Global smooth solutions of IBVP to nonlinear equation of suspended string

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Abstract

We shall consider IBVP to a nonlinear equation of suspended string with uniform density to which a nonlinear time-independent outer force works. The nonlinear term is smooth but not monotone. We shall show that IBVP has a unique time-global smooth solution. The regularity of the solutions shall be also studied.

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1. Introduction

Let \( \Omega \) be a cylindrical domain \((0, a) \times (0, T)\). We shall consider IBVP to a nonlinear equation of a flexible and heavy suspended string of finite length \(a\) with uniform density

\[
(P) \quad \begin{cases}
\partial_t^2 u(x, t) + Lu(x, t) + f(x, u(x, t)) = 0, & (x, t) \in \Omega, \\
u(a, t) = 0, & t \in (0, T), \\
u(x, 0) = \phi(x), \quad \partial_t u(x, 0) = \psi(x), & x \in (0, a),
\end{cases}
\]

where \(L = L_0(x, \partial_x)\) is a second order differential operator of the form

\[
L = L_0 = -(x \partial_x^2 + \partial_x),
\]

\(f(x, u)\) is smooth in \((x, u) \in [0, a] \times R^1\) and of order 3 in \(u\) at \(u = 0\), and \(\phi, \psi\) belong to suitable function spaces and lie in the potential well. The exact assumptions on \(\phi, \psi\) and \(f\) shall be given in (A) and (B) in Section 2.

The purpose of this paper is to show that IBVP \((P)\) has a time global smooth solution and to study the regularity of the solutions. Note that \(f(x, u)\) is not monotone with respect to \(u\). In the previous paper [4] it is shown that \((P)\) has a time-global weak solution under weaker conditions with less smoothness on \(f\) and \(\phi, \psi\). Also in [5] it is shown that \((P)\) with a monotone nonlinear term \(f(x, u) = u^3\) has a time-global classical solution under suitable smoothness on \(\phi, \psi\).
We shall prove the statement as follows. We shall set the generalized Lebesgue and Sobolev spaces with weights at $x = 0$, $L^2(0, a; x^m)$ and $H^s(0, a; x^m)$ respectively, as base spaces. (The definitions will be given in this section later. See also [6,7].) In this paper we shall take $m = 0$ that corresponds to the uniform density. (For general $m$, see [1,2,6].)

By using the Galerkin method in $L^2(0, a; x^0)$, we shall construct the sequence of approximate solutions that lie in the potential well for (P). The main part of the proof is to estimate the higher order time-derivatives of approximate solutions, by using the energy method, and to obtain the estimates of the higher order $x$-derivatives, by using the elliptic technique. In our case the norms of the function spaces have weights at $x = 0$, where $s \in N$ is the order of $x$-derivative. Therefore the properties of the usual Lebesgue and Sobolev spaces and the important inequalities in those spaces do not necessarily hold. The estimates shall be considered in $H^s(0, a; x^0)$ whose elements may have singularities at $x = 0$, which yields some difficulties to show several inequalities and to estimate general nonlinear terms. The estimates of nonlinear terms shall be obtained by careful calculations, by using some inequalities in the generalized Sobolev spaces. Then combining the energy estimates of the higher order time-derivatives of approximate solutions with the compactness arguments, we shall show that the approximate solution converges to a function that is a solution of (P) with higher order $t$-derivatives. The smoothness of the solutions shall be obtained by the above $t$-regularity of solutions and the $x$-regularity by the use of an elliptic technique shown in [8].

For linear IBVP with power density $\rho(x) = ax^m$, $m > -1$, see [1,2] and [6]. For the existence of time-periodic solutions of nonlinear equations, see [7] and [8].

This paper shall be organized as follows. In this section notation and definitions are given. Function spaces shall be defined. In Section 2 main theorem shall be formulated. In Section 3 the theorem shall be proved. In Section 3.1 the energy estimates of higher order $t$-derivatives of solutions shall be given. To this end the estimate and the Lipschitzness of higher order $t$-derivatives of nonlinear term $f(\cdot, u(\cdot, t))$ shall be established. In Section 3.2 the higher order estimates of approximate solutions constructed by the usual Galerkin method shall be derived, using the results of Section 3.1 and the estimates of the approximate initial data. In Section 3.3 the compactness argument shall be used to show the convergence of the sequence of the approximate solutions. In Section 3.4 the smoothness of solution shall be shown. In Appendix A the properties of function spaces and several inequalities used in this paper shall be shown. The eigenvalue problem for $L_0$, and the elliptic technique for $L_0$ shall be also given.

1.1. Notation and definitions

Let $R^1_+$ and $Z_+$ be the set of nonnegative numbers and nonnegative integers, respectively.

Let $O$ be any open set in $R^n$. Let $p \geq 1$ and $s \in Z_+$. $L^p(O)$ and $H^s(O)$ are the usual Lebesgue and Sobolev spaces, respectively.

We shall set function spaces defined in [6]. Let $m \in R^1_+$. $L^p(0, a; x^m)$ is a Banach space whose elements $f(x)$ are measurable in $(0, a)$ and satisfy $x^m/f(x) \in L^p(0, a)$, where the norm is defined by

$$ |f|_{L^p(0,a;x^m)} = \left( \int_0^a x^m |f(x)|^p \, dx \right)^{1/p}. $$

Clearly $L^p(0, a; x^0) = L^p(0, a)$. $L^2(0, a; x^m)$ is a Hilbert space with the inner product defined by

$$(f, g)_{L^2(0,a;x^m)} = \int_0^a x^m f(x)\overline{g(x)} \, dx.$$  

We simply write $(f, g)_{L^2(0,a;x^m)}$ as $(f, g)$. $H^s(0, a; x^m)$ is a Hilbert space whose elements $f$ and their weighted derivatives $x^{j/2} f^{(j)}$, $j = 1, \ldots, s$, belong to $L^2(0, a; x^m)$, where $f^{(j)}$ means the $j$th derivative of $f$. Its norm is defined by

$$ |f|_{H^s(0,a;x^m)} = \left( \sum_{j=0}^s \int_0^a x^{m+j} |f^{(j)}(x)|^2 \, dx \right)^{1/2}. $$
Let $T > 0$. Let $\Omega = (0, a) \times (0, T)$ and $D = (0, a) \times R^1$. $L^p(\Omega; x^m)$ is a Banach space whose elements $f(x, t)$ are measurable in $D$ and satisfy $x^{m/p} f(x, t) \in L^p(\Omega)$, where the norm is defined by
\[
|f|_{L^p(\Omega; x^m)} = \left( \int_{\Omega} x^m |f(x, t)|^p \, dx \, dt \right)^{1/p}.
\]
We see $L^p(\Omega; x^0) = L^p(\Omega)$.

$H^s(\Omega; x^m)$ is a Hilbert space whose elements $f$ and their weighted derivatives $x^{j+k} \partial_x^j \partial_t^k f$, $0 \leq j + k \leq s$, belong to $L^2(\Omega; x^m)$. Its norm is defined by
\[
|f|_{H^s(\Omega; x^m)} = \left( \sum_{j+k \leq s} \int_{\Omega} x^{m+j} \left| \partial_x^j \partial_t^k f(x, t) \right|^2 \, dx \right)^{1/2}.
\]
$H^1_0(0, a; x^m)$ is a subspace as Hilbert space of $H^1(0, a; x^m)$ whose elements $f$ satisfy $f(a) = 0$. In $H^1_0(0, a; x^0)$ the norm equivalent to $\| \cdot \|_{H^1_0(0, a; x^0)}$ is defined
\[
\|u\| = \left( \int_0^a \int_0^T x |f'(x)|^2 \, dx \, dt \right)^{1/2}.
\]
The equivalence is easily shown, by using Lemma A.1 of the Poincaré type. $H^1_0(\Omega; x^m)$ is a subspace of $H^1(\Omega; x^m)$ whose elements $f$ satisfy $f(a, t) = 0$ for almost all $t$. $K^s(0, a; x^m)$ is a subspace of $H^s(0, a; x^m)$ whose elements $f$ satisfy $L^j f \in H^1_0(0, a; x^m)$ for $j = 0, \ldots, \lfloor (s - 1)/2 \rfloor$. Note that
\[
\begin{align*}
K^0(0, a; x^m) &= L^2(0, a; x^m), \quad K^1(0, a; x^m) = H^1_0(0, a; x^m), \\
K^2(0, a; x^m) &= H^2(0, a; x^m) \cap H^1_0(0, a; x^m).
\end{align*}
\]
We set $K^\infty(0, a; x^m) = \bigcap_{s=0}^{\infty} K^s(0, a; x^m)$. Also note [6] that $H^s(\Omega)$ and $H^s(0, a)$ are densely embedded in $H^s(\Omega; x^m)$ and $H^s(0, a; x^m)$ respectively for $s \in Z_+$.

