Global solutions to a one-dimensional hyperbolic free boundary problem which arises in peeling phenomena

Kazuaki Nakane\textsuperscript{a,}\textsuperscript{*}, Tomoko Shinohara\textsuperscript{b}

\textsuperscript{a}Faculty of Engineering, Osaka Institute of Technology, Omiya, Asahi, Osaka 535-8585, Japan
\textsuperscript{b}Faculty of Science, Kanazawa University, Kanazawa 920-1192, Japan

Received 30 November 2001; received in revised form 31 May 2002

Abstract

This problem arises from the physical model “Peel a thin film from a domain”. The behavior of the peeling front is governed by a hyperbolic equation. As the first step, the one-dimensional case is treated. The local solutions have been already given. In this note, the global solution is constructed under the assumption the peeling speed is non negative.

\textcopyright 2002 Elsevier Science B.V. All rights reserved.

Keywords: Free boundary; Hyperbolic equation; Variational problem

1. Introduction

Let us consider the following one-dimensional free boundary problem:

\[(P) \begin{cases} u_{xx} - u_{tt} = 0 & \text{in } (0, \infty) \times \{t > 0\} \cap \{u > 0\}, \\ u_x^2 - u_t^2 = Q^2 & \text{on } (0, \infty) \times \{t > 0\} \cap \partial \{u > 0\} \end{cases}\]

with the initial conditions

\[(I) \begin{cases} u(x, 0) = e(x) & \text{on } (-l_0, 0), \\ u_t(x, 0) = g(x) & \text{on } (-l_0, 0) \end{cases}\]

\textsuperscript{*} Corresponding author. Tel.: +81-6-6954-4796; fax: +81-6-6975-2137.
E-mail address: nakane@ge.oit.ac.jp (K. Nakane).
and the boundary condition

\begin{equation}
(B) \quad u(-l_0, t) = f(t) \quad \text{on} \quad [0, \infty),
\end{equation}

where \( e(x), g(x) \) and \( f(t) \) are given functions, \( Q \) and \( l_0 \) are positive constants.

This problem arises from a variational problem which is related to the following physical model: “A thin film is pasted on the plate. By lifting up the edge of the film in the vertical direction, the film is peeled from the plate. We would like to know the behavior of the peeling front.” The shape of the film is described by the graph of a function \( u: \Omega \rightarrow \mathbb{R} \). It can be assumed that the effect of the peeling front on the Lagrangian is constant \( Q^2 \). Then, our problem is to find a stationary point of the functional

\begin{equation}
(J) \quad J(u) = \int_0^{T^*} \int_{\Omega} \left( \tau |\nabla u|^2 - \rho (D_t u)^2 + O^2 \right) dx dt
\end{equation}

where \( \Omega \) is a domain in \( \mathbb{R}^n \), \( T^* \) is a positive constant, \( \chi_{u>0} \) is a characteristic function of the set \( \{(x, t) \in \Omega \times (0, T^*); u(x, t) > 0\} \) and \( K \) is a suitable function space. Here the constants \( \tau \) and \( \rho \) are the tension and the line density, respectively. To approach this problem, we assume that a stationary point is sufficiently smooth. Then we can derive (1) and (2) as the Euler–Lagrange equations from the functional (J) (cf. [1,2,4]),

\begin{equation}
\tau \Delta u - \rho u_t = 0 \quad \text{in} \quad \Omega \times \{t > 0\} \cap \{u > 0\}, \quad (1)
\end{equation}

\begin{equation}
\frac{\tau}{2} |\nabla u|^2 - \frac{\rho}{2} u_t^2 = \frac{Q^2}{2} \quad \text{on} \quad \Omega \times \{t > 0\} \cap \{u > 0\}. \quad (2)
\end{equation}

In this note, as a first step, the one-dimensional problem will be analyzed. By changing of the variable \( t, \tau = 1 \) and \( \rho = 1 \) can be assumed. Therefore, we will consider the problem (P).

The initial condition (I) implies that the thin film has been already peeled from the plate on the interval \((-l_0, 0)\) and the boundary condition (B) corresponds to the situation in which the edge of the film is lifted up by \( f(t) \). Kikuchi and Omata [4] showed the existence of time-local solutions under several conditions which were imposed on the functions \( e(x), g(x) \) and \( f(t) \). On the global existence of solutions, however, they have not stated. In [3], numerical experiments were carried out. A sufficient condition for the global existence was given by Nakane [5] by using an iteration method. This method, however, can not be applied to the case the peeling speed is zero.

