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Cafiero approach to the Dieudonné type theorems

Paolo de Lucia [a,1], Endre Pap [b,*,2]

[a] Universita "Federico II," Dipartimento di Matematica e Applicazioni "Renato Caccioppoli," via Cinthia, 80126 Napoli, Italy
[b] Department of Mathematics and Informatics, University of Novi Sad, Trg D. Obradovića 4, 21000 Novi Sad, Serbia

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Abstract

An algebra of subsets of a normal topological space containing the open sets is considered and in this context the uniform exhaustivity and uniform regularity for a family of additive functions are studied. Based on these results the Cafiero convergence theorem with the Dieudonné type conditions is proved and in this way also the Nikodým–Dieudonné convergence theorem is obtained.

Keywords: Regular set function; Additive set function; Dieudonné theorem

1. Introduction

It is well known that the Nikodým convergence and the Nikodým boundedness theorems for measures fail, in general, in algebras of sets (see [4,5]) and that the same arrives for the Cafiero convergence theorem (see [6]). The classical Dieudonné conditions [8] ensure that the Nikodým theorems are, with suitable conditions, true in algebras. Many Dieudonné type theorems are known in literature [1–3,7,9–17].

In this paper we consider an algebra of subsets of a normal topological space containing the open sets and in this context we prove the Cafiero convergence theorem with the Dieudonné type conditions; in this way we obtain also the Nikodým–Dieudonné convergence theorem. The proofs are related with a study of uniform exhaustivity and uniform regularity for a family of additive functions following the pattern utilized in [3]. We are aware that this method works also in other more general situations, but we prefer for the sake of clarity to limit ourselves to the case of normal topological spaces.

* Corresponding author.
E-mail addresses: padeluci@unina.it (P. de Lucia), pap@im.ns.ac.yu, pape@eunet.yu (E. Pap).
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2. Regular set functions

Let (Ω, τ) be a Hausdorff normal topological space, i.e., for each disjoint pair of closed sets C_1, C_2 there are two disjoint open sets A_1, A_2 such that $C_1 \subseteq A_1, C_2 \subseteq A_2$. We will denote by $\mathcal{R}$ an algebra containing τ, and by $\mathcal{C}$ the set of all closed subsets of Ω.

Definition 2.1. A set function $\mu : \mathcal{R} \to \mathbb{R}$ is called $\mathcal{C}$-regular if for every $\varepsilon > 0$ and every $X \in \mathcal{R}$ there exists $H \in \mathcal{C}$ such that

$$H \subseteq X, \quad \left|\mu(Y)\right| < \varepsilon \quad \text{for every } Y \in \mathcal{R}_{X \setminus H}.$$

We denote by $\mathrm{ra}(\mathcal{R}, \Omega, \tau)$ the set of all finite additive $\mathcal{C}$-regular set functions from $\mathcal{R}$ into $\mathbb{R}$.

Let $\mathcal{G}$ be a subset of $\mathcal{R}$, we will say that μ is *$\mathcal{G}$-exhaustive*, *exhaustive* if $\mathcal{G} = \mathcal{R}$, if for every disjoint sequence $(X_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{G}$ we have $\lim_{k\to\infty} \mu(X_k) = 0$.

If $(\mu_n)_{n\in\mathbb{N}}$ is a sequence of functions from $\mathcal{R}$ into $\mathbb{R}$ and $\mathcal{G}$ is a subset of $\mathcal{A}$, then we say that $(\mu_n)_{n\in\mathbb{N}}$ is *uniformly $\mathcal{G}$-exhaustive* if for every disjoint sequence $(X_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{G}$ we have $\lim_{k\to\infty} \mu_n(X_k) = 0$ uniformly for $n \in \mathbb{N}$. It is clear what we intend saying that $(\mu_n)_{n\in\mathbb{N}}$ is uniformly $\mathcal{C}$-regular.

