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Book Review

Review of Matrices and Graphs in Geometry. Encyclopedia of Mathematics and its Applications, vol. 139 by Miroslav Fiedler, Cambridge University Press (2011). viii + 197 pp., ISBN: 978-0-521-46193-1

At some time each of us was initiated into the secrets of various “special points" of triangles, such as centroids, orthocenters, circumcenters, incenters, excenters, and possibly a few others. “Triangle geometry" had a flowering in the 19th century, and has had somewhat of a revival at the end of the 20th; witness the list of 400 “centers" in Kimberling’s book *Triangle Centers and Central Triangles* (Congr. Numerantium 129 (1998) 1–285) or the extended list with 3612 “centers" in his *Encyclopedia of Triangle Centers*, to be found at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html (accessed 8/18/2011). Most mathematicians can safely ignore most of these “centers", but still be interested in the analogues of at least some of them in dimensions higher than 2.

In the 3-dimensional case, some of the old texts occasionally mention a few “special points" associated with tetrahedra or other polyhedra. For example, Altshiller-Court discusses centroids in *College Geometry*" (2nd ed., Barnes & Noble, New York, 1952), as does Lines in *Solid Geometry*" (Dover, New York, 1965). The latter defines the orthocenter of certain tetrahedra, but does not use this concept.

Centroids of polytopes (especially convex ones) appear in numerous texts and there is no reason to list these here. Instead let us proceed to a description of what is Fiedler’s book about — material that cannot be found anywhere else in such detailed and accessible form. To quote the author, “The main subject is simplex geometry, a topic which has fascinated me since my student days… A large part of the content is concerned with qualitative properties of a simplex."

Here is a brief synopsis, according to the introduction:

The first chapter introduces matricial methods, that are one of the bases of the whole exposition. Simplices of all dimensions are defined, and their basic geometric properties are described in the second chapter. The third chapter studies the possible distributions of acute, right, and obtuse dihedral angles in a simplex, and extensions of that topic. Only special simplexes admit generalizations of some of the centers of triangles. Characterizations and properties of right, orthocentric, cyclic and other simplexes are the topic of the fourth chapter. The last two chapters deal with a number of special topics and applications.

Fiedler has been publishing a series of papers on simplices and their properties for over fifty years. Many of these are not easily available, and the present exposition is a welcome substitute for the often hard to get papers. Another advantage of the book is consistent terminology and notation — this also makes it easier to absorb the material. The presentation is entirely algebraic, and the description of simplices and their properties is almost exclusively formulated in terms of entities that depend on the edge-lengths of the simplices. Thus there are echoes of Blumenthal’s *Theory and Applications of Distance Geometry* (Clarendon Press, London, 1953), although the aims of the two books are quite different.

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To give a taste of the results of the book, consider the following (which is a slight reformulation of Theorem 3.1.3). Let an n-simplex have vertices $V_1, \ldots, V_{n+1}$ and opposite facets (that is, $(n-1)$-dimensional faces) $\omega_1, \ldots, \omega_{n+1}$; also let ω_{ij} be the dihedral (interior) angle determined by ω_i and ω_j. In the complete graph G with vertices $V_1, \ldots, V_{n+1}$ color edge V_iV_j red if ω_{ij} is acute, blue if ω_{ij} is obtuse, and green if ω_{ij} is right. Then the subgraph of G consisting of all the vertices and all the red edges is connected. Conversely, given a graph G with edges colored red, blue and green in such a way that the red edges form a connected subgraph that includes all $n+1$ vertices, there exists an n-simplex in which the dihedral angles that correspond to the red, blue, or green edges are acute, obtuse, or right, respectively.

As one of the very few graphical illustrations shown in the book, all 19 possible 3-colored graphs G that satisfy the above conditions for $n = 3$ are shown in Fig. 3.1.

Section 3.4 investigates the position of the circumcenter (that is, center of the circumsphere of the simplex) in dependence of the dihedral angles of the simplex. The results are generalizations of the observation that the circumcenter is an interior point for acute triangles, is on an edge for right triangles, and for obtuse ones it is outside the triangle for obtuse ones.

An n-simplex is said to be *right* if it has exactly n acute dihedral angles, and all the remaining $n(n-1)/2$ are right. One of the results (in Theorem 4.1.2) that is visually appealing states that the set of $n+1$ vertices of a right n-simplex can be completed to the set of the 2^n vertices of a rectangular n-box.

As is well known, the property of a triangle that the three altitudes are concurrent does not generalize to higher dimension. In Section 4.2 Fiedler provides characterizations and properties of *orthocentric* simplices, that is, simplices in which all altitudes have a common point. The characterizations are in terms of lengths of edges, and are too complicated to be detailed here. But several beautiful results are striking generalization of the 2-dimensional situation; one is (Theorem 4.2.12): The centroid T, the circumcenter S, and the orthocenter of an orthocentric n-simplex are collinear, with T between S and V and $2ST = (n-1)TV$.

In later sections of Chapter 4 several other special kinds of simplices are defined. These lead to the existence of various special points, that generalize traditionally named points (such as those named for Gergonne and Lemoine), and their properties.

The text has shortcoming of several kinds. One is the absence of meaningful references to further developments of subjects treated in the various chapters. This is highly regrettable, since some of the concepts introduced by Fiedler, and results about them, play a very important role in various applications. For example, the concept of M-matrices (considered in Appendix A of the book) is relevant to many contemporary works (see, for example, J. Hofbauer and J.W. So, Proc. AMS 128 (2000) 2675–2682, and many other papers which mention Fiedler). In the book there are 28 references to works by Fiedler (some with coauthors), but a total of only four references to other authors.

Some of the omissions are rather surprising. For example, Section 5.6 deals in part with Radon's well-known theorem about the existence of a partition of any $(n+2)$-point set in Euclidean n-space into two (disjoint) subsets, such that their convex hulls have a non-empty intersection. But this is presented — without any mention of Radon — as a result on objects called "n-bisimplexes", introduced in a roundabout way. Theorem 6.3.3 is just a weak version of the often-quoted result of H.W.E. Jung from 1901.

Aside from these problems, there are impediments that could have been easily avoided. For example, α is often used as an index of summation, but frequently denotes (with subscripts or without them) geometric entities. Many of the definitions of the numerous concepts introduced are not set off or marked as such. For these and other reasons, the reader should not expect that studying the book will be easy.

The book is a treasure trove for people interested in simplex geometry for its own sake. Those whose aims are towards applications would be better served by Fiedler's other book, *Special Matrices and Their Applications in Numerical Mathematics*, 2nd ed., Dover Publ., Mineola, NY, 2008.

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