The solutions can be constructed by pasting the local solutions, inductively. In this note, it is shown that if the peeling speed \( f' \) is nonnegative then the pasting interval does not decrease (cf. Main Theorem). It implies that time global solutions for problem (P), (I) and (B) are constructed. It is remarked here that this condition is an improvement on the previous one in [5].

2. Construction of solutions

To solve our problem, two variables \( \xi \) and \( \eta \) are introduced by

\begin{align*}
t &= (\xi + \eta)/2, \\
x &= (\xi - \eta)/2.
\end{align*}
Since the initial values $e$ and $g$ are given on the line $\{ (x, t); t = 0 \}$, they are given on the line $\{ (\xi, \eta); \xi + \eta = 0 \}$. We regard them as the functions of $\xi$ and rewrite them $e$ and $g$ again. Similarly, the boundary value $f$ is given on the line $\{ (\xi, \eta); \xi - \eta + 2l_0 = 0 \}$ and is a function of $\eta$. Therefore (P), (I) and (B) are transformed into

$$\begin{align*}
(P') & \quad \begin{cases}
  u_{\xi\eta} = 0 & \text{in } \{ u > 0 \}, \\
  -4u_{\xi\eta} = Q^2 & \text{on } \partial \{ u > 0 \},
\end{cases} \\
(I') & \quad \begin{cases}
  u(\xi, -\xi) = e(\xi) & \text{on } (-l_0, 0), \\
  u_\eta(\xi, -\xi) + u_\xi(\xi, -\xi) = g(\xi) & \text{on } (-l_0, 0),
\end{cases} \\
(B') & \quad u(\eta - 2l_0, \eta) = f(\eta - l_0) \quad \text{on } [l_0, \infty).
\end{align*}$$

For these equations, we consider the following problem:

**Problem 2.1.** Let $T$ be a positive constant. Find a pair of functions $u \in C^0(\{ (\xi, \eta); \xi \geq \eta - 2l_0, \xi \geq \eta \})$ and $l \in C^0([0, T)) \cap C^1((0, T))$ which satisfies (P'), (I') and (B') for $\eta < T$ and

1. $l(0) = 0$,
2. $u \in C^2(\{ (\xi, \eta); \eta - 2l_0 < \xi < l(\eta), \xi > \xi \} \cap C^1(\{ (\xi, \eta); \eta - 2l_0 < \xi \leq l(\eta), \xi \geq \eta \})$,
3. $u > 0$ in $\{ (\xi, \eta); \eta - 2l_0 \leq \xi < l(\eta), \xi \geq \eta \}$,
4. $u(\xi, \eta) = 0$ in $\{ (\xi, \eta); \xi \geq l(\eta) \} \cup \{ (\xi, \eta); \xi \geq -\eta, \eta < 0 \}$.

**Assumption 2.1.** The functions $f(\eta - l_0) \in C^2([l_0, \infty))$, $e(\xi) \in C^2([-l_0, 0])$ and $g(\xi) \in C^1([-l_0, 0])$ satisfy

- **(A.0)** $\begin{align*}
e(\xi) & > 0 \quad \text{on } [-l_0, 0), \\
g(\xi) & > 0 \quad \text{on } (-l_0, 0),
\end{align*}$
- **(A.1)** $\begin{align*}
f(0) & = e(-l_0), \\
f'(0) & = g(-l_0), \\
f''(0) & = e''(-l_0),
\end{align*}$
- **(A.2)** $\begin{align*}
e(0) & = 0, \\
& e'(0)^2 - g(0)^2 = Q^2, \\
& (e''(0) + g'(0))(e'(0) - g(0))^4 = Q^4(e''(0) - g'(0)).
\end{align*}$
- **(A.3)** $\begin{align*}
e'(\xi) & < g(\xi) \quad \text{on } [-l_0, 0], \\
f'(\xi + l_0) - (e'(\xi) + g(\xi))/2 & > 0 \quad \text{on } [-l_0, 0],
\end{align*}$
- **(A.4)** $f'(\eta - l_0) \geq 0 \quad \text{on } [l_0, \infty)$. 

Remark 2.1. (i) (A.1) is a compatible condition on the lifting edge and (A.2) is a condition on the peeling front at \( t = 0 \). These two conditions guarantee the regularity of the solution.

(ii) (A.4) plays an important role in the existence of global solutions.

The following lemma is originally given in [4]. To confirm the global existence of the solution, we will modify its proof.

