JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **129**, 231–242 (1988)

Some Global Generalizations of the Birkhoff–Kellogg Theorem and Applications

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Submitted by V. Lakshmikantham

Received July 3, 1986

The purpose of this paper is to investigate some global generalizations of the famous Birkhoff–Kellogg theorem [1] and give some applications. The main tool in this paper is the following lemma, which was proved in [2] by Guo Dajun:

LEMMA 1. *Let X be an infinite-dimensional Banach space, D a bounded open set in X, and $A\colon \bar{D} \to X$ a completely continuous operator. Suppose that $Ax \neq \mu x$ for $x \in \partial D$, $0 < \mu \leqslant 1$, and*

$$\inf_{x \in \partial D} \|Ax\| > 0. \tag{1}$$

Then the Leray–Schauder degree $\deg(I - A, D, 0) = 0$.

It should be noticed that Lemma 1 has been proved in [13] by the same method as [2]. Some applications and generalizations of Lemma 1 are discussed in [3]–[7].

In this paper, we suppose that X is an infinite-dimensional Banach space and $A\colon X \to X$ is a completely continuous operator. We shall denote by E the Banach space $R^1 \times X$ with norm $\|(\lambda, x)\| = (\lambda^2 + \|x\|^2)^{1/2}$. The closure of the set of nonzero solutions of the equation $x = \lambda Ax$ will be denoted by L, i.e.,

$$L = \overline{\{(\lambda, x) \mid (\lambda, x) \in E,\ x = \lambda Ax,\ x \neq \theta\}}.$$

0022-247X/88 $3.00

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THEOREM 1. *Suppose that* (i) *there exists a bounded open set* D *in* X, $\theta \in D$, *such that the condition* (1) *is satisfied, and* (ii) $A\theta = \theta$ *and* A *is Frechet differentiable at* θ. *Then* L *possesses a maximal subcontinuum* (*i.e., a maximal closed connected subet of* L) C *which is unbounded,* $C \subset (0, +\infty) \times X$, *and there exists* $\bar{\lambda} > 0$ *such that*

(i) $C \cap ([\bar{\lambda}, +\infty) \times D)$ *is unbounded*;

(ii) $C \cap (((0, +\infty) \times X) \backslash ((\bar{\lambda}, +\infty) \times \bar{D}))$ *is unbounded*;

(iii) $C \cap ([\bar{\lambda}, +\infty) \times \partial D) = \varnothing$.

To prove Theorem 1, we need some lemmas.

LEMMA 2. *Suppose that the conditions of Theorem* 1 *are satisfied, then there exists* $\bar{\lambda} > 0$ *such that*

$$L \cap ([\bar{\lambda}, +\infty) \times \partial D) = \varnothing, \tag{2}$$

$$\deg(I - \lambda A, D, 0) = 0 \qquad \text{for any } \lambda \geqslant \bar{\lambda}. \tag{3}$$

Proof. Let $\beta = \inf_{x \in \partial D} \|Ax\| > 0$, $M = \sup_{x \in \partial D} \|x\|$, $\bar{\lambda} > M/\beta$, then

$$\|\lambda A x\| \geqslant \bar{\lambda}\beta > \frac{M}{\beta} \cdot \beta = M \geqslant \|x\| \tag{4}$$

for $\lambda \geqslant \bar{\lambda}$, $x \in \partial D$. By Lemma 1, (3) holds for $\lambda \geqslant \bar{\lambda}$. That (2) holds is obvious. This completes the proof of the lemma.

We choose two sequences of real numbers λ_n and R_n, such that

$$\bar{\lambda} < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \qquad (n = 1, 2, 3, ...),$$

$$\sup_{x \in D} \|(\bar{\lambda}, x)\| = M_1 < R_1 < R_2 < \cdots < R_n < \cdots, \qquad (n = 1, 2, 3, ...)$$

$$\lim_{n \to \infty} \lambda_n = +\infty, \qquad \lim_{n \to \infty} R_n = +\infty, \tag{5}$$

and λ_n are not characteristic values of A'_θ. Let

$$F_n = \{(\lambda_n, x) \in E \mid x \in D\},$$

$$G_n = \{(\lambda, x) \in (E \backslash ((\bar{\lambda}, +\infty) \times \bar{D})) \mid \|(\lambda, x)\| = R_n\},$$

$$W_n = \{(\lambda, x) \in E \mid x \in D, \lambda \in [\bar{\lambda}, \lambda_n]\}$$
$$\cup \{(\lambda, x) \in (E \backslash ((\bar{\lambda}, +\infty) \times \bar{D})) \mid \|(\lambda, x)\| \leqslant R_n\}.$$

