad A_1 = A_0^T = A^T. \]

Since (5), and (6), it can be easy seen, that any symmetric matrix is the altruistic dual matrix of itself

\[ A = A^T(=) A_\lambda = A, \quad \lambda \in \langle 0, 1 \rangle \]

Altruism has obtained its mathematical expression by Bester and Güth [13]. In their paper is also shown that altruistic behavior (preferences of an individual reflect some concern for success of the others) can increase the fitness of a majority of individuals in an ecosystem. Formally, such preferences are described as

\[ E_\lambda(x, y) = (1 - \lambda)E(x, y) + \lambda E(y, x) \]

or in matrix form

\[ E_\lambda(x, y) = (1 - \lambda)x^TAy + \lambda y^TAx. \]

2. Results

The following two lemmas show next difference between \( s(A) \) and \( \det A \). They also imply that the proportion of types in the PE, given by formula (9), does not depend on addition any constant matrix to the given fitness matrix and does not depend on multiplication the fitness matrix by non zero number as well.

**Lemma 2.** The operator \( s(A) \) is invariant with respect to addition of constant matrices \( cE \), where \( c \in R \) and \( E \) denotes the matrix with all entries equal to one:

\[ s(A + cE) = s(A). \]  

(7)

**Proof.** Consider \( s_k(A + cE) \) for any fixed \( 1 \leq k \leq n \). If we subtract the \( k \)th row multiplied by the constant \( c \) from all other rows, then the value of the operator \( s_k \) will not change. Hence, the following equality is true:

\[
\begin{align*}
s_k(A + cE) &= \det \begin{bmatrix}
a_{11} + c & \ldots & a_{1n} + c \\
a_{k-1,1} + c & \ldots & a_{k-1,n} + c \\
1 & \ldots & 1 \\
a_{k+1,1} + c & \ldots & a_{k+1,n} + c \\
\vdots & \ddots & \vdots \\
a_{n1} + c & \ldots & a_{nn} + c \\
\end{bmatrix} \\
&= \det \begin{bmatrix}
a_{11} & \ldots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{k-1,1} & \ldots & a_{k-1,n} \\
1 & \ldots & 1 \\
a_{k+1,1} & \ldots & a_{k+1,n} \\
\vdots & \ddots & \vdots \\
a_{n1} & \ldots & a_{nn} \\
\end{bmatrix} = s_k(A)
\end{align*}
\]
and therefore
\[ s(A + cE) = \sum_{k=1}^{n} s_k(A + cE) = \sum_{k=1}^{n} s_k(A) = s(A). \]

**Lemma 3.** For any matrix \( A \in \mathbb{R}^{n \times n} \) and \( c \in \mathbb{R} \) we have:
\[ \det(A + cE) = \det A + c \cdot s(A). \] (8)

**Proof.** By properties of the determinant and (7) we obtain
\[
\det(A + cE) = \det \begin{bmatrix}
  a_{11} + c & a_{12} + c & \cdots & a_{1n} + c \\
  a_{21} + c & a_{22} + c & \cdots & a_{2n} + c \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} + c & a_{n2} + c & \cdots & a_{nn} + c
\end{bmatrix} + c \cdot s^1(A) \\
= \det \begin{bmatrix}
  a_{11} & a_{12} + c & \cdots & a_{1n} + c \\
  a_{21} & a_{22} + c & \cdots & a_{2n} + c \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} + c & \cdots & a_{nn} + c
\end{bmatrix} + c \cdot s^1(A) + c \cdot s^2(A) \\
= \cdots = \det \begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix} \\
+ c \cdot \sum_{k=1}^{n} s_k(A) = \det A + c \cdot s(A). \quad \square
\]

The theorem below shows that under some conditions there must exist population equilibrium in mixed strategies, with support containing all pure strategies.