Lemma 2.1. Let \( c \) be a positive number and \( I = [0, c) \) an interval on the \( \eta \)-axis. Let \( \gamma(\eta) \in C^2(I) \) be a function which satisfies

(i) \( \gamma(0) = 0 \),

(ii) \( \gamma'(\eta) > 0 \) on \( I \).

Then the following functions are well-defined

\[
\psi(\eta) = \gamma(\eta) \quad \text{on} \quad I,
\]

\[
l(\eta) = \frac{4}{Q^2} \int_0^\eta \psi'(s)^2 \, ds \quad \text{on} \quad I,
\]

\[
\phi(\xi) = -\psi(l^{-1}(\xi)) \quad \text{on} \quad [0, l(c))
\]

and a pair of functions \( u(\xi, \eta) = \phi(\xi) + \psi(\eta) \in C^2(\{(\xi, \eta); \eta \in I, 0 < \xi < l(\eta)\}) \cap C^1(\{(\xi, \eta); \eta \in I, 0 \leq \xi \leq l(\eta)\}) \) and \( l(\eta) \in C^2(I) \) satisfies

\[
\begin{cases}
  u_{\xi \eta} = 0 & \text{in} \quad \{(\xi, \eta); 0 < \xi < l(\eta), \eta \in I\}, \\
  -4u_{\xi \eta} = Q^2 & \text{on} \quad \{(l(\eta), \eta); \eta \in I\}, \\
  u(l(\eta), \eta) = 0 & \text{on} \quad I,
\end{cases}
\]

(PL)

\[
u(0, \eta) = \gamma(\eta) \quad \text{on} \quad I,
\]

\[l(0) = 0.
\]

Moreover, a pair of functions \((u, l)\) which satisfies (PL) are uniquely determined.

Proof. Because of (ii), \( l^{-1}(\xi) \) exists in \([0, l(c))\). Then the function \( \phi \) is well defined. It is not difficult to make sure that \( u \) and \( l \) satisfy (PL).

To show the uniqueness of the solution, we suppose that there exists another pair of functions \((\tilde{u}(\xi, \eta), \tilde{l}(\eta))\) satisfying (PL) which has the form \( \tilde{u}(\xi, \eta) = \tilde{\phi}(\xi) + \tilde{\psi}(\eta) \). It follows that:

\[
\tilde{u}(\tilde{l}(\eta), \eta) = \tilde{\phi}(\tilde{l}(\eta)) + \tilde{\psi}(\eta) = 0. \tag{3}
\]

By differentiating both sides of (3) and from the fact \(-4u_{\xi \eta} = Q^2\), we have

\[
\tilde{l}'(\eta) = \frac{4}{Q^2} \tilde{\psi}'(\eta)^2.
\]
Since \( l(0) = 0 \), it holds that
\[
\tilde{l}(\eta) = \frac{4}{Q^2} \int_0^\eta \tilde{\psi}'(s)^2 \, ds.
\]

Because of \( \tilde{u}(0, \eta) = \gamma(\eta) \), we obtain
\[
\tilde{\psi}(\eta) = \gamma(\eta) + \tilde{\phi}(0) \quad \text{and} \quad \tilde{\psi}'(\eta) = \gamma'(\eta).
\]

It implies that \( \tilde{l} \) is equal to \( l \). Moreover, since \( l^{-1}(\xi) \) exists, we have from (3)
\[
\tilde{\phi}(\xi) = -\tilde{\psi}(l^{-1}(\xi)) = -\gamma(l^{-1}(\xi)) + \tilde{\phi}(0).
\]

Therefore, \( \tilde{u}(\xi, \eta) = \tilde{\phi}(\xi) + \tilde{\psi}(\eta) \) coincides with \( u(\xi, \eta) = \phi(\xi) + \psi(\eta) \), it is a contradiction proving our assertion. \( \square \)

**Main Theorem.** For any \( T > 0 \), there exists a unique solution to Problem 2.1.

**Proof.** Firstly, let \( \phi_0 \) and \( \psi_1 \) be functions such that
\[
\phi_0(\xi) = \frac{1}{2} \left( e(\xi) + \int_0^\xi g(s) \, ds \right) \quad \text{on} \quad [-\lambda_1, 0),
\]
\[
\psi_1(\eta) = \frac{1}{2} \left( e(-\eta) + \int_{-\eta}^0 g(s) \, ds \right) \quad \text{on} \quad [0, \lambda_1),
\]
where
\[
\lambda_1 = l_0.
\]