For properties of the function spaces, see Appendix A for use in this paper. See also [4,6–8] for details.

2. Formulation of theorem

In this section we shall consider the IBVP
\[
(P) \quad \begin{cases} \\
\partial_x^2 u(x, t) + Lu(x, t) + f(x, u(x, t)) = 0, & (x, t) \in \Omega, \\
\partial_t u(a, t) = 0, & t \in (0, T), \\
\partial_t u(x, 0) = \phi(x), & \partial_x u(x, 0) = \psi(x), & x \in (0, a),
\end{cases}
\]
and formulate the main theorem.

The potential of $f$ is given by $F(x, y) = \int_0^y f(x, \rho) \, d\rho$. The potential energy, the kinetic energy and the total energy for (P) are defined as follows
\[
J(w) = \int_0^a \left( \frac{1}{2} x \left( \partial_x w(x) \right)^2 + F(x, w(x)) \right) \, dx = \frac{1}{2} \|w\|^2 + \int_0^a F(x, w(x)) \, dx
\]
for $w \in H^1_0(0, a; x^0)$,
\[
K(w) = \frac{1}{2} \int_0^a \left( \partial_t w(x, t) \right)^2 \, dx
\]
for $w \in H^1_0(\Omega; x^0)$, and
\[
E(t; u) = \hat{E}(u(\cdot, t), \partial_x u(\cdot, t)) = K(u(\cdot, t)) + J(u(\cdot, t))
\]
for $u \in H_0^1(\Omega; x^0)$. Here $\hat{E}(u, v)$ is denoted by

$$\hat{E}(u, v) = \frac{1}{2} \int_0^a v(x)^2 \, dx + J(u).$$

We also define the total free energy

$$e(t; u) = K(u(\cdot, t)) + \frac{1}{2} \int_0^a x(\partial_t u(x, t))^2 \, dx = \frac{1}{2} \int_0^a \left\{ \left( \partial_t u(x, t) \right)^2 + x(\partial_x u(x, t))^2 \right\} \, dx = \hat{e}(u(\cdot, t), \partial_t u(\cdot, t)),
$$

where $\hat{e}$ is defined by

$$\hat{e}(v, w) = \frac{1}{2} \int_0^a w(x)^2 \, dx + \frac{1}{2} \int_0^a x(\partial_x v(x))^2 \, dx.$$

The following fact is seen in Proposition 3.1 in [4].

**Remark 2.1.** Assume that $f(x, u)$ is continuous in $(x, u) \in [0, a] \times R^1$. Then $J(u)$ is continuous in $H_0^1(0, a; x^0)$.

**2.1. Assumptions on $f$ and $\phi, \psi$**

We shall assume the following conditions on $f$.

(A) $f(x, u)$ is of $C^s$-class in $(x, u) \in [0, a] \times R^1$, and monotone decreasing in $u \in R^1$, and satisfies

$$f(x, 0) = \partial_u f(x, 0) = \partial_u f(x, 0) = 0, \quad \partial^2_u f(x, u) \leq -C, \quad \gamma f(x, u) - F(x, u) \leq M \quad \text{(2.1)}$$

for any $x \in [0, a]$ and $u \in R^1$. Here $\gamma \in (0, 1/2)$ and $M$ and $C$ are constants independent of $x$ and $u$.

It should be noted that (A) implies the condition (A) in [4].

**Remark 2.2.** From (2.1) $f$ is written of the form

$$f(x, u) = u^3 h(x, u), \quad \text{(2.3)}$$

where $h$ is of $C^s$ in $(x, u) \in [0, a] \times R^1 \setminus \{0\}$, of $C^{s-3}$ in $(x, u) \in [0, a] \times R^1$ and satisfies $h(x, u) \leq -C/6$. Clearly $f$ is not a monotone operator in $L^2(0, a; x^0)$.

From (2.3) there exists a constant $\lambda > 0$ such that $J(\lambda u)$ is monotone increasing in $\lambda \in (0, \lambda)$ for any fixed $u \neq 0$. The potential well $W$ for (P) around the origin is defined in the same way as [4].

$$W = \left\{ u \in H_0^1(0, a; x^0); \ 0 \leq J(u) < d, \ 0 \leq \lambda \leq 1 \right\}. \quad \text{(2.4)}$$

Here $d$ is the depth of the potential well $W$ defined by

$$d = \inf_{u \in H_0^1(0, a; x^0)} J(\lambda_0(u)u), \quad \text{(2.5)}$$

where $\lambda = \lambda_0(u) > 0$ is the first value of $\lambda$ at which $J(\lambda u)$ starts to decrease strictly. We see [4] that $0 < d < +\infty$ and $W$ is bounded and open in $H_0^1(0, a; x^0)$.

We assume the following conditions on the initial data $\phi, \psi$.

(B) $\phi \in W \cap K^{2s+1}(0, a; x^0)$ and $\psi \in K^{2s}(0, a; x^0)$. $\phi$ and $\psi$ satisfy the following condition

$$\int_0^a \left\{ \frac{1}{2} \left( \psi(x)^2 + x\phi'(x)^2 \right) + F(x, \phi(x)) \right\} \, dx < d. \quad \text{(2.6)}$$
2.2. Main theorem

**Theorem 2.1.** Assume (A) and (B). Then IBVP (P) has a solution \( u \) in

\[
C^{s+1}((0, T); L^2(0, a; x^0)) \cap \bigcap_{i=1}^s C^i((0, T); H^{s+1-i}(0, a; x^0) \cap H_0^1(0, a; x^0)).
\]  

(2.7)

\( u(\cdot, t) \) lies in \( W \) for all \( t \in (0, T) \).

**Remark 2.3.** If \( s \geq 3 \), it follows from Proposition 3.5 in Section 3 that the solution in Theorem 2.1 is of \( C^2 \) in \( \Omega \). Therefore IBVP (P) has the solution in classical sense.

**Remark 2.4.** It follows from [4] that under (A) and (B) IBVP (P) has a weak solution in \( H^1_0(\Omega; x^0) \).

3. Proof of Theorem 2.1

The theorem is proved by dividing into several steps.

3.1. Energy estimates of higher order \( t \)-derivatives

We shall obtain the necessary energy estimates of equation

\[
\partial_t^2 u + Lu + f(x, u) = 0.
\]  

(3.1)

Here we shall assume (A) on \( f \).

First we shall consider IBVP for linear equation

\[
\text{(LP)} \quad \begin{cases}
(\partial_t^2 + L)u = g(x, t), & (x, t) \in \Omega, \quad t \in R^1, \\
u(a, t) = 0, & x \in (0, a), \quad \partial_t u(x, 0) = q(x),
\end{cases}
\]

where \( g \in L^2(\Omega) \cap C((0, T); L^2(0, a; x^0)) \) and \( (p, q) \in K^1(0, a; x^0) \times L^2(0, a; x^0) \). We derive the standard energy estimate for (LP) in

\[
X = C^1((0, T); L^2(0, a; x^0)) \cap C((0, T); H^1_0(0, a; x^0)).
\]  

(3.2)

**Proposition 3.1.** IBVP (LP) has a unique solution in \( X \) satisfying the energy estimate

\[
e(t; u) \leq e^t \left( e(0; u) + \int_0^t \| g(\cdot, \tau) \|_{L^2(0, a; x^0)}^2 d\tau \right).
\]  

(3.3)

Here \( e(0, u) = (1/2) \int_0^a (q(x))^2 + xp'(x)^2 \, dx \).