LEMMA 3. *Suppose that the conditions of Theorem* 1 *are satisfied, then for each* n, *there exists a maximal subcontinuum* C_n *of* $L \cap W_n$ *such that*

$$C_n \cap F_n \neq \varnothing, \qquad C_n \cap G_n \neq \varnothing. \tag{6}$$

Proof. Let n be fixed and $T = \{(\lambda_n, x) \mid x \in D, x = \lambda_n Ax, x \neq \theta\}$. Then T is a compact set, For any $u \in T$, we denote by $C(u)$ the maximal subcontinuum, containing u, of $L \cap W_n$. We claim that there exists a $u \in T$ such that $C(u)$, corresponding to u, satisfies $C(u) \cap G_n \neq \varnothing$. In fact, if this is false, then for any $u \in T$, $C(u) \cap G_n = \varnothing$. By the same method as the proof of Lemma 1.2 in [8], we can prove that there exists a relatively open subset $U(u)$ of W_n such that $\partial U(u) \cap L = \varnothing$, $(\lambda_n, \theta) \bar{\in} \overline{U(u)}$, $d(U(u), G_n) > 0$, $d(U(u), [\bar{\lambda}, +\infty) \times \partial D) > 0$, where $\partial U(u)$ is the boundary of $U(u)$ in W_n and $d(\cdot, \cdot)$ denotes the distance between two sets. Obviously, in the metric space $\{\lambda_n\} \times D$, $\{U(u) \cap (\{\lambda_n\} \times D) \mid u \in T\}$ is an open covering of T. Since T is a compact set in $\{\lambda_n\} \times D$, so there exist $u_i \in T$ $(i = 1, 2, ..., k)$ such that $\{U(u_i) \cap (\{\lambda_n\} \times D) \mid i = 1, 2, ..., k\}$ is also an open covering of T. Let

$$U = \bigcup_{i=1}^{k} U(u_i), \tag{7}$$

then U is an open subset of W_n,

$$\partial U \cap L = \varnothing, \qquad (\lambda_n, \theta) \bar{\in} \bar{U}, \tag{8}$$

and $d(U, [\lambda, +\infty) \times \partial D) > 0$, $d(U, G_n) > 0$. Obviously, $T \subset U$.

By the same method as the proof of Theorem 1 in [10], we can prove

$$\deg(I - \lambda_n A, U(\lambda_n), 0) \equiv 0 \qquad (\mathrm{mod}\ 2), \tag{9}$$

$$\deg(I - \lambda_n A, U_1(\lambda_n), 0) = 0, \tag{10}$$

where $U(\lambda_n) = U \cap (\{\lambda_n\} \times X)$, $U_1(\lambda_n) = U \cap ((\{\lambda_n\} \times X) \backslash (\{\lambda_n\} \times \bar{D}))$. Let $U_2(\lambda_n) = U(\lambda_n) \backslash U_1(\lambda_n)$, then by (9), (10) we have

$$\deg(I - \lambda_n A, U_2(\lambda_n), 0) \equiv 0 \qquad (\mathrm{mod}\ 2). \tag{11}$$

Since λ_n is not a characteristic value of A'_θ, so

$$|\mathrm{ind}(I - \lambda_n A, \theta)| = 1. \tag{12}$$

From (11), (12), we obtain $\deg(I - \lambda_n A, D, 0) \equiv 1 \pmod 2$, which contradicts Lemma 2. Lemma 3 is proved.

For each n, we take a maximal subcontinuum C_n of $L \cap W_n$ such that

$$C_n \cap F \neq \varnothing, \qquad C_n \cap G_n \neq \varnothing \qquad (n = 1, 2, 3, ...). \tag{13}$$

Define the superior limit H of $\{C_n \mid n = 1, 2, 3, ...\}$ as

$$H = \overline{\lim_{n \to \infty}}\, C_n = \{z \mid \text{there exist a subsequence } \{n_k\} \text{ of } \{n\} \text{ and } z_{n_k} \in C_{n_k} \text{ such that } \lim_{k \to \infty} z_{n_k} = z\}. \tag{14}$$

For any $z \in H$, we denote by $C(z)$ the maximal subcontinuum, containing z, of H. Let

$$\lambda(z) = \sup\{\lambda \mid (\lambda, x) \in (C(z) \cap ([\bar{\lambda}, +\infty) \times D))\}, \tag{15}$$

$$R(z) = \sup\{\|(\lambda, x)\| \mid (\lambda, x) \in (C(z) \cap (E \backslash ((\bar{\lambda}, +\infty) \times \bar{D})))\}. \tag{16}$$

LEMMA 4. *Suppose that the conditions of Theorem* 1 *are satisfied, then for any* $z \in H$, *either* $\lambda(z) = +\infty$ *or* $R(z) = +\infty$.