Evidently, \( u(\xi, \eta) = \phi_0(\xi) + \psi_1(\eta) \) is a unique solution to the initial value problem in \( \{(\xi, \eta); -l_0 < -\eta < \xi < 0\} \). Let us define the free boundary \( l_1 \) by
\[
l_1(\eta) = \frac{4}{Q^2} \int_0^\eta \psi_1'(s)^2 \, ds.
\]

By (A.3), \( \psi_1'(\eta) \) is positive on \([0, \lambda_1)\). Therefore there exists \( l_1^{-1}(\xi) \) on \([0, l_1(\lambda_1))\) and we can define \( \phi_1 \) by
\[
\phi_1(\xi) = -\psi_1(l_1^{-1}(\xi)) \quad \text{on} \quad [0, l_1(\lambda_1)).
\]

By using these functions, we define the functions \( u \) and \( l \) by
\[
u(\xi, \eta) = \begin{cases} 
\phi_0(\xi) + \psi_1(\eta) & \text{in} \ D_{0,1}, \\
\phi_1(\xi) + \psi_1(\eta) & \text{in} \ D_{1,1}, \\
0 & \text{in} \ D_{c,1},
\end{cases}
\]
\[
l(\eta) = l_1(\eta) \quad \text{on} \quad [0, \lambda_1),
\]
where
\[ D_{0,1} = \{ (\xi, \eta); \xi \geq -\eta, \xi < 0, 0 \leq \eta < \lambda_1 \}, \]
\[ D_{1,1} = \{ (\xi, \eta); 0 \leq \xi < \lambda_1(\eta), 0 \leq \eta < \lambda_1 \}, \]
\[ D_{\varepsilon,1} = \{ (\xi, \eta); \xi \geq \lambda_1(\eta), 0 \leq \eta < \lambda_1 \}. \]

By direct calculation, we have
\[
\begin{align*}
\phi_0(0) &= \frac{1}{2} e(0), \\
\phi_1(0) &= -\frac{1}{2} e(0), \\
\phi_0'(0) &= \frac{1}{2} (e'(0) + g(0)), \\
\phi_1'(0) &= -\frac{Q^2}{4 \psi_1'(l_1^{-1}(0))} = \frac{Q^2}{2} \left( \frac{1}{(e'(0) - g(0))} \right), \\
\phi_0''(0) &= \frac{1}{2} (e''(0) + g'(0)), \\
\phi_1''(0) &= \frac{Q^4}{16 (\psi_1'(l_1^{-1}(0)))^4} = \frac{Q^4}{2} \left( \frac{e''(0) - g'(0))}{(e'(0) - g(0))^2} \right).
\end{align*}
\]

Therefore (A.2) guarantees the regularity at \( \xi = 0 \) of \( \phi_0 \) and \( \phi_1 \). By Lemma 2.1, \((u(\xi, \eta), l(\eta))\) is a unique solution to Problem 2.1 for \( T := \lambda_1 \).

Nextly, we define \( \psi_2 \) by
\[
\psi_2(\eta) = \begin{cases} 
  f(\eta - l_0) - \phi_0(\eta - 2l_0) & \text{on } [\lambda_1, 2\lambda_1), \\
  f(\eta - l_0) - \phi_1(\eta - 2l_0) & \text{on } [2\lambda_1, \lambda_2),
\end{cases}
\]

where
\[
\lambda_2 = \lambda_1(\lambda_1) + 2l_0.
\]

Since \( \phi_0 \) and \( \phi_1 \) are connected smoothly, \( \psi(\eta) \) is of \( C^2 \)-class. Because the part of the boundary known so far ends in \((\xi, \eta) = (l_1(\lambda_1), \lambda_1)\), \( l_2 \) is defined by
\[
l_2(\eta) = \frac{4}{Q^2} \int_{\lambda_1}^{\eta} \psi_2(s)^2 ds + l_1(\lambda_1) \quad \text{on } [\lambda_1, \lambda_2).
\]

Obviously, \( l_2 \) is of \( C^2 \)-class. Because it yields that
\[
\psi_2'(\eta) = \begin{cases} 
  f'(\eta - l_0) - \frac{1}{2} (e'(\eta - 2l_0) + g(\eta - 2l_0)) & \text{on } [\lambda_1, 2\lambda_1), \\
  f'(\eta - l_0) + \frac{Q^2}{4 \psi_1'(l_1^{-1}(\eta - 2l_0))} & \text{on } [2\lambda_1, \lambda_2),
\end{cases}
\]