**Proof.** Let \( Y \) be the following function space

\[
Y = \bigcap_{i=0}^2 C^i((0, T); K^{2-i}(0, a; x^0)).
\]  

(3.4)

Note that if we assume that the solutions exist in \( Y \), we obtain (3.3) by the standard technique. In fact, multiplying equation in (LP) by \( \partial_t u \), integrating with respect to \( x \) and using Proposition A.1(i) and the Schwarz inequality, we obtain the differential inequality to (LP)

\[
\frac{d}{dt} e(t; u) \leq e(t; u) + \int_0^a | g(\cdot, t) |_{L^2(0, a; x^0)}^2 d\tau.
\]  

(3.5)

Using the Gronwall inequality, we obtain (3.3).
We shall show the existence of the solution in $X$. Let expand $g$ and $(p, q)$ into the Bessel–Fourier series and formally expand $u$ into the Bessel–Fourier series

$$
g = \sum_{i=1}^{\infty} g_i(t) \phi_i, \quad p = \sum_{i=1}^{\infty} p_i \phi_i, \quad q = \sum_{i=1}^{\infty} q_i \phi_i, \quad u = \sum_{i=1}^{\infty} u_i(t) \phi_i,
$$

and define

$$
h_n = \sum_{i=1}^{n} g_i(t) \phi_i, \quad v_n = \sum_{i=1}^{n} u_i(t) \phi_i. \tag{3.6}
$$

Since $u_i$ satisfies a second order ODE with initial conditions

$$
\dot{u}_i + \lambda_i u_i = g_i, \quad u_i(0) = p_i, \quad \dot{u}_i = q_i, \tag{3.7}
$$

$u_i$ has the form

$$
u_i(t) = p_i \cos a_i t + \frac{q_i}{a_i} \sin a_i t + \frac{1}{a_i} \int_{0}^{t} g_i(\tau) \sin a_i (t - \tau) d\tau, \tag{3.8}
$$

where $a_i = \sqrt{\lambda_i}$. Since $g_i(t) = (g(\cdot), t, \phi_i)$ is continuous in $t$, $u_i(t)$ is of $C^2$-class in $t \in (0, T)$. Therefore $v_n$ belongs to $C^2((0, T); K^{\infty}(0, a; x^0))$. Using (3.8) and the Parseval equality, it follows that

$$
|v_n - u|_{H^1(\Omega; x^0)} \rightarrow 0, \\
\sup_{t \in (0, T)} |v_n(\cdot, t) - u(\cdot, t)|_{H^1(0, a; x^0)} \rightarrow 0, \\
\sup_{t \in (0, T)} \left| \partial_t (v_n(\cdot, t) - u(\cdot, t)) \right|_{L^2(0, a; x^0)} \rightarrow 0 \tag{3.9}
$$

as $n \rightarrow +\infty$ and hence $u$ belongs to

$$
H_0^1(\Omega; x^0) \cap \bigcap_{i=0}^{1} C^i((0, T); K^{1-i}(0, a; x^0)).
$$

Therefore $u$ is the solution of (LP). Since $v_n$ belongs to $Y$, we obtain

$$
e(t; v_n) \leq e^t \left( e(0; v_n) + \int_{0}^{t} \left| h_n(\cdot, t) \right|_{L^2(0, a; x^0)}^2 dt \right). \tag{3.10}
$$

Therefore taking $n \rightarrow +\infty$ in (3.10), it follows from (3.9) that (3.3) holds. \qed

We shall show the following fundamental proposition on the energy estimate on (P).

**Proposition 3.2.** Assume that $u$ is a solution of (3.1) with $t$-derivatives of $s$-order that belong to (3.2), and that $u(\cdot, t)$ belongs to $W$. Let $(\phi, \psi)$ belong to $K^2s+1(0, a; x^0) \times K^{2s}(0, a; x^0)$. Then higher $t$-derivative estimates of $u$ are obtained:

$$
e(t; \partial_t^i u) \leq e^t \left( e(0; \partial_t^i u) + B_i \right) \tag{3.11}
$$

for $i = 0, \ldots, s$, where $B_i > 0$ depends on $T$, $\sup_{\Omega} |\partial_t^j f(x, u(x, t))|$ for $j = 0, \ldots, i$, and $\partial_t^i u(x, 0)$ and $\partial_t \partial_t^j u(x, 0)$ for $j = 1, \ldots, i$. Here we define

$$
\begin{cases}
u(0, 0) = \phi, \quad \partial_t u(0, 0) = \psi, \\
\partial_t^i u(0, 0) = -(L \partial_t^{i-2} u(0, 0) + \partial_t^{i-2} f(\cdot, u(\cdot, t)))_{t=0}
\end{cases} \tag{3.12}
$$

for $i = 2, \ldots, s + 1$. 

\[\]
Assume that \( u_l, l = 1, 2 \), satisfy the same conditions as \( u \) and (3.12) replaced \( u, \phi, \psi \) by \( u_l, \phi_l, \psi_l \), respectively.

Then

\[
e(t; \partial_t^j(u_1 - u_2)) \leq \hat{C} e^j \sum_{j=0}^i e(0; \partial_t^j(u_1 - u_2)),
\]

where \( \hat{C} \) depends on \( T \), \( \sup_{\Omega} |\partial_t^j f(x, u(x,t))| \) for \( j = 0, \ldots, i \), and \( \partial_t^j u_l(x,0) \) and \( \partial_x \partial_t^{j-1} u_l(x,0) \) for \( j = 1, \ldots, i \).

We shall prepare a lemma on the estimates of the nonlinear function to show this proposition.

**Lemma 3.1.** Let \( k \in \mathbb{Z}_+ \) with \( k \leq s \). Let \( u, v \) be solutions of \( \text{(P)} \) with \( t \)-derivatives \( \partial_t^j u(\cdot, t), \partial_t^j v(\cdot, t) \) in (3.2), \( j = 0, \ldots, k \), and satisfy (3.12) with each initial values in \( K^{2s+1}(0, a; x^0) \times K^{2s}(0, a; x^0) \). Let \( u(\cdot, t), v(\cdot, t) \) belong to \( W \). Assume that (3.11) with \( u, v \) holds for \( i = 0, \ldots, k - 1 \). Then the following inequalities hold:

\[
|\partial_t^k f(x,u)|^2_{L^2(0,a;x^0)} \leq C_0,
\]

where \( C_0 \) depends on \( \sup_{\Omega} |\partial_t^j f(x,u(x,t))| \) for \( j = 0, \ldots, k \), and \( \partial_t^j u(x,0) \) and \( \partial_x \partial_t^{j-1} u(x,0) \) for \( j = 1, \ldots, k \), and

\[
|\partial_t^k (f(x,u) - f(x,v))|^2_{L^2(0,a;x^0)} \leq C_1 \sum_{j=0}^{k-1} e(t; \partial_t^j (u-v)),
\]

where \( C_1 \) depends on \( \sup_{\Omega} |\partial_t^j f(x,u(x,t))| \), \( \sup_{\Omega} |\partial_t^j f(x,v(x,t))| \), \( \partial_t^j u(x,0) \), \( \partial_t^j v(x,0) \) for \( j = 0, \ldots, k \), and \( \partial_x \partial_t^{j-1} u(x,0), \partial_x \partial_t^{j-1} v(x,0) \) for \( j = 1, \ldots, k \).

**Proof.** We shall show (3.14) by induction with \( i = 0, \ldots, k \). First note that the following statement holds: Since \( u(\cdot, t) \) belongs to \( W \) for \( t \in (0, T) \), it follows from [4] that \( u(x,t) \) is bounded in \( \Omega \). Hence \( (\partial^\beta f)(x,u(x,t)), \beta = 0, 1, \ldots, k \), are bounded in \( \Omega \).

The case \( i = 0 \) is clear. Let \( i \geq 1 \) and assume that

\[
|\partial_t^j f(x,u)|^2_{L^2(0,a;x^0)} \leq C_0
\]

hold for \( i = 1, \ldots, k - 1 \). Using the differentiation formula of a composite function, we have

\[
\partial_t^k f(x,u) = \sum_{\alpha=0}^k C_{\alpha} (\partial^\alpha u)(\partial_\alpha u)^{\alpha_1} \cdots (\partial^k u)^{\alpha_k},
\]

where the summation is taken in \( (\alpha_1, \ldots, \alpha_k) \in \mathbb{Z}_+^k \) and \( \beta \in \mathbb{Z}_+ \) satisfying \( \alpha_1 + \cdots + \alpha_k = \beta \) and \( 1\alpha_1 + \cdots + k\alpha_k = k \). We write \( \partial_t^k f \) in the following form

\[
\partial_t^k f(x,u) = \partial_t^k u(\partial_t f)(x,u) + \sum_{\alpha_k=0}^k C_{\alpha} (\partial^\alpha u)(\partial_\alpha u)^{\alpha_1} \cdots (\partial^{k-1} u)^{\alpha_{k-1}} \equiv I_1(u) + I_2(u).
\]

We see

\[
\left| \prod_{l=1}^p w_l \right|_{L^2(0,a;x^0)} \leq C \prod_{l=1}^p \| w_l \|
\]

for \( w_l \in H^1_0(0, a; x^0), 1 \leq l \leq p \). In fact, applying the Hölder inequality to the left-hand side of (3.18) and using Proposition A.2(iii), we easily obtain (3.18).