Proof. If the lemma is false, then there exists $z_0 \in H$ such that $\lambda(z_0) < +\infty$, $R(z_0) < +\infty$. Obviously, $C(z_0)$ is a compact set. By the same method as the proof of Lemma 1.2 in [8], we can prove that there exists a bounded open subset U of E such that

$$C(z_0) \subset U, \qquad \partial U \cap H = \varnothing. \tag{17}$$

Since U is bounded, so there exists n' such that

$$\sup\{\lambda \mid (\lambda, x) \in (U \cap ([\bar{\lambda}, +\infty) \times D))\} < \lambda_{n'},$$
$$\sup\{\|(\lambda, x)\| \mid (\lambda, x) \in (U \cap (E \backslash ((\bar{\lambda}, +\infty) \times \bar{D})))\} < R_{n'}.$$

By the definition of H, there exist a subsequence $\{n_k\}$ of $\{n\}$ and $z_{n_k} \in C_{n_k}$ such that $\lim_{k \to \infty} z_{n_k} = z_0$. Without loss of generality we can assume that $n_k \geqslant n'$ and $z_{n_k} \in U$ for all k. By the connectivity of C_{n_k} and (6) we have $C_{n_k} \cap \partial U \neq \varnothing$ for all n_k. Take $y_{n_k} \in C_{n_k} \cap \partial U$, then there exist a subsequence $\{y_{n_{k_i}}\}$ of $\{y_{n_k}\}$ and $y^* \in \partial U$ such that $\lim_{i \to \infty} y_{n_{k_i}} = y^*$. By the definition of H, $y^* \in H$, so $y^* \in H \cap \partial U$, which contradicts (17). This contradiction proves the lemma.

Let $H(\bar{\lambda}) = H \cap (\{\bar{\lambda}\} \times D)$, then obviously $H(\bar{\lambda})$ is a nonempty compact set. We can prove the following lemma:

LEMMA 5. *Suppose that the conditions of Theorem* 1 *are satisfied, then there exists a* $z^* \in H(\bar{\lambda})$ *such that* $\lambda(z^*) = +\infty$, $R(z^*) = +\infty$.

Proof. Let

$$M = \{z \in H(\bar{\lambda}) \mid \lambda(z) = +\infty\}, \qquad N = \{z \in H(\bar{\lambda}) \mid R(z) = +\infty\},$$
$$M_1 = \{z \in H(\bar{\lambda}) \mid R(z) < R_1\}, \qquad N_1 = \{z \in H(\bar{\lambda}) \mid \lambda(z) < \lambda_1\},$$
$$M_n = \{z \in H(\bar{\lambda}) \mid R_{n-1} \leqslant R(z) < R_n\} \quad \text{for} \quad n = 2, 3, ...,$$
$$N_n = \{z \in H(\bar{\lambda}) \mid \lambda_{n-1} \leqslant \lambda(z) < \lambda_n\} \quad \text{for} \quad n = 2, 3, ...$$

If the lemma is false, then $M \cap N = \emptyset$. By Lemma 4, we have

$$M \cup N = H(\bar{\lambda}), \qquad M = \bigcup_{n=1}^{\infty} M_n, \qquad N = \bigcup_{n=1}^{\infty} N_n. \tag{18}$$

Let $z \in M$, then $R(z) < +\infty$, and by the same method as the proof of (7) in Lemma 3 we can prove that there exists a bounded open subset $U(z)$ of $(E \backslash ((\bar{\lambda}, +\infty) \times \bar{D})) \cup ([\bar{\lambda}, \lambda_1] \times D)$ such that

$$(C(z) \cap ((E \backslash ((\bar{\lambda}, +\infty) \times \bar{D})) \cup ([\bar{\lambda}, \lambda_1] \times D))) \subset U(z),$$

$\partial U(z) \cap H = \emptyset$, $d(U(z), [\bar{\lambda}, +\infty) \times \partial D) > 0$, and

$$\sup\{\|(\lambda, x)\| \mid (\lambda, x) \in (U(z) \cap (E \backslash ((\bar{\lambda}, +\infty) \times \bar{D})))\} < +\infty,$$

where $\partial U(z)$ is the boundary of $U(z)$ in $(E \backslash ((\bar{\lambda}, \infty) \times \bar{D})) \cup ([\bar{\lambda}, \lambda_1] \times D)$.