Lemma 2.2. *Let μ be a finite additive set function from $\mathcal{R}$ into $\mathbb{R}$, then the following statements are equivalent*:

(i) $\mu \in \mathrm{ra}(\mathcal{R}, \Omega, \tau)$;

(ii) *for every $\varepsilon > 0$ and $X \in \mathcal{R}$ there exist $A \in \mathcal{O}$ such that*

$$X \subseteq A, \quad \left|\mu(Y)\right| < \varepsilon \quad \textit{for every } Y \in \mathcal{R}_{A \setminus X};$$

(iii) *for every $\varepsilon > 0$ and $X \in \mathcal{R}$ there exist $A \in \tau$ and $H \in \mathcal{C}$ such that*

$$H \subseteq X \subseteq A, \quad \left|\mu(Y)\right| < \varepsilon \quad \textit{for every } Y \in \mathcal{R}_{A \setminus H}.$$

We need some properties of $\mathcal{C}$-regular set functions.

Lemma 2.3. *Let $\mu \in \mathrm{ra}(\mathcal{R}, \Omega, \tau)$ and let $\varepsilon > 0$, $B \in \mathcal{C}$ and $\Gamma \in \tau$ with $B \subseteq \Gamma$. Then there exist $A_1, A_2 \in \tau$ such that*

$$B \subseteq A_2 \subseteq \overline{A}_2 \subseteq A_1 \subseteq \Gamma,$$
$$\left|\mu(Y)\right| < \varepsilon \quad \textit{for every } Y \in \mathcal{R}_{A_1 \setminus B}, \qquad \left|\mu(Y)\right| < \frac{\varepsilon}{2} \quad \textit{for every } Y \in \mathcal{R}_{A_2 \setminus B}. \tag{1}$$

Proof. Let A_1, A_1' be two open sets such that

$$B \subseteq A_1' \subseteq A_1 \subseteq \Gamma,$$
$$\left|\mu(Y)\right| < \varepsilon \quad \text{for every } Y \in \mathcal{R}_{A_1 \setminus B}, \qquad \left|\mu(Y)\right| < \frac{\varepsilon}{2} \quad \text{for every } Y \in \mathcal{R}_{A_1' \setminus B}.$$

Then B and $\setminus A_1$ are two disjoint closed sets and so there exist two disjoint open sets A' and A'' such that

$$B \subseteq A', \qquad \setminus A_1 \subseteq A''.$$

Put $A_2 = A_1' \cap A'$, then we have $A_2 \subseteq A_1'$, $A_2 \subseteq \setminus A'' \subseteq A_1$, and this completes the proof. □

Lemma 2.4. *Let $\mu \in \mathrm{ra}(\mathcal{R}, \Omega, \tau)$. Then we have for every $A \in \tau$*

$$\mu(\mathcal{R}_A) \subseteq \left(\mu(\tau_A) - \mu(\tau_A)\right)^{-}.$$

Proof. Let $X \in \mathcal{R}_A$. For every $\varepsilon > 0$ there exist $G \in \tau$ and $H \in \mathcal{C}$ such that

$$H \subseteq X \subseteq G \subseteq A, \quad \left|\mu(Y)\right| < \varepsilon \quad \text{for every } Y \in \mathcal{R}_{G \setminus H},$$

then we have

$$\left|\mu(X) - \left(\mu(G) - \mu(G \setminus H)\right)\right| = \left|\mu(X \setminus H)\right| < \varepsilon$$

with $\mu(G), \mu(G \setminus H) \in \mu(\tau_A)$. □

Lemma 2.5. *A sequence $(\mu_n)_{n\in\mathbb{N}}$ of* $\operatorname{ra}(\mathcal{R}, \Omega, \tau)$ *uniformly τ-exhaustive is also uniformly $\mathcal{C}$-regular on every $H \in \mathcal{C}$.*

Proof. Let $\varepsilon > 0$ and $H \in \mathcal{C}$. Suppose that $A_1, \dots, A_n$ are τ-open sets such that

(i) $H \subseteq A_{i+1} \subseteq \overline{A}_{i+1} \subseteq A_i$ for every $i \in \{1, \dots, n-1\}$;
(ii) for every $i \in \{1, \dots, n\}$

$$\left|\mu_j(Y)\right| < \frac{\varepsilon}{2^i} \quad \text{for every } Y \in \mathcal{R}_{A_i \setminus H} \text{ and for every } j \in \{1, \dots, i\}.$$