We shall proceed to estimate \( I_j = I_j(u) \). We have by Proposition A.2(iii) and the assumption (3.11) with \( i = k - 1 \),

\[
||I_1||_{L^2(0,a;x^0)} \leq \sup_{(x,t) \in \Omega} |\partial_t f(x,u(x,t))| \left| \partial_t^k u \right|_{L^2(0,a;x^0)} \leq C e(t; \partial_t^{k-1} u)^{1/2} \leq \hat{C}.
\]
where \( \tilde{C} \) depends on \( \sup_{\Omega} |\partial_\alpha f(x,u)|, \partial_t^j u(x,0) \) and \( \partial_x \partial_t^{k-1} u(x,0) \). Next we estimate \( I_2 \). \( I_2 \) is a polynomial of \( \partial_t u, \ldots, \partial_t^{k-1} u \) with coefficients \( C_{\alpha\beta}(\partial_\beta u f)(x,u) \). Using (3.18), we have

\[
\left| (\partial_t u)^{a_1} \cdots (\partial_t^{k-1} u)^{a_{k-1}} \right|_{L^2((0,t);x)} \leq \|\partial_t u\|_{L^2((0,t);x)}^{2a_1} \cdots \|\partial_t^{k-1} u\|_{L^2((0,t);x)}^{2a_{k-1}} \leq C e(t; \partial_t u)^{a_1} \cdots \alpha(e; \partial_t^{k-1} u)^{a_{k-1}}.
\]

Therefore using the assumption that (3.11) holds for \( i = 1, \ldots, k-1 \), we obtain

\[
|I_2|^2_{L^2((0,t);x)} \leq C \tag{3.19}
\]

for \( t \in (0, T) \), where \( C \) depends on \( \sup_{\Omega} |\partial_\alpha f(x,u(x,t))|, \partial_t^j u(x,0) \) and \( \partial_x \partial_t^{j-1} u(x,0) \) for \( j = 1, \ldots, k \). Therefore we have

\[
|\partial_t^k f(x,u)|^2_{L^2((0,t);x)} \leq 2\left( |I_1|^2_{L^2((0,t);x)} + |I_2|^2_{L^2((0,t);x)} \right) \leq C_0. \tag{3.20}
\]

This shows (i).

(3.15) is shown in the similar way. We have

\[
\left| I_1(u) - I_1(v) \right|^2_{L^2((0,t);x)} \leq 2 \left( \sup_{\Omega} \left| (\partial_t f(x,u)) \right|^2_{L^2((0,t);x)} \right) + \sup_{x \in (0,t)} \left| (\partial_t f(x,u) - (\partial_t f(x,v))^2_{L^2((0,t);x)} \right| \alpha e(t; \partial_t^{k-1} v)
\]

using the mean-value theorem to the second term and Proposition A.2(iii) in order, we have

\[
\leq C e(t; \partial_t^{k-1} (u - v)) + C_2 \sup_{\Omega} \left| \partial_t^k f(x,\hat{u}) \right|^2 e(t; u - v) e(t; \partial_t^{k-1} v)
\]

\[
\leq C e(t; \partial_t^{k-1} (u - v)) + C_3 e(t; u - v) e(t; \partial_t^{k-1} v),
\]

where \( \hat{u} \) is an intermediate value between \( u \) and \( v \). By (3.11) we have

\[
\left| I_1(u) - I_1(v) \right|^2_{L^2((0,t);x)} \leq C e(t; \partial_t^{k-1} (u - v)) + C_4 e(t; u - v), \tag{3.21}
\]

where \( C_4 \) depends on \( \partial_t^k v(x,0) \) and \( \partial_x \partial_t^{j-1} v(x,0) \). Next we estimate \( I_2(u) - I_2(v) \). We set

\[
F(w_1, \ldots, w_{k-1}) = w_1^{a_1} \cdots w_{k-1}^{a_{k-1}}.
\]

By the mean value theorem we have

\[
F(\partial_t u, \ldots, \partial_t^{k-1} u) - F(\partial_t v, \ldots, \partial_t^{k-1} v) = \sum_{p=1}^{k-1} F_{u,p} (\text{intermediate point}) \partial_t^p (u - v). \tag{3.22}
\]

Using (3.18) and (3.11), we have

\[
\left| F_{u,p} (\text{intermediate point}) \right|_{L^2((0,t);x)} \leq \text{Const.} \tag{3.23}
\]

where \( \text{Const.} \) depends on \( \partial_t^j u(x,0), \partial_t^j v(x,0), \partial_x \partial_t^{j-1} u(x,0) \) and \( \partial_x \partial_t^{j-1} v(x,0) \) for \( j = 1, \ldots, k \). We see

\[
\left| (\partial_t^\alpha f)(x,u) \right| \leq \left| (\partial_t^\alpha f)(x,v) \right| \leq \left| (\partial_t^\alpha f)(x,u) \right| \left| F(\partial_t u, \ldots, \partial_t^{k-1} u) - F(\partial_t v, \ldots, \partial_t^{k-1} v) \right|
\]

\[
\leq \left| (\partial_t^\alpha f)(x,u) \right| \left| F(\partial_t u, \ldots, \partial_t^{k-1} u) \right| + \left| (\partial_t^\alpha f)(x,v) \right| \left| F(\partial_t v, \ldots, \partial_t^{k-1} v) \right|
\]

\[
= J_1 + J_2.
\]

Since \( (\partial_t^\alpha f)(x,u) \) is bounded in \( \Omega \), it follows from (3.23), (3.22) and Proposition 4.2(iii) that

\[
|I_1|^2_{L^2((0,t);x)} \leq C \sum_{p=1}^{k-1} \sup_{x} |\partial_t^p (u - v)(x,t)| \leq C \sum_{p=0}^{k-1} e(t; \partial_t^p (u - v))^{1/2}. \tag{3.24}
\]

Also we have
where $\hat{u}$ is an intermediate value between $u$ and $v$. Since we have
\[
\|u - v\| \leq \hat{C}e(t; u - v)^{1/2}, \quad \|F(\partial_t v, \ldots, \partial_t^{k-1} v)\|_{L^2(0,a;x^0)} \leq C,
\]
where $C$ depends on $\partial_t^j v(x, 0)$, $j = 0, \ldots, k$, we have
\[
|J_2|_{L^2(0,a;x^0)} \leq \hat{C}e(t; u - v)^{1/2}.
\]
From (3.24) and (3.25) we obtain
\[
|I_2(u) - I_2(v)|^2_{L^2(0,a;x^0)} \leq C \sum_{j=0}^{k-1} e(t; \partial_t^j (u - v)).
\]
From (3.21) and (3.26) we have the conclusion. $\square$

**Proof of Proposition 3.2.** We shall show (3.11) by induction. The case $i = 0$ is clear. In fact, applying (3.3) to equation in (P), we obtain (3.11) for $i = 0$. Assuming that (3.11) are true for $i = 1$ to $i = k - 1$, we shall show that (3.11) is true for $i = k$. We differentiate equation $k$-times ($k = 0, \ldots, s$) with respect to $t$. Then we have
\[
\partial_t^2 (\partial_t^k u) + L(\partial_t^k u) + \partial_t^k f(x,u) = 0.
\]
Applying (3.3) to Eq. (3.27), we have
\[
e(t; \partial_t^k u) \leq e^t \left( e(0; \partial_t^k u) + \int_0^t \|\partial_t^k f(x,u)\|^2_{L^2(0,a;x^0)} d\tau \right).
\]
From Lemma 3.1 (3.11) holds for $i = k$. This completes the induction.

Next we shall show (3.13). We use induction. Subtracting (3.27) on $v$ from that on $u$, we have
\[
\partial_t^2 \partial_t^k (u - v) + L\partial_t^k (u - v) + \partial_t^k (f(x,u) - f(x,v)) = 0.
\]
We show the case $i = 0$. Setting $k = 0$ in (3.28) and using (3.3), we have
\[
e(t; u - v) \leq e^t \left( e(0; u - v) + \int_0^t \|f(x,u) - f(x,v)\|^2_{L^2(0,a;x^0)} d\tau \right).
\]
As we see, using the mean-value theorem and Lemma A.1,
\[
\|f(x,u) - f(x,v)\|_{L^2(0,a;x^0)} \leq C\|u - v\|_{L^2(0,a;x^0)} \leq C\|u - v\| \leq C e(t; u - v)^{1/2},
\]
we have
\[
e(t; u - v) \leq e^t \left( e(0; u - v) + C \int_0^t e(\tau; u - v) d\tau \right).
\]
By applying the Gronwall inequality, we obtain
\[
e(t; u - v) \leq C e(0; u - v).
\]
This shows that (3.13) holds for $i = 0$. Next we assume that (3.13) holds for $i = 1, \ldots, k - 1$. It follows from (3.28) and (3.3) that
\[
e(t; \partial_t^k (u - v)) \leq e^t \left( e(0; \partial_t^k (u - v)) + \int_0^t \|\partial_t^k (f(x,u(\cdot, \tau)) - f(x,v(\cdot, \tau)))\|^2_{L^2(0,a;x^0)} d\tau \right).
\]
Applying (3.15) in Lemma 3.1 to the integrand in the right-hand side and using the assumption of the induction in order, we have

\[ e(t; \partial_t^k (u - v)) \leq e^d \left( e(0; \partial_t^k (u - v)) + C_1 \sum_{j=0}^{k-1} \int_0^t e(\tau; \partial_t^j (u - v)) \, d\tau \right) \]

\[ \leq e^d \left( e(0; \partial_t^k (u - v)) + \tilde{C} \sum_{j=0}^{k-1} e(0; \partial_t^j (u - v)) \right) \]

\[ \leq \tilde{C} e^d \sum_{j=0}^k e(0; \partial_t^j (u - v)). \]

This proves (3.13). Thus the proof of the proposition is completed. \( \Box \)

3.2. Approximate solutions of \((P)\)

We shall construct approximate solutions of \((P)\), using the Galerkin method. We shall show that there exists a subsequence of the approximate sequence such that it converges to a time-global classical solution of IBVP \((P)\).