By the same method, for $z \in N$ there exists a bounded open subset $V(z)$ of $([\bar{\lambda}, +\infty) \times D) \cup \{(\lambda, x) \in (E \backslash ((\bar{\lambda}, +\infty) \times \bar{D})) \mid \|(\lambda, x)\| \leqslant R_1\}$ such that

$$\begin{gathered}(C(z) \cap (([\bar{\lambda}, +\infty) \times D) \\ \cup \{(\lambda, x) \in (E \backslash ((\bar{\lambda}, +\infty) \times \bar{D})) \mid \|(\lambda, x)\| \leqslant R_1\})) \subset V(z), \\ \partial V(z) \cap H = \emptyset, \qquad d(V(z), [\bar{\lambda}, +\infty) \times \partial D) > 0, \\ \sup\{\lambda \mid (\lambda, x) \in (V(z) \cap ([\bar{\lambda}, +\infty) \times D))\} < +\infty.\end{gathered}$$

Obviously, $\{U(z) \cap (\{\bar{\lambda}\} \times D) \mid z \in M\} \cup \{V(z) \cap (\{\bar{\lambda}\} \times D) \mid z \in N\}$ is an open covering of $H(\bar{\lambda})$. Since $H(\bar{\lambda})$ is a compact set, so there exist z_i $(1 \leqslant i \leqslant p)$, such that $z_i \in M$ $(1 \leqslant i \leqslant m)$, $z_i \in N$ $(m+1 \leqslant i \leqslant p)$, and

$$\{U(z_i) \cap (\{\bar{\lambda}\} \times D) \mid i = 1, 2, ..., m)\} \cup \{V(z_i) \cap (\{\bar{\lambda}\} \times D) \mid i = m+1, ..., p\}$$

is also an open covering of $H(\bar{\lambda})$. Let

$$U_1 = \bigcup_{i=1}^{m} U(z_i), \qquad V_1 = \bigcup_{i=m+1}^{p} V(z_i),$$

then U_1 is a bounded open subset of $(E \backslash ((\bar{\lambda}, +\infty) \times \bar{D})) \cup ([\bar{\lambda}, \lambda_1] \times D)$, $\partial U_1 \cap H = \emptyset$, and

$$\sup\{\|(\lambda, z)\| \mid (\lambda, x) \in (U_1 \cap (E \backslash ((\bar{\lambda}, +\infty) \times \bar{D})))\} < +\infty. \tag{19}$$

Similarly, V_1 is also a bounded open subset of $([\bar{\lambda}, +\infty) \times D) \cup \{(\lambda, x) \in (E \backslash ((\bar{\lambda}, +\infty) \times \bar{D})) \mid \|(\lambda, x)\| \leqslant R_1\}$, $\partial V_1 \cap H = \emptyset$, and

$$\sup\{\lambda \mid (\lambda, x) \in (V \backslash ([\bar{\lambda}, +\infty) \times D))\} < +\infty. \tag{20}$$

Let $J = \partial U_1 \cup \partial V_1 \cup ((\{\bar{\lambda}\} \times D) \backslash (U_1 \cup V_1)) \cup ([\bar{\lambda}, +\infty) \times \partial D)$. Obviously, $J \cap H = \emptyset$ and there exists n' such that $C_n \cap J = \emptyset$ for $n \geqslant n'$. By (19), (20), there exists n'' such that

$$R_n > \sup\{\|(\lambda, x)\| \mid (\lambda, x) \in (U_1 \cap (E \backslash ((\bar{\lambda}, +\infty) \times \bar{D})))\},$$
$$\lambda_n > \sup\{\lambda \mid (\lambda, x) \in (V_1 \cap (\lambda_1, +\infty) \times D)\}$$

for $n \geqslant n''$. Take $n^* = \max\{n', n''\}$. Since $C_n \cap J = \emptyset$ for $n \geqslant n^*$, C_n is connective, so $C_n \cap ((\{\bar{\lambda}\} \times D) \cap U_1 \cap V_1) \neq \emptyset$ for $n \geqslant n^*$. Hence