By Lemma 2.3 there exists $A_{n+1} \in \tau$ such that

$$H \subseteq A_{n+1} \subseteq \overline{A}_{n+1} \subseteq A_n,$$
$$\left|\mu_j(Y)\right| < \frac{\varepsilon}{2^{n+1}} \quad \text{for every } Y \in \mathcal{R}_{A_{n+1} \setminus H} \text{ and for every } j \in \{1, \dots, n+1\}.$$

Therefore we can construct, by induction, a sequence $(A_n)_{n\in\mathbb{N}}$ of open sets such that

$$H \subseteq A_{n+1} \subseteq \overline{A}_{n+1} \subseteq A_n,$$
$$\left|\mu_j(Y)\right| < \frac{\varepsilon}{2^n} \quad \text{for every } Y \in \mathcal{R}_{A_n \setminus H} \text{ and every } j \in \{1, \dots, n\}. \tag{2}$$

To complete the proof it is enough to see that for every $\sigma > 0$ there exists $m \in \mathbb{N}$ such that

$$\left|\mu_i(Y)\right| < \varepsilon \quad \text{for every } Y \in \tau_{A_m \setminus H} \text{ and every } i \in \mathbb{N}.$$

Suppose the contrary. Then there exists $\sigma > 0$ such that for every $p \in \mathbb{N}$ there exists $Y \in \tau_{A_p \setminus H}$ and $i \in \mathbb{N}$ such that $|\mu_i(Y)| > \sigma$. Since, for every $n, i \in \mathbb{N}$ it is

$$\mu_i(Y) = \mu_i(Y \setminus \overline{A}_n) + \mu_i(Y \cap \overline{A}_n)$$

and $Y \cap \overline{A}_n$ belongs to $\mathcal{R}_{A_{n-1} \setminus H}$, for every $n > i$, we have

$$\left|\mu_i(Y \cap \overline{A}_n)\right| < \frac{\varepsilon}{2^{n-1}}.$$

Then

$$\lim_{n\to\infty} \mu_i(Y \setminus \overline{A}_n) = \mu_i(Y) \quad \text{for every } i \in \mathbb{N},$$

therefore there exists $q_i \in \mathbb{N}$ such that

$$\left|\mu_i(Y \setminus \overline{A}_r)\right| > \sigma \quad \text{for every } r \geqslant q_i.$$

We can now construct, by induction, two sequences of natural numbers, $(i_r)_{r\in\mathbb{N}}$ and $(q_r)_{r\in\mathbb{N}}$, the second of them increasing, and a sequence $(Y_r)_{r\in\mathbb{N}}$ of open sets such that

$$Y_r \subseteq A_{q_{r-1}} \setminus H, \quad \left|\mu_{i_r}(Y_r \setminus \overline{A}_{q_r})\right| > \sigma \quad \text{for every } r \in \mathbb{N} \setminus \{1\}.$$

From (2) it follows that $(Y_r \setminus \overline{A}_{q_r})_{r\in\mathbb{N}}$ is a sequence of pairwise disjoint open sets and this contradicts the uniform τ-exhaustivity of $(\mu_n)_{n\in\mathbb{N}}$. □

We can observe that if $(\mu_i)_{i\in\mathbb{N}}$ is a sequence of $\operatorname{ra}(\mathcal{R}, \Omega, \tau)$ uniformly τ-exhaustive form Lemma 2.5 it follows that for every $\varepsilon > 0$ and $H \in \mathcal{C}$ there exists $A \in \tau$ such that $H \subseteq A$ and $|\mu_i(Y)| < \varepsilon$ for every $Y \in \mathcal{R}_{A\setminus H}$ and $i \in \mathbb{N}$. But (Ω, τ) is normal then there exist two disjoint open sets Γ_1 and Γ_2 such that $H \subseteq \Gamma_1$, $\setminus A \subseteq \Gamma_2$ and then