Throughout this and next subsections we assume the conditions (A) and (B).

First we shall construct the sequence of approximate solutions by the similar way to [4]. We can expand \( \phi \) and \( \psi \) into the Bessel series

\[ \phi = \sum_{i=1}^{\infty} a_i \phi_i, \quad \psi = \sum_{i=1}^{\infty} b_i \phi_i \]

in \( K_{2s+1}(0, a; x^0) \) and \( K_{2s}(0, a; x^0) \), respectively. We shall define a sequence of approximate solutions of the form

\[ u_n(x, t) = \sum_{i=1}^{n} a_i^n(t) \phi_i(x) \]

that satisfy

\[ (\partial_t^2 u_n + L u_n + f(x, u_n), \phi_k) = 0, \quad k = 1, \ldots, n, \] (3.29)

with initial condition

\[ u_n(\cdot, 0) = \alpha_n, \quad \partial_t u_n(\cdot, 0) = \beta_n. \] (3.30)

Here \( \alpha_n(x) \) and \( \beta_n(x) \) are of the form

\[ \alpha_n(x) = \sum_{i=1}^{n} a_i^n \phi_i(x), \quad \beta_n(x) = \sum_{i=1}^{n} b_i^n \phi_i(x) \] (3.31)

satisfying

\[ |\alpha_n - \phi|_{H^{2s+1}(0, a;x^0)} \to 0, \quad |\beta_n - \psi|_{H^{2s}(0, a;x^0)} \to 0 \] (3.32)

as \( n \to +\infty \). Clearly \( \alpha_n \) and \( \beta_n \) belong to \( K^\infty(0, a; x^0) \). Let \( P_n \) be the projector of \( L^2(0, a; x^0) \) onto the linear subspace spanned by \{\( \phi_1, \ldots, \phi_n \)\}. Clearly \( |P_n f|_{L^2(0, a;x^0)} \leq |f|_{L^2(0, a;x^0)} \). Also note that \( P_n \) is commutative with \( L \) and \( \partial_t \). Eq. (3.29) is equivalent to

\[ \partial_t^2 u_n + L u_n + P_n f(x, u_n) = 0. \] (3.33)

(3.29)–(3.30) is IVP to a system of ODEs with \( a_1^n(t), \ldots, a_n^n(t) \). Note that

\[ \begin{cases}
  u_n(x, 0) = \alpha_n(x), & \partial_t u_n(x, 0) = \beta_n(x), \\
  \partial_t^2 u_n(x, 0) = -(L \partial_t^{j-2} u_n(x, 0) + P_n \partial_t^{j-2} f(x, u_n(x, 0)))
\end{cases} \] (3.34)

for \( i = 2, \ldots, s \).
According to [4] we can take a subsequence of \( \{u_n\} \), again written by \( \{u_n\} \), such that it converges to some \( u \) in \( H_0^1(\Omega; x^0) \) which is a weak solution of (P).

The following lemma is due to [4].

**Lemma 3.2.** (See [4].) There exists \( n_0 \in \mathbb{N} \) such that for any \( n \geq n_0 \) the following holds:

(i) IVP (3.29)–(3.30) has a solution \( u_n \) of \( C^s \)-class for \( t \in (0, T) \).

(ii) \( u_n(\cdot, t) \in W \) for \( t \in (0, T) \).

(iii) \( u_n \) satisfies
\[
E(t; u_n) < d 
\]
for all \( t \in (0, T) \).

We shall estimate the higher order \( t \)-derivatives of \( u_n \).

**Proposition 3.3.** \( \{u_n\} \) satisfies
\[
e(t; \partial_t^i u_n) \leq C, \quad t \in (0, T), \tag{3.36}
\]
for \( i = 0, \ldots, s \), where \( C \) is independent of \( n \).

**Lemma 3.3.**
\[
e(0; \partial_t^i u_n) \leq \hat{C} \tag{3.37}
\]
for \( i = 0, \ldots, s \), where \( \hat{C} \) is independent of \( n \).

**Proof.** We shall show this lemma by induction. First we note the following. \( L\partial_t^i u_n(x, 0) \) is represented by the form:

If \( j \) is even,
\[
L\partial_t^j u_n(x, 0) = -(-L)^{j/2+1} \alpha_n + A_j(x, \alpha_n(x), \alpha_n'(x), \ldots, \alpha_n^{(j)}(x), \beta_n(x), \beta_n'(x), \ldots, \beta_n^{(j-2)}(x)), \tag{3.38}
\]
and if \( j \) is odd,
\[
L\partial_t^j u_n(x, 0) = -(-L)^{(j+1)/2} \beta_n(x) + B_j(x, \alpha_n(x), \alpha_n'(x), \ldots, \alpha_n^{(j-1)}(x), \beta_n(x), \beta_n'(x), \ldots, \beta_n^{(j-1)}(x)), \tag{3.39}
\]
where \( A_j(x, y_0, \ldots, y_j, z_0, \ldots, z_{j-2}) \) and \( B_j(x, y_0, \ldots, y_{j-1}, z_0, \ldots, z_{j-2}) \) are of \( C^{s-j} \) in \( (x, y_0, \ldots, y_j, z_0, \ldots, z_{j-2}) \) and \( (x, y_0, \ldots, y_{j-1}, z_0, \ldots, z_{j-1}) \) in \( [0, a] \times R^{j+1} \times R^{j-1} \), respectively.

By the assumption (B) we have \( \hat{e}(\phi, \psi) \leq M \) (see [4]), which leads to the case \( i = 0 \). For \( i = 1 \) we have
\[
e(0; \partial_t u_n) = \frac{1}{2} \int_0^a \left( |\partial_t^2 u_n(x, 0)|^2 + x |\partial_x \partial_t u_n(x, 0)|^2 \right) dx.
\]

By equation we have
\[
\int_0^a |\partial_t^2 u_n(x, 0)|^2 dx \leq 2 \int_0^a \left( |L u_n(x, 0)|^2 + |f(x, u_n(x, 0))|^2 \right) dx = 2 \int_0^a \left( |L \alpha_n(x)|^2 + |f(x, \alpha_n(x))|^2 \right) dx \leq \text{Const}.
\]

We have by (3.32),
\[
\int_0^a x |\partial_x \partial_t u_n(x, 0)|^2 dx = \int_0^a x |\partial_x \beta_n(x)|^2 dx \leq \text{Const}.
\]
From these inequalities we obtain (3.37) for \( i = 1 \).