\( \psi_2'(\eta) > 0 \) holds on \([\lambda_1, \lambda_2)\) by (A.3) and (A.4). Then there exists \( l_2^{-1}(\xi) \) on \([l_2(\lambda_1), l_2(\lambda_2))\). Hence we can define
\[
\phi_2(\xi) = -\psi_2(l_2^{-1}(\xi)) \quad \text{on } [l_2(\lambda_1), l_2(\lambda_2)).
\]
The functions \((u(\xi, \eta), l(\eta))\) are extended as the following:

\[
u(\xi, \eta) = \begin{cases} 
  \phi_0(\xi) + \psi_2(\eta) & \text{in } D_{0,2}, \\
  \phi_1(\xi) + \psi_2(\eta) & \text{in } D_{1,2}, \\
  \phi_2(\xi) + \psi_5(\eta) & \text{in } D_{2,2}, \\
  0 & \text{in } D_{e,2},
\end{cases}
\]

\[l(\eta) = l_2(\eta) \quad \text{on } [\lambda_1, \lambda_2),\]

where

\[
D_{0,2} = \{ (\xi, \eta); \xi \geq \eta - 2l_0, \xi < 0, \lambda_1 \leq \eta < 2\lambda_1 \},
\]

\[
D_{1,2} = \{ (\xi, \eta); \xi \geq \eta - 2l_0, 0 \leq \xi < l_1(\lambda_1), \lambda_1 \leq \eta < \lambda_2 \},
\]

\[
D_{2,2} = \{ (\xi, \eta); l_1(\lambda_1) \leq \xi < l_2(\eta), \lambda_1 \leq \eta < \lambda_2 \},
\]

\[
D_{e,2} = \{ (\xi, \eta); \xi \geq l_2(\eta), \lambda_1 \leq \eta < \lambda_2 \}.
\]

It follows from (A.1) that \(\psi_1\) and \(\psi_2\) are connected smoothly at \(\eta = \lambda_1\), concurrently \(\phi_1\) and \(\phi_2\) are smooth at \(\xi = \lambda_2 - 2l_0\). By applying Lemma 2.1 again, we can see the pair of functions \((u(\xi, \eta), l(\eta))\) is a unique solution to Problem 2.1 for \(T := \lambda_2\).

Inductively, for \(j \geq 3\), we define the functions \(\psi_j, l_j\) and \(\phi_j\) by

\[\psi_j(\eta) = f(\eta - l_0) - \phi_{j-1}(\eta - 2l_0) \quad \text{on } [\lambda_{j-1}, \lambda_j),\]

\[l_j(\eta) = \frac{4}{Q^2} \int_{\lambda_{j-1}}^{\eta} \psi_j(s)^2 \, ds + l_{j-1}(\lambda_{j-1}) \quad \text{on } [\lambda_{j-1}, \lambda_j),\]

\[\phi_j(\xi) = -\psi_j(l_j^{-1}(\xi)) \quad \text{on } [l_j(\lambda_{j-1}), l_j(\lambda_j)),\]

where

\[\lambda_j = l_j(\lambda_{j-1}) + 2l_0.\]

It can be seen that \(\psi_3(\eta)\) and \(l_3(\eta)\) are well defined and of \(C^2\)-class. Because it holds that

\[
\psi_3(\eta) = f'(\eta - l_0) - \phi_2'(\eta - 2l_0)
= f'(\eta - l_0) + \frac{Q^2}{4} \frac{1}{\psi_3'(l_2^{-1}(\eta - 2l_0))},
\]

we obtain \(\psi_3'(\eta) > 0\) on \([\lambda_2, \lambda_3)\) by (A.4) and the fact \(\psi_2'(\eta) > 0\) on \([\lambda_1, \lambda_2)\). So the function \(l_3^{-1}\) exists on \([l_3(\lambda_2), l_3(\lambda_3))\), \(\phi_3\) is well defined and of \(C^2\)-class. Suppose that it has been already shown that \(\psi_j(\eta), l_j(\eta)\) and \(\phi_j(\xi)\) are well defined and of \(C^2\)-class, and \(\psi_j'(\eta) > 0\) on \([\lambda_{j-1}, \lambda_j)\). It can be obtained that \(\psi_{j+1}\) and \(l_{j+1}\) are well defined and of \(C^2\)-class, immediately. We have

\[
\psi_{j+1}(\eta) = f'(\eta - l_0) - \phi_j'(\eta - 2l_0)
= f'(\eta - l_0) + \frac{Q^2}{4} \frac{1}{\psi_j'(l_j^{-1}(\eta - 2l_0))},
\]

(4)
it follows $\psi'_{j+1}(\eta) > 0$ on $[\lambda_j, \lambda_{j+1}]$ by (A.4) and induction assumption. Therefore, there exists $l_{j+1}^{-1}$ on $[l_{j+1}(\lambda_j), l_{j+1}(\lambda_{j+1}))$, it implies that $\phi_{j+1}$ is well defined and of $C^2$-class.