$$H \subseteq \Gamma_1 \subseteq \setminus \Gamma_2 \subseteq A.$$

Therefore we have

Lemma 2.6. *If $(\mu_i)_{i\in\mathbb{N}}$ is a sequence of* $\mathrm{ra}(\mathcal{R},\Omega,\tau)$ *uniformly τ-exhaustive, then for every $\varepsilon>0$, $H\in\mathcal{C}$, and $A\in\tau$ containing H, there exists $F\in\mathcal{C}$ and $G\in\tau$ such that*

$$H\subseteq G\subseteq F\subseteq A,\quad \left|\mu_i(Y)\right|<\varepsilon\quad \textit{for every } Y\in\mathcal{R}_{F\setminus H} \textit{ and every } i\in\mathbb{N}.$$

Lemma 2.7. *A sequence $(\mu_n)_{n\in\mathbb{N}}$ of* $\mathrm{ra}(\mathcal{R},\Omega,\tau)$ *uniformly τ-exhaustive is also uniformly $\mathcal{C}$-exhaustive.*

Proof. Suppose that there exist $\varepsilon>0$, a disjoint sequence $(H_k)_{k\in\mathbb{N}}$ of closed sets and a subsequence of $(\mu_i)_{i\in\mathbb{N}}$, that for simplicity of notations we will yet denote it by $(\mu_i)_{i\in\mathbb{N}}$, such that

$$\left|\mu_k(H_k)\right|>\varepsilon\quad\text{for every } k\in\mathbb{N}. \tag{3}$$

Put

$$B_n=\bigcup_{k=1}^{n}H_k\quad\text{for every } n\in\mathbb{N},$$

and we choose $\sigma>0$, by Lemmas 2.5 and 2.6, we can find $A_1\in\tau$ such that

$$B_1\subseteq A_1,\quad \left|\mu_i(Y)\right|<\frac{\sigma}{2}\quad\text{for every } Y\in\mathcal{R}_{\overline{A}_1\setminus B_1}\text{ and for every } i\in\mathbb{N}.$$

In the same way we can find an open set A_2 such that $B_2\cup\overline{A}_1\subseteq A_2$,

$$\left|\mu_i(Y)\right|<\frac{\sigma}{2^2}\quad\text{for every } Y\in\mathcal{R}_{\overline{A}_2\setminus(B_2\cup\overline{A}_1)}\text{ and } i\in\mathbb{N}.$$

We observe now that

$$\overline{A}_2\setminus B_2\subseteq\big(\overline{A}_2\setminus(B_2\cup\overline{A}_1)\big)\cup(\overline{A}_1\setminus B_2)\subseteq\big(\overline{A}_2\setminus(B_2\cup\overline{A}_1)\big)\cup(\overline{A}_1\setminus B_1).$$

Then, if $Y\in\mathcal{R}_{\overline{A}_2\setminus B_2}$, there exist Y_1,Y_2 such that

$$Y_1\in\mathcal{R}_{\overline{A}_1\setminus B_1},\qquad Y_2\in\mathcal{R}_{\overline{A}_2\setminus(B_2\cup\overline{A}_1)},\qquad Y=Y_1\cup Y_2$$

and we have $B_2\cup\overline{A}_1\subseteq A_2$,

$$\left|\mu_i(Y)\right|<\frac{\sigma}{2}+\frac{\sigma}{2^2}\quad\text{for every } Y\in\mathcal{R}_{\overline{A}_2\setminus B_2}\text{ and } i\in\mathbb{N}.$$

Suppose now that we have $A_1,\dots,A_n$ in τ such that

$$B_p\cup\overline{A}_{p-1}\subseteq A_p\quad\text{for every } p\in\{1,\dots,n\}\ (A_0=\emptyset),$$

$$\left|\mu_i(Y)\right|<\sum_{q=1}^{p}\frac{\sigma}{2^q}\quad\text{for every } Y\in\mathcal{R}_{\overline{A}_p\setminus B_p}\text{ and every } i\in\mathbb{N},$$

and let (see Lemmas 2.5 and 2.6) A_{n+1} be an open set such that

$$B_{n+1}\cup\overline{A}_n\subseteq A_{n+1},$$

$$\left|\mu_i(Y)\right|<\frac{\sigma}{2^{n+1}}\quad\text{for every } Y\in\mathcal{R}_{\overline{A}_{n+1}\setminus B_{n+1}\cup\overline{A}_n}\text{ and every } i\in\mathbb{N}.$$