Let (3.37) be satisfied for \( i = 2, \ldots, 2k - 1 \), where \( 2k \leq s \). Then we have

\[
\int_0^a |\partial_t^{2k+1} u_n(x, 0)|^2 \, dx = \int_0^a |\partial_t^{2k-1} \partial_t^2 u_n(x, 0)|^2 \, dx \\
\leq 2 \left\{ \int_0^a |L \partial_t^{2k-1} u_n(x, 0)|^2 \, dx + \int_0^a (\partial_t^{2k-1} f(x, u_n(x, t)))^2 \, dx \right\} \\
= 2(J_1 + J_2).
\]

\( J_2 \) is directly estimated by the same way as the proof of (3.14) in Lemma 3.1:

\[
|\partial_t^j f(x, u_n(\cdot, t))|_{t=0} \leq C.
\]  

(3.40)

For \( j = 2k - 1 \) the constant of the right-hand side of (3.40) is uniformly bounded with respect to \( n \) by the assumption of induction. Therefore \( J_2 \leq C \). For \( J_1 \) we use (3.39). By using Proposition A.2(iv) and (3.32), it is shown that \( \alpha_n, \ldots, \alpha_n^{(j-1)}, \beta_n, \ldots, \beta_n^{(j-1)} \) are uniformly bounded with respect to \( n \). Hence \( B_j(\alpha_n, \ldots, \beta_n^{(j-1)})(x) \) is also uniformly bounded with respect to \( n \). Since \( |L^k \beta_n|_{L^2(0, a; x^0)} \) is estimated by \( |\beta_n|_{H^2(0, a; x^0)} \), \( J_1 \) is bounded in \( n \). We shall estimate the second integral in \( e(0; \partial_t^i u_n) \). We have

\[
\int_0^a x |\partial_t \partial_t^k u_n(x, 0)|^2 \, dx = \int_0^a L \partial_t^{2k} u_n(x, 0) \partial_t^2 u_n(x, 0) \, dx \\
\leq |L \partial_t^{2k} u_n(x, 0)|_{L^2(0, a; x^0)} |\partial_t^2 u_n(x, 0)|_{L^2(0, a; x^0)} \\
\leq C
\]

by (3.38), Propositions A.5 and A.2(iv).

Next let (3.37) be satisfied for \( i = 2k - 2 \). Then using (3.40) and (3.38) instead of (3.39), we shall be able to show in the same way as above that (3.37) holds for \( i = 2k - 1 \). This leads to the completion of the induction. Thus the lemma is proved.

**Proof of Proposition 3.3.** We shall show (3.36) in the similar way to the energy estimate (3.11) in Proposition 3.2. We shall use induction. Differentiating Eq. (3.29) \( i \)-times (\( 0 \leq i \leq s \)), we have

\[
(\partial_t^2 \partial_t^i u_n + L \partial_t^i u_n + \partial_t^j f(x, u_n, \phi_k) = 0.
\]

We multiply the above equation by \( (d^i / dt^i)u_n^0 \) and sum from \( k = 1 \) to \( k = i \). Then using the energy estimate (3.5), we obtain

\[
\frac{d}{dt} e(t; \partial_t^i u_n) \leq e(t; \partial_t^i u_n) + 2 |\partial_t^j f(\cdot, u_n)|_{L^2(0, a; x^0)}^2.
\]

The case \( i = 0 \) is clear from the fact that \( \sup f(\cdot, u_n)|_{L^2(0, a; x^0)} \) is uniformly bounded with respect to \( n \) as \( u_n(x, t) \) is uniformly bounded in \( \Omega \). Let (3.36) hold for \( i = 1, \ldots, s - 1 \). Similarly to (3.20) in the proof of Proposition 3.2 we see

\[
|\partial_t^i f(x, u_n)|_{L^2(0, a; x^0)}^2 \leq C(e(t; \partial_t^{i-1} u_n)^2 + 1) \leq \text{Const.}
\]

(3.41)

Here in the last inequality we have used the assumption of the induction. Therefore we have

\[
e(t; \partial_t^i u_n) \leq C e^T (e(0; \partial_t^i u_n) + \gamma).
\]

(3.42)

Here \( \gamma > 0 \) is a constant independent of \( n \). From Lemma 3.3 we have the conclusion.

Using (3.15) in Lemma 3.1, we can show the following proposition, similarly to the proof of Propositions 3.2 and 3.3.
Proposition 3.4. \( \{u_n\} \) satisfies

\[
e(t; \partial_t^i (u_n - u_m)) \leq C e^T \sum_{k=0}^{i} e(0; \partial_t^k (u_n - u_m)), \quad t \in (0, T),
\]

for \( i = 1, \ldots, s \), where \( C \) is independent of \( n, m \).

Corollary 3.1. Let \( \partial_t^i u_n(\cdot, 0) \), \( i = 0, \ldots, s \), be a Cauchy sequence with respect to the energy norm. Then there exists \( u \) with \( \sup_{t \in (0, T)} e(t; \partial_t^i u) < +\infty \) such that

\[
\sup_{t \in (0, T)} e(t; \partial_t^i (u_n - u)) \to 0 \quad (n \to +\infty)
\]

for \( i = 0, \ldots, s \) and \( t \in (0, T) \). Here \( \partial_t^i u \) means the derivative in the distribution sense.

3.3. Compactness method

In this subsection we shall extract subsequences of \( \{u_n\} \), but for brevity of notation we shall again write them by \( \{u_n\} \).

Lemma 3.4. There exist a subsequence \( \{u_n\} \) such that \( \partial_t^i u_n \) converges to \( \partial_t^i u \) in \( L^2(\Omega; x^0) \) and converges a.e. in \( \Omega \) for \( i = 0, \ldots, s \). \( \partial_t^i u \) belong to \( H^1_0(\Omega; x^0) \).

Proof. From Proposition 3.3 and Lemma A.1 \( \{\partial_t^i u_n\} \) is bounded in \( H^1_0(\Omega; x^0) \). Hence by Proposition A.3 (the Rellich type theorem) we can extract a subsequence that converges in \( L^2(\Omega; x^0) \). The limit element is equal to \( \partial_t^i u \) by the standard distribution argument. Also by the boundedness of \( \{\partial_t^i u_n\} \) in \( H^1_0(\Omega; x^0) \) we can extract a subsequence of \( \{\partial_t^i u_n\} \) that converges weakly in \( H^1_0(\Omega; x^0) \). Therefore \( \partial_t^i u \) belongs to \( H^1_0(\Omega; x^0) \). It is well known that a bounded sequence in \( L^2(\Omega) \) contains a subsequence which converges a.e. in \( \Omega \). This proves the lemma. \( \square \)

From this lemma we have the following lemma.

Lemma 3.5. Let \( n_0 \) be the integer taken in Lemma 3.2. Let \( n \geq n_0 \). Then there exists a subsequence \( \{u_n\} \) such that

\[
|(L \partial_t^j u_n, \zeta) - (L \partial_t^j u, \zeta)| \to 0
\]

for any \( \zeta \in H^1_0(\Omega; x^0) \) and \( j = 1, \ldots, s - 1 \) as \( n \to +\infty \).

Proof. It follows from Proposition 3.3 that \( \{\sqrt{n} \partial_t^j u_n\} \) is bounded in \( L^2(\Omega; x^0) \). Hence \( \{(L \partial_t^j u_n, \zeta)\} \) is bounded so that by Proposition A.1(i), we can extract a subsequence of \( \{(L \partial_t^j u_n, \zeta)\} \) that converges to \( (x \partial_x \partial_t^j u, \partial_x \zeta) \) for any \( \zeta \in H^1_0(\Omega; x^0) \). \( \square \)

The following proposition assures that the solution is of \( C^2 \) in \( \Omega \).

Proposition 3.5. Let \( s \geq 3 \). Then

\[
\bigcap_{i=0}^{s} C^i \left( (0, T); H^{3-i} (0, a; x^0) \right) \subset C^2 \left( (0, a] \times [0, T] \right).
\]

Proof. It is enough to show (3.45) for \( s = 3 \). Let \( u \) belong to \( \bigcap_{i=0}^{3} C^i \left( (0, T); H^{3-i} (0, a; x^0) \right) \). Then \( \partial_t^l u \) belong to \( H^{3-l} (0, a; x^0) \) for \( l = 0, \ldots, i \). Hence it follows from Proposition A.2(ii) that \( \partial_t^l u(x, t) \) is \( (2-i) \)-times differentiable in \( x \in (0, a] \) in the classical sense and

\[
|x^{l/2} \partial_t^l \partial_t^i u(x, t)| \leq C |\partial_t^l u(\cdot, t)|_{H^{3-l}(0, a; x^0)}
\]

(3.46)
holds for \( j = 0, \ldots, 2 - i \) and \( t \in (0, T) \). On the other hand we have, by (3.46),

\[
\frac{x^{j/2}}{h} \left| \frac{\partial^j_{t} \partial^l_{t}^{-1} u(x, t + h) - \partial^j_{t} \partial^l_{t}^{-1} u(x, t)}{h} - \partial^j_{t} \partial^l_{t} u(x, t) \right| \leq C \left| \frac{\partial^j_{t} \partial^l_{t}^{-1} u(\cdot, t + h) - \partial^j_{t} \partial^l_{t}^{-1} u(\cdot, t)}{h} - \partial^j_{t} u(\cdot, t) \right|_{H^{3-i}(0, a; \xi^0)}.
\]

The right-hand side tends to 0 as \( h \to 0 \). This means that \( \partial^j_{t} \partial^l_{t}^{-1} u(x, t) \) is differentiable in \( t \in [0, T] \) in the classical sense and the strong derivative \( \partial^j_{t} \partial^l_{t} u \) coincides with the partial derivative with respect to \( t \) in the classical sense.