$$H_1(\bar{\lambda}) = H(\bar{\lambda}) \cap U_1 \cap V_1 \neq \emptyset.$$

It is easy to see that $H_1(\bar{\lambda})$ is a compact set. Let $z \in H_1(\bar{\lambda})$, if $z \in M$, then by $\partial V_1 \cap H = \emptyset$, we have $R(z) > R_1$, hence $z \bar{\in} M_1$. Similarly, if $z \in N$, then $z \bar{\in} N_1$. Thus

$$H_1(\bar{\lambda}) \cap (M_1 \cup N_1) = \emptyset. \tag{21}$$

Substituting separately R_2, λ_2, $H_1(\bar{\lambda})$ for R_1, λ_1, $H(\bar{\lambda})$ as above, we can similarly prove that there exists a nonempty compact set $H_2(\bar{\lambda}) \subset H_1(\bar{\lambda})$ such that

$$H_2(\bar{\lambda}) \cap (M_2 \cup N_2) = \emptyset. \tag{22}$$

Similarly for each $n = 3, 4, ...,$ there exists a nonempty compact set $H_n(\bar{\lambda}) \subset H_{n-1}(\bar{\lambda})$ such that

$$H_n(\bar{\lambda}) \cap (M_n \cup N_n) = \emptyset. \tag{23}$$

Let $H^* = \bigcap_{n=1}^{\infty} H_n(\bar{\lambda})$, then $H^* \subset H(\bar{\lambda})$ is a nonempty compact set by Theorem 1.2.18 in [14]. From (21), (22), (23) it follows that $H^* \cap (M_n \cup N_n) = \emptyset$ for all n. Hence $H^* \cap (M \cup N) = \emptyset$, which contradicts (18) since H^* is a nonempty subset of $H(\bar{\lambda})$. This contradiction proves the lemma.

Proof of Theorem 1. By Lemma 5 there exists a maximal subcontinuum C^* of H such hat $C^* \cap ([\bar{\lambda}, +\infty) \times D)$ and $C^* \cap (E \backslash ((\bar{\lambda}, +\infty) \times \bar{D}))$ are all unbounded. From the definition of H it is obvious that $C^* \subset L$. Let C be the maximal subcontinuum of L containing C^*, then $C \cap ([\bar{\lambda}, +\infty) \times D)$ and $C \cap (E \backslash ((\bar{\lambda}, +\infty) \times \bar{D}))$ are all unbounded. The conclusion (iii) follows from (2). By the same method as the proof of the conclusion (ii) of Theorem 1 in [2] we can prove $C \subset (0, +\infty) \times X$. The proof of Theorem 1 is completed.

COROLLARY 1. *Suppose that the conditions of Theorem* 1 *are satisfied. Then L possesses a maximal subcontinuum C' which is unbounded, $C' \subset ((-\infty, 0) \times X)$, and there exists $\lambda' < 0$ such that*

(i) $C' \cap ((-\infty, \lambda') \times D)$ *is unbounded;*

(ii) $C' \cap (((-\infty, 0) \times X) \backslash ((-\infty, \lambda') \times \bar{D}))$ *is unbounded;*

(iii) $C' \cap ((-\infty, \lambda'] \times \partial D) = \varnothing$.

Proof. Apply Theorem 1 to the operator $-A$.

Remark 1. In [11], Sun Jingxian point out that the proof of Theorem 2.1 in [12] is false. By the same method as the proof of Theorem 1 in this paper we can give a correct proof for Theorem 2.1 in [12].

In Theorem 1, we suppose that the operator A is differentiable at θ. If we do not suppose that the operator A is differentiable at θ, then we can prove

THEOREM 2. *Suppose that there exists a bounded open subset D in X, $\theta \in D$, such that the condition* (1) *is satisfied. Then L possesses a maximal subcontinuum C which is unbounded and there exists $\bar{\lambda} > 0$ such that*

(i) $C \cap (((0, +\infty) \times X) \backslash ((\bar{\lambda}, +\infty) \times \bar{D}))$ *is unbounded;*

(ii) $C \cap ([\bar{\lambda}, +\infty) \times \partial D) = \varnothing$, $C \cap (\{0\} \times (X \backslash \{\theta\})) = \varnothing$;

and either

(iii) $C \cap ([\lambda, +\infty) \times D)$ *is unbounded, or*

(iii)* $C \cap ([0, +\infty) \times \{\theta\}) \neq \theta$.