We observe now that

$$\begin{aligned}\overline{A}_{n+1}\setminus B_{n+1}&\subseteq(\overline{A}_{n+1}\setminus B_{n+1}\cup\overline{A}_n)\cup(\overline{A}_n\setminus B_{n+1})\\&\subseteq(\overline{A}_{n+1}\setminus B_{n+1}\cup\overline{A}_n)\cup(\overline{A}_n\setminus B_n);\end{aligned}$$

for every $Y\in\mathcal{R}_{\overline{A}_{n+1}\setminus B_{n+1}}$ we have $Y=Y_1\cup Y_2$ with

$$Y_1\in\mathcal{R}_{\overline{A}_n\setminus B_n},\qquad Y_2\in\mathcal{R}_{\overline{A}_{n+1}\setminus B_{n+1}\cup\overline{A}_n},$$

and so

$$B_{n+1}\cup\overline{A}_n\subseteq A_{n+1},$$

$$\left|\mu_i(Y)\right| < \sum_{q=1}^{n+1} \frac{\sigma}{2^q} \quad \text{for every } Y \in \mathcal{R}_{\overline{A}_{n+1} \setminus B_{n+1}} \text{ and every } i \in \mathbb{N}.$$

In this way we can construct by induction, a sequence $(A_n)_{n\in\mathbb{N}}$ of open sets such that

$$B_n \subseteq A_n \subseteq \overline{A}_n \subseteq A_{n+1},$$

$$\left|\mu_i(Y)\right| < \sum_{q=1}^{n} \frac{\sigma}{2^q} \quad \text{for every } Y \in \mathcal{R}_{\overline{A}_n \setminus B_n} \text{ and every } i \in \mathbb{N}. \tag{4}$$

By (4) we have, for every $n \in \mathbb{N}$,

$$H_{n+1} \subseteq A_{n+1} \setminus B_n \subseteq (A_{n+1} \setminus \overline{A}_n) \cup (\overline{A}_n \setminus B_n),$$

then there exists $Y_n \in \mathcal{R}_{A_{n+1}\setminus \overline{A}_n}$ such that

$$\left|\mu_i(H_{n+1})\right| \leqslant \left|\mu_i(Y_n)\right| + \sum_{q=1}^{n} \frac{\sigma}{2^q} < \left|\mu_i(Y_n)\right| + \sigma \quad \text{for every } i \in \mathbb{N}. \tag{5}$$

By Lemma 2.4 we can also find two elements A'_n, A''_n of $\tau_{A_{n+1}\setminus\overline{A}_n}$ contained in $A_{n+1} \setminus \overline{A}_n$ such that

$$\left|\mu_i(Y_n)\right| \leqslant \left|\mu_i(A'_n)\right| + \left|\mu_i(A''_n)\right| + \sigma;$$

by the disjointness of the sequence $(A_{n+1} \setminus \overline{A}_n)_{n\in\mathbb{N}}$ it follows that also the sequences $(A'_n)_{n\in\mathbb{N}}$ and $(A''_n)_{n\in\mathbb{N}}$ are disjoint and then, by the uniform τ-exhaustivity of $(\mu_i)_{i\in\mathbb{N}}$ we have that there exists $m \in \mathbb{N}$ such that

$$\left|\mu_i(Y_n)\right| \leqslant 3\sigma \quad \text{for every } i \in \mathbb{N} \text{ and } n \geqslant m.$$

From (5) we have

$$\left|\mu_i(H_{n+1})\right| < 4\sigma \quad \text{for every } i \in \mathbb{N} \text{ and } n \geqslant m,$$

a contradiction with (3) if we choose $\sigma < \varepsilon/4$. □

We arrive now to the main result of this section.