**Theorem.** Let \([A, A^T]\) be an evolutionary game, where \( A \in \mathbb{R}^{n \times n} \), \( s(A) \neq 0 \) and all \( s^i(A) \) are negative (positive). If \( f_i \) denotes the proportion of the \( i \)th type \( (i = 1, \ldots, n) \) in the PE of the given population, then:
\[ f_i : f_j = s^i(A) : s^j(A); \quad i, j = 1, \ldots, n \] (9)
and the fitness \( E(A) \) of the entire population per capita is given by formula:
\[ E(A) = \frac{\det A}{s(A)}. \] (10)

**Proof.** When Column is playing her strategies with probabilities \( y_1, \ldots, y_n \), then Row’s expected payoffs, when he plays his \( i \)th strategy \( (i = 1, \ldots, n) \), are
\[ E_i = \sum_{j=1}^{n} a_{ij} y_j \quad (i = 1, \ldots, n). \] (11)
If Row is indifferent among these strategies, all his expected payoffs are the same. This means that Column must solve the following system of linear equations:

\[ Ay = c; \quad \text{where } c = [c, \ldots, c]^T \text{ (c is unknown value)} \] (12)

We will consider two cases, because if \( s(A) \neq 0 \), then it has to be \( \text{rank}(A) = n \) or \( \text{rank}(A) = n - 1 \). Really, assume that \( s(A) \neq 0 \), and at the same time \( \text{rank}(A) < n - 1 \). But in this case all cofactors of \( A \) with respect of any row (column) are equal to zero. Since for rows we have \( s_k(A) = \det A(k) = 0 \) (for columns we have \( s_k(A) = \det A(k) = 0 \)), then from (3), (4) we get \( s(A) = 0 \) (contradiction).

**Case 1** (\( A \) is non-singular). If \( A \) is non-singular, then the system (12) has the unique solution, where

\[ y_j = \frac{c}{\det A} s^j(A); \quad j = 1, \ldots, n. \]

Hence, by (3) we get

\[ \sum_{j=1}^{n} y_j = 1 \Rightarrow \frac{c}{\det A} \sum_{j=1}^{n} s^j(A) = 1 \Rightarrow c = \frac{s(A)}{\det A} = 1. \]

So \( y_j = \frac{s^j(A)}{s(A)} \) and (9) follows, as \( p_i : p_j = y_i : y_j = s^i(A) : s^j(A); \ i, j = 1, \ldots, n. \)

Moreover, the determinant of \( A \) can be evaluated by \( s^j \) or \( s_i \) to obtain the following quasi-Laplacian expansion:

\[ \sum_{j=1}^{n} a_{kj} s^j(A) = \sum_{i=1}^{n} a_{ik} s_i(A) = \det A. \] (13)

In fact, if we consider the \( k \)th equation in the system \( Ay = c \), then we have:

\[ \sum_{j=1}^{n} a_{kj} s^j(A) = \frac{1}{s(A)} \sum_{j=1}^{n} a_{kj} s^j(A) = \frac{\det A}{s(A)} \Rightarrow \sum_{j=1}^{n} a_{kj} s^j(A) = \det A. \]

Similarly we get the second part of the expansion:

\[ \sum_{i=1}^{n} a_{ik} s_i(A) = \frac{1}{s(A)} \sum_{i=1}^{n} a_{ik} s_i(A) = \frac{\det A}{s(A)} \Rightarrow \sum_{i=1}^{n} a_{ik} s_i(A) = \det A. \]

**Case 2** (\( A \) is singular and \( \text{rank}(A) = n - 1 \)). Obviously, if \( \text{rank}(A) < n \), then \( \det A = 0 \). Without loss of generality we can assume that in the system \( Ay = c \) at least two coefficients of the \( n \)th equation, say \( a_{n1} \neq a_{n2} \), have to be different. Then, by the Kronecker–Capelli theorem, the system above is consistent if \( \text{rank}(A, c) = \text{rank}(A) = n - 1 \). This means that the system of the vectors

\[ v_i = [2a_{i1}, \ldots, 2a_{in}, c], \quad i = 1, \ldots, n \] (14)

is linear dependent. Assume that

\[ v_n = \sum_{i=1}^{n-1} \beta_i v_i, \quad \text{where } \sum_{i=1}^{n-1} \beta_i^2 > 0. \]

By comparing the last coefficients of (14), we have

\[ \sum_{i=1}^{n-1} \beta_i c = c. \]
We need to show now that \( c \) must be equal to zero.

Really, if \( c \neq 0 \), then \( \sum_{i=1}^{n-1} \beta_i = 1 \). As rank \((A) = n - 1\), the subsystem of (13)

\[
\begin{align*}
    a_{11} \beta_1 + \cdots + a_{n-1,1} \beta_{n-1} &= a_{n1} \\
    \cdots \\
    a_{1,n-1} \beta_1 + \cdots + a_{n-1,n-1} \beta_{n-1} &= a_{n,n-1}
\end{align*}
\]

is the Cramer’s system.