To prove Theorem 2, we first prove a lemma. Let $\beta = \inf_{x \in \partial D} \|Ax\|$, $M = \sup_{x \in \partial D} \|x\|$, $\bar{\lambda} > M/\beta$, $M_1 = \sup_{x \in \bar{D}} \|(\bar{\lambda}, x)\|$; we choose three sequences λ_n, R_n, and r_n,

$$\bar{\lambda} < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots,$$

$$M_1 < R_1 < R_2 < \cdots < R_n < \cdots,$$

$$r_1 > r_2 > r_3 > \cdots > r_n > \cdots$$

$(n = 1, 2, 3, \ldots)$ such that $r_1 < \inf_{x \in \partial D} \|x\|$ and

$$\lim_{n \to \infty} \lambda_n = +\infty, \qquad \lim_{n \to \infty} R_n = +\infty, \qquad \lim_{n \to \infty} r_n = 0.$$

Let $B_n = \{x \in X \mid \|x\| < r_n\}$,

$$F_n = \{(\lambda_n, x) \in E \mid x \in (D \backslash B_n)\} \cup \{(\lambda, x) \mid \lambda \in [0, \lambda_n], x \in \partial B_n\},$$

$$G_n = \{(\lambda, x) \in (E \backslash ((\bar{\lambda}, +\infty) \times \bar{D})) \mid \|(\lambda, x)\| = R_n\},$$

$$W_n = \{(\lambda, x) \in (E \backslash ((\bar{\lambda}, +\infty) \times \bar{D})) \mid \|(\lambda, x)\| \leqslant R_n\}$$
$$\cup \{(\lambda, x) \in E \mid x \in (D \backslash B_n), \lambda \in [0, \lambda_n]\}.$$

LEMMA 6. *Suppose that the conditions of Theorem 2 are satisfied, then for each n, there exist maximal subcontinuums C_n of $L \cap W_n$ such that*

$$C_n \cap F_n \neq \emptyset, \qquad C_n \cap G_n \neq \emptyset. \tag{24}$$

Proof. Let n be fixed. We define the operator A_n on X by

$$\begin{aligned} A_n x &= Ax & &\text{when} \quad \|x\| \geqslant r_n \\ &= \left(\frac{2}{r_n}\|x\| - 1\right) A\left(\frac{r_n}{\|x\|} x\right) & &\text{when} \quad \frac{r_n}{2} < \|x\| < r_n \\ &= 0 & &\text{when} \quad \|x\| \leqslant \frac{r_n}{2}. \end{aligned}$$

Obviously, A_n is also completely continuous. Let

$$L_n \overline{\{(\lambda, x) \in E \mid x = \lambda A_n x, x \neq \theta\}},$$

then by Theorem 1, L_n possesses a maximal subcontinuum C'_n which satisfies

(i) $C'_n \cap ([\bar{\lambda}, +\infty) \times D$ is unbounded;
(ii) $C'_n \cap (((0, +\infty) \times X) \backslash ((\bar{\lambda}, +\infty) \times \bar{D}))$ is unbounded;
(iii) $C'_n \subset (0, +\infty) \times X$ and $C'_n \cap ([\bar{\lambda}, +\infty) \times \partial D) = \emptyset$.

For $z \in C'_n \cap F_n$, we denote by $C'_n(z)$ the maximal subcontinuum, containing z, of $C'_n \cap W_n$. If for all $z \in C'_n \cap F_n$, $C'_n(z) \cap G_n = \emptyset$, then by the same method as the proof of Lemma 3, we can prove that there exists a bounded open subset U_n of W_n such that

$$F_n \cap C'_n \subset U, \qquad \partial U_n \cap L_n = \emptyset, \qquad d(U_n, G_n) > 0.$$

It is obvious that $C'_n \subset (U_n \cup ([\bar{\lambda}, +\infty) \times D) \backslash ([0, \bar{\lambda}] \times B_n))$, which contradicts the property (ii) of C'_n mentioned above. This contradiction shows that there exists $z^* \in F_n \cap C_n$ such that $C'_n(z^*) \cap G_n \neq \emptyset$. Let $C_n = C'_n(z^*)$, then from the definition of A_n it follows that $C_n \subset L$, which implies C_n is also a maximal subcontinuum of $W_n \cap L$ and (24) holds. This completes the proof.