We show that \( \partial^p_{t} \partial^q_{t} u(x, t), 0 \leq p + q \leq 2 \) are continuous in \( (x, t) \in (0, a] \times [0, T] \). We have

\[
\left| x^{p/2} \partial^p_{t} \partial^q_{t} u(x, t) - x_0^{p/2} \partial^p_{t} \partial^q_{t} u(x_0, t_0) \right| \leq \left| x^{p/2} (\partial^p_{t} \partial^q_{t} u(x, t) - \partial^p_{t} \partial^q_{t} u(x_0, t_0)) \right| + \left| x^{p/2} \partial^p_{t} \partial^q_{t} u(x, t_0) - x_0^{p/2} \partial^p_{t} \partial^q_{t} u(x_0, t_0) \right|. \tag{3.47}
\]

The first term is estimated

\[
\left| x^{p/2} \partial^p_{t} \partial^q_{t} u(x, t) - \partial^p_{t} u(x, t_0) \right| \leq C \left| \partial^q_{t} u(\cdot, t) - \partial^q_{t} u(\cdot, t_0) \right|_{H^{p+1}(0, a; \xi^0)} \leq C \left| \partial^q_{t} u(\cdot, t) - \partial^q_{t} u(\cdot, t_0) \right|_{H^{3-q}(0, a; \xi^0)}.
\]

The right-hand side tends to 0 as \( t \to t_0 \). The second term in the right-hand side of (3.47) clearly tends to 0 as \( x \to x_0 \).

For, \( \partial^p_{t} \partial^q_{t} u(x, t_0) \) is continuous in \( x \in (0, a) \). This means that \( x^{p/2} \partial^p_{t} \partial^q_{t} u(x, t) \) is continuous in \( (x, t) \in \Omega \). This leads to the conclusion. \( \Box \)

### 3.4. Smooth Solutions of IBVP (P)

In this subsection we shall show that \( u \) obtained in Section 3.3 is the global smooth solution of IBVP (P), and study the regularity.

We shall again consider (3.33) with (3.34)

\[
\partial^2_{t} u_n + L u_n + P_n f(x, u_n) = 0, \tag{3.33}
\]

\[
\begin{align*}
\{ u_n(x, 0) &= \alpha_n(x), \\
\partial^l_{t} u_n(x, 0) &= \beta_n(x), \\
\partial^l_{t} u_n(x, 0) &= -(L \partial^l_{t}^{-2} u_n(x, 0) + P_n \partial^l_{t}^{-2} f(x, u_n(x, 0)))
\end{align*} \tag{3.34}
\]

for \( i = 2, \ldots, s \). (3.33) is equivalent to (3.29)

\[
\partial^2_{t} u_n + L u_n + f(x, u_n), \phi_k = 0, \quad k = 1, \ldots, n. \tag{3.29}
\]

Differentiating (3.29) \( j \)-times with respect to \( r \) for \( j = 0, \ldots, s - 1 \), we have

\[
\partial^2_{t} \partial^{j}_{r} u_n + L \partial^j_{r} u_n + \partial^j_{r} f(x, u_n), \phi_k = 0, \quad k = 1, \ldots, n. \tag{3.48}
\]

Let \( V_m \) be the set of linear combinations of \( \phi_1, \ldots, \phi_m \). Then we have

\[
\partial^2_{t} \partial^{j}_{r} u_n + L \partial^j_{r} u_n + \partial^j_{r} f(x, u_n), \xi = 0
\]

for any \( \xi \in V_m \) if \( n \geq m \). Using Corollary 3.1, Lemma 3.5 and (3.15) in Lemma 3.1, we obtain

\[
\partial^2_{t} \partial^{j}_{r} u + L \partial^j_{r} u + \partial^j_{r} f(x, u), \xi = 0
\]

as \( n \to +\infty \). Since \( \bigcup_{m=1}^{\infty} V_m \) is dense in \( L^2(0, a; \xi^0) \), it follows that

\[
\partial^2_{t} \partial^{j}_{r} u_n + L \partial^j_{r} u_n + \partial^j_{r} f(x, u) = 0 \quad \text{in} \quad L^2(0, a; \xi^0)
\]

for \( j = 0, \ldots, s - 1 \). We shall show that \( u \) belongs to

\[
C^{s+1}((0, T); L^2(0, a; \xi^0)) \cap \bigcap_{i=1}^{s} C^i((0, T); H^{s+1-i}(0, a; \xi^0) \cap H^1_0(0, a; \xi^0)). \tag{3.50}
\]
From Corollary 3.1 and Lemma A.1 the convergence is uniform with respect to \( t \in (0, T) \) in energy norm with \( i = s \). Hence

\[
  u \in C^{s+1}((0, T); L^2(0, a; x^0)) \cap C^s((0, T); H^1_0(0, a; x^0)).
\]

We show that \( u \) belongs to \( C^{s-1}((0, T); K^2(0, a; x^0)) \). Setting \( j = s - 1 \) in (3.49), we see

\[
  L\partial^s_t u(\cdot, t) = -\partial^{s+1}_t u(\cdot, t) - \partial^{s-1}_t f(\cdot, u(\cdot, t)).
\]

(3.51)

Since from (3.15) in Lemma 3.1 \( \partial_s^{s-1} f(\cdot, u(\cdot, t)) \) is \( L^2 \)-continuous in \( t \), the right-hand side is \( L^2 \)-continuous in \( t \). By applying Proposition A.8 to (3.51), it follows that \( \partial^s_t u(\cdot, t) \) belongs to \( K^2(0, a; x^0) \). Applying (A.9) in Proposition A.8 to the difference \( u(\cdot, t) - u(\cdot, t_0) \), we have the \( H^2 \)-continuity of \( \partial^s_t u(\cdot, t) \). This means

\[
  u \in C^{s-1}((0, T); H^2(0, a; x^0) \cap H^1_0(0, a; x^0)).
\]

Next for \( j = s - 2 \) Eq. (3.49) becomes

\[
  L\partial^{s-2}_t u(\cdot, t) = -\partial^s_t u(\cdot, t) - \partial^{s-2}_t f(\cdot, u(\cdot, t)).
\]

(3.52)

We shall show that the right-hand side belongs to \( H^1_0(0, a; x^0) \) and \( H^1 \)-continuous in \( t \in (0, T) \). From (3.16)

\[
  \partial_x \partial^s_t f(x, u) = \sum C_{\alpha \beta} \partial_x \{ (\partial^\beta_u f)(x, u)(\partial_t u)^{\alpha_1} \cdots (\partial^k_t u)^{\alpha_k} \}
\]

\[
  = \sum C_{\alpha \beta} \left\{ (\partial^{\beta+1}_u f)(x, u)(\partial_t u)^{\alpha_1} \cdots (\partial^k_t u)^{\alpha_k} + \sum_{p=1}^k (\partial^\beta_u f)(x, u)\alpha_p(\partial_x \partial^p_t u)(\partial_t u)^{\alpha_1} \cdots (\partial^p_t u)^{\alpha_p-1} \cdots (\partial^k_t u)^{\alpha_k} \right\}.
\]

We estimate each term of the second term with \( k = s - 2 \) as follows:

\[
  \int_0^a x |(\partial_x \partial^s_t u)(\partial_t u)^{\alpha_1} \cdots (\partial^p_t u)^{\alpha_p-1} \cdots (\partial^{s-2}_t u)^{\alpha_{s-2}}|^2 \, dx
\]

\[
  \leq \sup_x (\partial_t u)^{\alpha_1} \cdots (\partial^p_t u)^{\alpha_p-1} \cdots (\partial^{s-2}_t u)^{\alpha_{s-2}} \int_0^a x |\partial_x \partial^p_t u|^2 \, dx.
\]

Since each \( \partial^l_t u(\cdot, t) \), \( l = 1, \ldots, s - 2 \) belongs to \( H^1_0(0, a; x^0) \) and

\[
  \sup_x |\partial^l_t u(\cdot, t)| \leq C |\partial^l_t u(\cdot, t)|_{H^1(0, a; x^0)}.
\]

Also

\[
  \int_0^a x |\partial_x \partial^p_t u|^2 \, dx \leq C |\partial^p_t u(\cdot, t)|_{H^1(0, a; x^0)}.
\]

The first term is estimated in the similar way. Hence it follows that

\[
  \int_0^a x |\partial_x \partial^{s-2}_t f(x, u(x, t))|^2 \, dx < +\infty.
\]

Since

\[
  \int_0^a |\partial^{s-2}_t f(x, u(x, t))|^2 \, dx < +\infty
\]
is clear, $\partial_t^s \tilde{f}(\cdot, u(\cdot, t))$ belongs to $H_0^1(0, a; x^0)$. The similar calculations to prove (3.15) in Lemma 3.1 show the $H^1$-continuity of $\partial_t^s \tilde{f}(\cdot, u(\cdot, t))$. Therefore the right-hand side of (3.52) is $H^1$-continuous. By (A.9) $\partial_t^s \tilde{u}(\cdot, t)$ is $H^3$-continuous. This means

$$u \in C^{s-2}_t((0, T); H^3(0, a; x^0) \cap H_0^1(0, a; x^0)).$$

Repeating this process, we can show that $u$ belongs to (3.50).