Theorem 2.8. *A sequence $(\mu_n)_{n\in\mathbb{N}}$ of* $\mathrm{ra}(\mathcal{R}, \Omega, \tau)$ *uniformly τ-exhaustive is uniformly exhaustive and uniformly $\mathcal{C}$-regular.*

Proof. Suppose that there exist $\varepsilon > 0$, a disjoint sequence $(X_k)_{k\in\mathbb{N}}$ of $\mathcal{R}$ and a subsequence of $(\mu_i)_{i\in\mathbb{N}}$, that for simplicity of notations we will yet denote it by $(\mu_i)_{i\in\mathbb{N}}$, such that

$$\left|\mu_k(X_k)\right| > \varepsilon \quad \text{for every } k \in \mathbb{N}.$$

Let, for every $k \in \mathbb{N}$, H_k be a closed set contained in X_k such that

$$\left|\mu_k(Y)\right| < \frac{\varepsilon}{2} \quad \text{for every } Y \in \mathcal{R}_{X_k\setminus H_k},$$

it follows

$$\left|\mu_k(H_k)\right| \geqslant \left|\mu_k(X_k)\right| - \left|\mu_k(X_k \setminus H_k)\right| > \frac{\varepsilon}{2} \quad \text{for every } k \in \mathbb{N}.$$

A contradiction because by Lemma 2.7 the $(\mu_i)_{i\in\mathbb{N}}$ is uniformly $\mathcal{C}$-exhaustive. So $(\mu_i)_{i\in\mathbb{N}}$ is uniformly exhaustive.

We need now to prove that $\mu_i, i \in \mathbb{N}$, are uniformly $\mathcal{C}$-regular. Let $X \in \mathcal{R}$. We can construct by induction an increasing sequence $(H_n)_{n\in\mathbb{N}}$ of $\mathcal{C}$ and a decreasing sequence $(A_n)_{n\in\mathbb{N}}$ of τ such that

$$H_n \subseteq X \subseteq A_n,$$

$$\left|\mu_i(Y)\right| < \frac{1}{2^n} \quad \text{for every } Y \in \mathcal{R}_{A_n\setminus H_n} \text{ and } i \in \{1, \ldots, n\}.$$

Since $(A_n \setminus H_n)_{n\in\mathbb{N}}$ is a decreasing sequence of $\mathcal{R}$ and we have proved that $(\mu_i)_{i\in\mathbb{N}}$ are uniformly exhaustive, then by (2.2) of [5] we have that for every $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that for every $n \geqslant m$

$$\left|\mu_i(Y)\right| < \varepsilon \quad \text{for every } Y \in \mathcal{R}_{A_n\setminus H_n} \text{ and } i \in \mathbb{N},$$

and this completes the proof. □

3. Cafiero–Dieudonné theorem

We are now able to prove the main results of the paper.

Theorem 3.1 *(Cafiero–Dieudonné). Let (Ω, τ) be a Hausdorff normal topological space and let $\mathcal{R}$ be an algebra containing τ. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of τ-exhaustive elements of* $\mathrm{ra}(\mathcal{R}, \Omega, \tau)$. *Then $(\mu_n)_{n\in\mathbb{N}}$ is uniformly exhaustive and uniformly regular if (and only if) for every disjoint sequence $(A_k)_{k\in\mathbb{N}}$ of τ and for every $\varepsilon > 0$ there exist $\bar{k}, n_0 \in \mathbb{N}$ such that*

$$\left|\mu_i(A_{\bar{k}})\right| < \varepsilon \quad \textit{for every } i \geqslant n_0.$$

Proof. Let $(X_k)_{k\in\mathbb{N}}$ be a disjoint sequence of τ and let

$$\mathcal{G} = \left\{X \in \tau \;\middle|\; \exists \Delta \in \mathcal{P}(\mathbb{N}) \text{ such that } X = \bigcup_{k\in\Delta} X_k\right\} \quad \left(\bigcup_{k\in\emptyset} X_k = \emptyset\right).$$