Proof of Theorem 2. From Lemma 6 it follows that for each n there exist subcontinuums C_n of L such that (24) holds. By the same method of the proofs of Lemma 4 and Lemma 5, we can prove that there exists a maximal subcontinuum C of L such that

$$C \cap F_n \neq \emptyset, \qquad C \cap G_n \neq \emptyset \qquad (n = 1, 2, 3, ...).$$

It is obvious that the conclusions (i), (ii) hold. Finally, if $C \cap ([\bar{\lambda}, +\infty) \times D)$ is bounded, then from the definition of F_n it is easy to see that there exists n^* such that for all $n \geqslant n^*$, $(C \cap ([0, \lambda_{n^*}] \times \partial B_n)) \neq \varnothing$. Take $(\mu_n, x_n) \in (C \cap ([0, \lambda_{n^*}] \times \partial B_n))$, and we can assume without loss of generality that $\lim_{n \to \infty} \mu_n = \mu^*$. Since $\lim_{n \to \infty} \|x_n\| = 0$, so $\lim_{n \to \infty} (\mu_n, x_n) = (\mu^*, \theta)$, which implies $(\mu^*, \theta) \in C$. This completes the proof.

Remark 2. Theorem 1 and Theorem 2 are obviously global generalizations of the Brikhoff–Kellogg theorem. They give the main features of the global structure of L under the conditions of the Birkhoff–Kellogg theorem.

Remark 3. If A is a set-contraction operator, then we can prove results similar to Theorem 1 and Theorem 2 by the conclusions obtained in [7]. If A is a cone mapping, we can also prove similar results.

THEOREM 3. *Suppose that A is differentiable at θ, $A\theta = \theta$, and*

$$\lim_{\|x\| \to \infty} \frac{\|Ax\|}{\|x\|} = +\infty. \tag{25}$$

Then

(i) *0 is a unique asymptotic bifurcation point of A;*

(ii) *L possesses a maximal subcontinuum C^+, passing through* $(0, \infty)$ *(i.e., for any $\delta > 0$, $M > 0$, there exists $(\lambda, x) \in C^+$ such that $|\lambda| < \delta$, $\|x\| > M$), $C^+ \subset (0, +\infty) \times X$, and for any $\lambda > 0$, there exists x_λ such that $(\lambda, x_\lambda) \in C^+$;*

(iii) *L possesses a maximal subcontinuum C^-, passing through* $(0, \infty)$, *$C^- \subset (-\infty, 0) \times X$, and for any $\lambda < 0$, there exists x_λ such that $(\lambda, x_\lambda) \in C^-$;*

(iv) $\lim_{\lambda \to 0, (\lambda, x_\lambda) \in C^+ \cup C^-} \|x_\lambda\| = +\infty$.

Proof. Suppose that $\bar{\lambda}$ is a given positive number. By (25), there exists $R > 0$ such that

$$\|Ax\| \geqslant \frac{2}{\bar{\lambda}} \|x\|, \qquad \text{when} \quad \|x\| \geqslant R. \tag{26}$$

Let $D = x \mid x \in X, \|x\| < R\}$, $\beta = \inf_{x \in \partial D} \|Ax\|$, $M = \sup_{x \in \partial D} \|x\|$, then from (26) it follows that $\beta \geqslant (2/\bar{\lambda}) M$, i.e., $\bar{\lambda} > M/\beta$. By Theorem 1 (from the proof of Theorem 1 we know that for any $\bar{\lambda} > M/\beta$ the conclusion of Theorem 1 hold) L possesses a maximal subcontinuum $C^+ \subset (0, +\infty) \times X$ such that

(i)* $C^+ \cap ([\bar{\lambda}, +\infty) \times D)$ is unbounded;

(ii)* $C^+ \cap (((0, +\infty) \times X) \backslash ((\bar{\lambda}, +\infty) \times \bar{D}))$ is unbounded;

(iii)* $C^+ \cap ([\bar{\lambda}, +\infty) \times \partial D) = \varnothing$.

It is obvious that for any $\lambda \geqslant \bar{\lambda}$ there exists $x_\lambda \in D$ such that $(\lambda, x_\lambda) \in C^+$. By (26) we have $\|\lambda Ax\| \geqslant 2\|x\|$ for $\lambda \geqslant \bar{\lambda}$, $\|x\| \geqslant R$, i.e.,

$$L \cap ([\bar{\lambda}, +\infty) \times (X \backslash D)) = \varnothing. \tag{27}$$

From (27) and (ii)* it is obvious that $C^+ \cap ((0, \bar{\lambda}) \times X)$ is unbounded. Hence, by (25) we have

$$\lim_{(\lambda, x) \in C^+, \|x\| \to \infty} \lambda = \lim_{(\lambda, x) \in C^+, \|x\| \to \infty} \frac{\|x\|}{\|Ax\|} = 0,$$

which implies that 0 is an asymptotic bifurcation point of A and C^+ passes through $(0, \infty)$. By the connectivity of C^+ the conclusion (ii) is proved. Similarly, we can prove the conclusion (iii). The conclusion (i) follows from (25). The conclusion (iv) can be proved by the same method as the proof of conclusion (ii) of Theorem 1 in [2]. This completes the proof of Theorem 3.