Appendix A

A.1. Properties of function spaces and inequalities

**Proposition A.1.** (See [6], Lemma 2.1) $L$ has the following properties.

(i) For $f \in K^2(0, a; x^m)$ and $g \in K^1(0, a; x^m)$

$$(L f, g)_{L^2(0, a; x^m)} = \int_0^a \frac{x^{m+1}}{m+1} \partial_x f(x) \partial_x g(x) \, dx.$$

(ii) $L$ is a positive definite self-adjoint elliptic operator in $L^2(0, a; x^m)$ with domain $D(L) = K^2(0, a; x^m)$, i.e.,

$$(L f, f)_{L^2(0, a; x^m)} \geq 0, \quad (L f, g)_{L^2(0, a; x^m)} = (L g, f)_{L^2(0, a; x^m)}$$

hold for $f, g \in K^2(0, a; x^m)$.

**Proposition A.2.** The following holds:

(i) If $s \geq 1$, $H^s(0, a; x^m)$ is embedded continuously in $L^2(0, a; x^{m-\delta})$ for any $\delta$ with $0 < \delta < m+1$.

(ii) If $s \geq 1$ and $1 \leq r \leq +\infty$, $H^s(0, a; x^0)$ is embedded in $L^r(0, a) \cap C^{s-1}((0, a])$. For $u \in H^s(0, a; x^0)$

$$|u|_{L^r(0, a; x^0)} \leq C |u|_{H^s(0, a; x^0)},$$

where $C$ depends on $r, a$ and is independent of $s$ and $k$ is any nonnegative number.

(iii) If $1 \leq r \leq +\infty$, $H_0^1(0, a; x^0)$ is embedded continuously in $L^r(0, a) \cap C((0, a])$. For $u \in H_0^1(0, a; x^0)$

$$|u|_{L^r(0, a; x^0)} \leq C \|u\|,$$

where $C$ depends on $r, a$, and $k$ is any nonnegative number.

(iv) Let $s \geq 1$. If $u$ belongs to $H^s(0, a; x^0)$, $\partial_x^j u(x)$ is uniformly bounded in $[0, a]$ for $j = 0, \ldots, [(s-1)/2]$.

**Proof.** (i) is seen in [6, Lemma 2.1].

We prove (ii). First let $r = +\infty$. From [7] we have

$$\sup_{x \in (0, a)} f(x)^2 \leq C \left( \int_0^a x f(x)^2 \, dx + \int_0^a x f'(x)^2 \, dx \right).$$

From this $H^1(0, a; x^0)$ is continuously embedded in $L^\infty(0, a)$. It is clear by the Sobolev lemma that $H^s(0, a; x^m)$ is continuously embedded in $C^{s-1}((0, a])$. The case $r < +\infty$ is clear from $|f|_{L^r} \leq C |f|_{L^\infty}$.

(iii) is directly proved from (ii) with $s = 1$ and Lemma A.1.

We show (iv). From Lemma 2.1 in [7] we have

$$\sup_{x \in [0, a]} |u^{(j)}(x)| \leq C \left( \int_0^a x |u^{(j)}(x)|^2 \, dx + \int_0^a x |u^{(j+1)}(x)|^2 \, dx \right).$$
Applying Lemma 2.1 in [7] to the right-hand side, we have, for any \( p \in \mathbb{N}, \ p \leq s - (j + 1) \),

\[
\int_0^a x \left| u(j+1)(x) \right|^2 \, dx \leq C \sum_{k=0}^p \int_0^a x^{2k+1} \left| u(j+k+1)(x) \right|^2 \, dx.
\]

Taking \( p = j \), we obtain

\[
\sup_{x \in [0,a]} \left| u(j)(x) \right| \leq C \| u \|_{H^s(0,a;x^0)}
\]

for \( 2j + 1 \leq s \). This proves (iv). \( \square \)

**Corollary A.1.** Let \( s \geq 1 \) and \( k \geq 0 \). Let \( 1 \leq r < +\infty \). For \( u \in H^s(\Omega; x^m) \)

\[
|u|_{L^r(\Omega; x^{m+k})} \leq C|u|_{H^s(\Omega; x^m)},
\]

where \( C \) depends on \( r, m, a \) and is independent of \( s, T \).

**Lemma A.1.** (See [6].) For \( u \in H^1_0(0, a; x^m) \)

\[
|u|_{L^2(0, a; x^m)} \leq a|\partial_x u|_{L^2(0, a; x^{m+1})}.
\]

This is the Poincaré type inequality in \( H^1_0(0, a; x^m) \).

**Proposition A.3.** Any bounded set contained in \( H^1_0(\Omega; x^m) \) is relatively compact in \( L^2(\Omega; x^m) \).

Proof is done in the similar way to [8, Proposition 4.1]. So it is omitted here.

**Proposition A.4.** If \( s \geq m + 1 \), \( H^s(0, a; x^m) \) is embedded continuously in \( L^\infty(0, a) \cap C^{s-1}(0, a) \).

This is a corollary of Proposition 2.3 in [7].

**Proposition A.5.** (See [6].) Let \( p, q \in \mathbb{Z}_+ \). Then

\[
C_1 \| f \|_{H^{2p+q}(0, a; x^m)} \leq \| L^p f \|_{H^q(0, a; x^m)} \leq C_2 \| f \|_{H^{2p+q}(0, a; x^m)}
\]

(A.4)

for \( f \in K^{2p+q}(0, a; x^m) \), where \( C_1, C_2 > 0 \) depends on \( p, q, m \).

**A.2. Eigenvalue problem for \( L_m \)**

We shall consider the eigenvalue problem for \( L_m \)

\[
L_m \phi(x) = \lambda \phi(x), \quad x \in (0, a),
\]

\[
\phi(a) = 0.
\]

(A.5)

We obtain the eigenvalues and the corresponding eigenfunctions ([6, Section 3], [3])

\[
\lambda_j = \frac{\mu_j^2}{4(m+1)a}, \quad \phi_j(x) = \frac{1}{a^{1/2}J_{m+1}(\mu_j)} \frac{J_m(\mu_j \sqrt{x/a})}{x^{m/2}},
\]

where \( j \in \mathbb{N} \). Here \( \{ \mu_j : j \in \mathbb{N} \} \) is the set of all positive zero points of the \( m \)-order Bessel function \( J_m(x) \) with \( \mu_1 < \mu_2 < \cdots \).

**Proposition A.6.** (See [6].) \( \{ \phi_j \} \) is CONS in \( L^2(0, a; x^m) \) and a complete and orthogonal system in \( H^1_0(0, a; x^m) \).
Proposition A.7. (See [6].) Let \( u \in K^s(0, a; x^0) \) and its Fourier expansion be \( u = \sum_{k \geq 1} u_k \phi_k \). Then
\[
c_0 |u|_{H^s(0, a; x^0)}^2 \leq \sum_{k \geq 1} \lambda_k^s |u_k|^2 \leq c_1 |u|_{H^s(0, a; x^0)}^2,
\]
where \( c_0 \) and \( c_1 \) are independent of \( u \).

A.3. Elliptic estimates

We shall need a result on an elliptic BVP
\[
\begin{cases}
Lu = h(x), & x \in (0, a), \\
u(a) = 0,
\end{cases}
\]
where \( h \in L^2(0, a; x^0) \).

Proposition A.8. (See [8].) BVP (A.7) has a solution \( u \in K^2(0, a; x^0) \) satisfying
\[
|u|_{H^2(0, a; x^0)} \leq C|h|_{L^2(0, a; x^0)}.
\]
More generally, if \( h \in H^p(0, a; x^0) \cap H^1_0(0, a; x^0) \), \( p \in \mathbb{N} \), then \( u \in H^{p+2}(0, a; x^0) \cap H^1_0(0, a; x^0) \) satisfying
\[
|u|_{H^{p+2}(0, a; x^0)} \leq C|h|_{H^p(0, a; x^0)}.
\]

References