Therefore we obtain

\[
a_{n1} = \cdots = a_{n,n-1},
\]

which is in contrary to the assumption \( a_{n1} \neq a_{n2} \).

Hence \( c = \det A = 0 \). So in the case under consideration the system becomes the homogeneous system \( Ay = 0 \), i.e. the \( r \)th equation \((r = 1, \ldots, n)\) has the form:

\[
\sum_{j=1}^{n} a_{rj} y_j = 0,
\]

which, by (13), we can write as

\[
\sum_{j=1}^{n} a_{rj} s_j^i (A)t = 0; \quad t \in \mathbb{R}.
\]

Hence \( y_j = s_j^i (A)t, \quad j = 1, \ldots, n \).

Similarly as in Case 1, by (3), we get

\[
\sum_{j=1}^{n} y_j = \sum_{j=1}^{n} s_j^i (A)t = s(A)t = 1 \Rightarrow t = \frac{1}{s(A)}.
\]

Therefore

\[
y_j = s_j^i (A)t = \frac{s_j^i (A)}{s(A)}, \quad j = 1, \ldots, n.
\]

Similarly, when Row is playing his strategies with probabilities \( x_1, \ldots, x_n \) then Column’s expected payoffs, when she plays her \( j \)th strategy \((j = 1, \ldots, n)\), are equal to

\[
E_j = \sum_{i=1}^{n} a_{ij} x_i.
\]

If Column is indifferent among these strategies, all her expected payoffs are the same. This means that Row has to solve the following system of linear equations:

\[
A^T x = d, \quad \text{where } d = [d, \ldots, d]^T.
\]

But we have

\[
d = \frac{\det A^T}{s(A^T)} = \frac{\det A}{s(A)} = c,
\]

and by Lemma 1

\[
s(A^T) = s(A), \quad s_k(A^T) = s^k(A),
\]

so we obtain
\[ x_i = \frac{s_i(A^T)}{s(A^T)} = \frac{s^i(A)}{s(A)}; \quad i = 1, \ldots, n. \]

Therefore \[ f_i : f_j = x_i : x_j = s^i(A) : s^j(A); \quad i, j = 1, \ldots, n. \]

Note that if we consider the population of \( N \) individuals, then we can interpret \( f_i \) either as there is the set of \( f_i \cdot N \) individuals with the \( i \)th type, or alternatively, as the probability that an individual will play the \( i \)th type equals \( f_i \).

Bishop and Cannings [14] showed a useful result that if \( p \) is a mixed evolutionarily stable strategy (ESS) with support \( p_1, \ldots, p_n \), then

\[ E(p_1, p) = E(p_2, p) = \cdots = E(p_n, p) = E(p, p). \]

In other words all supported strategies must have the same fitness when played against the ESS. Following theorem, we can complete Bishop–Cannings result as follows in the corollary below:

**Corollary 1.** If in the evolutionary bi-matrix game \([A, A^T]\) a strategy \( p \) is a mixed ESS with support \( p_1, \ldots, p_n \), then

\[ E(p_i, p) = \frac{\det A}{s(A)}; \quad i = 1, \ldots, n \quad (15) \]

or, using (8), we can write

\[ E(p_i, p) = \frac{\det A}{\det(A + E) - \det A}; \quad i = 1, \ldots, n. \quad (16) \]

**Corollary 2.** From Lemmas 2, 3, and Theorem, and Property 3 of \( s(A) \), we immediately obtain well known properties of matrix games:

\[ E(A + cE) = \frac{\det(A + cE)}{s(A + cE)} = \frac{\det A + cs(A)}{s(A)} = E(A) + c, \quad (17) \]

\[ E(cA) = \frac{\det(cA)}{s(cA)} = \frac{c^n \det A}{c^{n-1}s(A)} = c \frac{\det A}{s(A)} = cE(A). \quad (18) \]

**Corollary 3.** Cheon’s Formula [15] says, that at the PE the fitness of the game with the parameter \( \lambda \) and the fitness its altruistic dual game are equal:

\[ E(A_{1-\lambda}) = E(A_\lambda). \quad (19) \]

Since (10), (5), and (6) we can give the short proof of (19):

\[ E(A_{1-\lambda}) = \frac{\det A_{1-\lambda}}{s(A_{1-\lambda})} = \frac{\det A_{1-\lambda}^T}{s(A_{1-\lambda}^T)} = \frac{\det A_\lambda}{s(A_\lambda)} = E(A_\lambda). \]

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