Remark 4. Theorem 3 is an improvement and generalization of the main results in [2, 9].

Next we give some applications of the general results in this paper to integral equations.

Suppose that G is a bounded closed domain in R^n and $k(x, y)$ is continuous on $G \times G$. Consider Hammerstein integral equations

$$u(x) = \lambda \int_G k(x, y) f(y, u(y)) \, dy = \lambda Au(x) \tag{28}$$

in Banach space L^n.

THEOREM 4. *Suppose that* (i)

$$f(x, u) = \sum_{i=1}^{n} a_i(x) \, u^i,$$

where n is an even number, $a_n(x)$ is a bounded measurable function defined on G, $\inf_{x \in G} a_n(x) > 0$, and $a_i(x) \in L^{n/(n-i)}$ $(i = 1, 2, ..., n-1)$;

(ii) *there exists $c(x) \in L^{n/(n-1)}$ such that*

$$\int_G c(x) \, k(x, y) \, dx > 0 \qquad \textit{for} \quad y \in G.$$

Then

(i) 0 *is a unique asymptotic bifurcation of* A *defined by* (28);

(ii) L *possesses a maximal subcontinuum* C^+, *passing through* $(0, \infty)$, $C^+ \subset ((0, +\infty) \times L^n)$, *and for any* $\lambda > 0$, *there exists* u_λ *such that* $(\lambda, u_\lambda) \in C^+$;

(iii) L *possesses a maximal subcontinuum* C^-, *passing through* $(0, \infty)$, $C^- \subset ((-\infty, 0) \times L^n)$, *and for any* $\lambda < 0$, *there exists* u_λ *such that* $(\lambda, u_\lambda) \in C^-$;

(iv) $\lim_{\lambda \to 0, (\lambda, u_\lambda) \in C^+ \cup C^-} \|u_\lambda\| = +\infty$.

Proof. It is easy to prove that A is a completely continuous operator, acting from L^n into itself, $A\theta = \theta$, and A is differentiable at θ. Thus to complete the proof of the theorem, it is sufficient to show

$$\lim_{\|u\| \to \infty} \frac{\|Au\|}{\|u\|} = +\infty. \tag{29}$$

We define a bounded linear functional on L^n by

$$h(u(x)) = \int_G c(x)\, u(x)\, dx$$

and denote its norm by $\|h\|$. Then

$$\begin{aligned}
\|Au\| &\geqslant \frac{1}{\|h\|} |h(Au)| = \frac{1}{\|h\|} \left| \int_G c(x) \left[\int_G k(x, y) f(y, u(y))\, dy \right] dx \right| \\
&= \frac{1}{\|h\|} \left| \int_G f(y, u(y))\, dy \int_G c(x)\, k(x, y)\, dx \right| \\
&\geqslant \frac{1}{\|h\|} \left| \int_G a_n(y)[u(y)]^n\, dy \int_G c(x)\, k(x, y)\, dx \right| \\
&\quad - \frac{1}{\|h\|} \left| \int_G \sum_{i=1}^{n-1} a_i(y)[u(y)[y)]^i\, dy \int_G c(x)\, k(x, y)\, dx \right| \\
&\geqslant \frac{\tau\beta}{\|h\|} \|u\|^n - \frac{M}{\|h\|} \left| \int_G c(x)\, dx \right| \sum_{i=1}^{n-1} \|a_i\|^{n/(n-i)} \|u\|^i,
\end{aligned} \tag{30}$$

where $\tau = \inf_{x \in G} a_n(x)$, $\beta = \inf_{y \in G} \int_G c(x)\, k(x, y)\, dx$, and $M = \max_{(x,y) \in G \times G} |k(x, y)|$. From (30) it follows that (29) holds. This completes the proof.

ACKNOWLEDGMENT

This research was supported by the Science and Technique Foundation of the People's Republic of China.

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