Then $\mathcal{G}$ is a σ-ring so the restrictions of the μ_i to $\mathcal{G}$ verifies the hypothesis of 2.11 of [5], then they are uniformly exhaustive and we have

$$\lim_k \mu_i(X_k) = 0$$

uniformly for $n \in \mathbb{N}$. It follows that the $(\mu_n)_{n\in\mathbb{N}}$ is uniformly τ-exhaustive. The thesis is a consequence of Theorem 2.8. □

Theorem 3.2 *(Dieudonné convergence theorem). Let (Ω, τ) be a Hausdorff normal topological space and let $\mathcal{R}$ be an algebra containing τ. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of* $\mathrm{ra}(\mathcal{R}, \Omega, \tau)$ *which is τ-exhaustive and pointwise convergent in τ. Then $(\mu_n)_{n\in\mathbb{N}}$ is uniformly exhaustive, uniformly $\mathcal{C}$-regular and pointwise convergent to an exhaustive element of* $\mathrm{ra}(\mathcal{R}, \Omega, \tau)$.

Proof. If $(\mu_n)_{n\in\mathbb{N}}$ is pointwise convergent to 0 then by Theorem 3.1 $(\mu_n)_{n\in\mathbb{N}}$ is uniformly exhaustive and uniformly $\mathcal{C}$-regular.

We consider now the general case, if $(\mu_n)_{n\in\mathbb{N}}$ is not uniformly exhaustive, then there exist $\varepsilon > 0$, a sequence $(A_k)_{k\in\mathbb{N}}$ of disjoint open sets, and a subsequence of $(\mu_n)_{n\in\mathbb{N}}$, that yet we denote by $(\mu_n)_{n\in\mathbb{N}}$, such that

$$\left|\mu_k(A_k)\right| > \varepsilon \quad \text{for every } k \in \mathbb{N}.$$

But every μ_n is τ-exhaustive then it is possible to construct a subsequence $(\mu_{n_r})_{r\in\mathbb{N}}$ of $(\mu_n)_{n\in\mathbb{N}}$ such that for every $r \in \mathbb{N}$ we have

$$\left|\mu_{n_r}(A_k)\right| < \frac{\varepsilon}{2} \quad \text{for every } k \geqslant n_{r+1}.$$

The sequence $(\mu_{n_{r+1}} - \mu_{n_r})_{r\in\mathbb{N}}$ is a sequence of $\mathrm{ra}(\mathcal{R}, \Omega, \tau)$, τ-exhaustive and pointwise convergent to 0 in τ, then, by our previous observation, it is also uniformly exhaustive, but we have also, for every $r \in \mathbb{N}$,

$$\left|(\mu_{n_{r+1}} - \mu_{n_r})(A_{n_{r+1}})\right| \geqslant \left|\mu_{n_{r+1}}(A_{n_{r+1}})\right| - \left|\mu_{n_r}(A_{n_{r+1}})\right| > \frac{\varepsilon}{2} \quad \text{for every } r \in \mathbb{N},$$

a contradiction. Therefore also in the general case the $(\mu_n)_{n\in\mathbb{N}}$ is uniformly exhaustive and then, by Theorem 2.8 it is also uniformly $\mathcal{C}$-regular.

To prove the pointwise convergence of $(\mu_n)_{n\in\mathbb{N}}$, let $X \in \mathcal{R}$ and $\varepsilon > 0$, let A be an open set such that

$$X \subseteq A, \quad \left|\mu_i(A \setminus X)\right| < \varepsilon \quad \text{for every } i \in \mathbb{N}.$$

Then we have, for $p, q \in \mathbb{N}$,

$$\begin{aligned}\left|\mu_p(X) - \mu_q(X)\right| &\leqslant \left|\mu_p(A \setminus X)\right| + \left|\mu_q(A \setminus X)\right| + \left|\mu_p(A) - \mu_q(A)\right| \\ &\leqslant 2\varepsilon + \left|\mu_p(A) - \mu_q(A)\right|,\end{aligned}$$

and this completes the proof. □

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