The regularity of $\psi_2$ and $\psi_3$ at $\eta = \lambda_2$ can be shown, since $\phi_1$ and $\phi_2$ are connected smoothly. Accordingly, we obtain the regularity $\phi_2$ and $\phi_3$ at $\zeta = \lambda_3 - 2l_0$. It is supposed that $\phi_{j-1}$ and $\phi_j$ are connected smoothly at $\zeta = \lambda_j - 2l_0$. From the definition, $\psi_j$ and $\psi_{j+1}$ are connected smoothly at $\eta = \lambda_j$. Therefore $\phi_j$ and $\phi_{j+1}$ are connected smoothly at $\zeta = \lambda_{j+1} - 2l_0$.

By combining these functions for $j \geq 3$, the pair of functions $(u(\xi, \eta), l(\eta))$ can be extended as the following

$$u(\xi, \eta) = \begin{cases} 
\phi_{j-1}(\xi) + \psi_j(\eta) & \text{in } D_{j-1,j}, \\
\phi_j(\xi) + \psi_j(\eta) & \text{in } D_{j,j}, \\
0 & \text{in } D_{k,j},
\end{cases}$$

$$l(\eta) = l_j(\eta) \text{ on } [\lambda_{j-1}, \lambda_j),$$

where

$$D_{j-1,j} = \{(\xi, \eta); \xi \geq \eta - 2l_0, \xi < l_{j-1}(\lambda_{j-1}), \lambda_{j-1} \leq \eta < \lambda_j\},$$

$$D_{j,j} = \{(\xi, \eta); l_{j-1}(\lambda_{j-1}) \leq \xi < l_j(\eta), \lambda_{j-1} \leq \eta < \lambda_j\},$$

$$D_{k,j} = \{(\xi, \eta); \xi \geq l_j(\eta), \lambda_{j-1} \leq \eta < \lambda_j\}.$$

By the same argument as for $j = 2$, $(u(\xi, \eta), l(\eta))$ is a unique solution to Problem 2.1 for $T := \lambda_j$.

Finally, we shall show the sequence $\{\lambda_j\}_{j=1}^\infty$ goes to infinity. We remark that $l_{j-1}(\lambda_{j-1}) = l_j(\lambda_{j-1})$, it holds that $\lambda_{j+1} - \lambda_j = l_j(\lambda_j) - l_j(\lambda_{j-1})$. Then the following inequation holds:

$$\lambda_{j+1} - \lambda_j = \frac{4}{Q^2} \int_{\lambda_{j-1}}^{\lambda_j} \psi_j'(s)^2 \, ds$$

$$= \frac{4}{Q^2} \int_{\lambda_{j-1}}^{\lambda_j} \left( f'(s - l_0) + \frac{Q^2}{4} \frac{1}{\psi'_{j-1}(l_{j-1}^{-1}(s - 2l_0))} \right)^2 \, ds$$

$$\geq \frac{4}{Q^2} \int_{\lambda_{j-1}}^{\lambda_j} \left( \frac{Q^2}{4} \frac{1}{\psi'_{j-1}(l_{j-1}^{-1}(s - 2l_0))} \right)^2 \, ds. \quad (5)$$

By using the change of variable

$$y = l_{j-1}^{-1}(s - 2l_0),$$

the right-hand side of (5) is reduced

$$\frac{4}{Q^2} \int_{\lambda_{j-1}}^{\lambda_j} \left( \frac{Q^2}{4} \frac{1}{\psi'_{j-1}(l_{j-1}^{-1}(s - 2l_0))} \right)^2 \, ds = \int_{\lambda_{j-2}}^{\lambda_{j-1}} dy = \lambda_{j-1} - \lambda_{j-2}.$$

Then we obtain

$$\lambda_{j+1} - \lambda_j \geq \lambda_{j-1} - \lambda_{j-2},$$

it implies that $\{\lambda_j\}$ diverges. □
Acknowledgements

The authors would like to express sincere thanks to Professor Omata (Kanazawa Univ.) and Professor Kikuchi (Shizuoka Univ.) for introducing this problem and their valuable comments on this article.

References