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Flood Routing Based on Diffusion Wave Equation Using Lattice Boltzmann Method

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Abstract

One-dimensional diffusion wave equation is a simplified form of the full Saint Venant equations by neglecting the inertia terms. In this study, the Lattice Boltzmann method for the linear diffusion wave equation was developed. In order to verify the calculation accuracy of it, the analytical solution and Muskingum method were also introduced. Excellent agreement was obtained between observed data and numerical prediction. The results show that the Lattice Boltzmann method is a very competitive method for solving diffusion wave equation in terms of computational efficiency and accuracy.

Keywords: flood routing; Saint Venant equations; diffusion wave equation; Lattice Boltzmann method; Muskingum method

1. Introduction

Flood routing has a series of methods to estimate the future flood in downstream cross-section of the river which relates to the flood in upstream cross-section, the theory of it is the flood wave law in the river. These methods can be divided into hydraulic way and hydrology way [1]. For the hydraulic way, flood wave movement can be described by full Saint Venant equations which has rigorous theoretical basis. However, these equations are too complicated to be calculated, and this way requires a lot of detailed river topography information. Therefore, a simplified form of full Saint Venant equations should be chosen. A lot of studies have found that the diffusion wave equation can reflect the characteristics of

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flood wave movement well [2]. Diffusion wave equation solution is a method which has sufficient accuracy and easy to be solved, it is a good selection to describe the flood wave movement in the river.

The Lattice Boltzmann method was proposed in recent years as a new numerical method for simulating fluid flows [3]. The fundamental idea of it is to construct simplified kinetic models that incorporate the essential physics of mesoscopic processes so that the macroscopic averaged properties obey the desired macroscopic equations. It provides an indirect way to solve flow equations and brings certain advantages over conventional numerical methods, such as parallel computation, easy handing of complex geometries, easy programming and easy simulating complex flows [4, 5]. In addition, the Lattice Boltzmann method has been developed to solve nonlinear partial differential equations, such as, shallow water flow equation and Burgers equation, etc. [6, 7]. These features make the Lattice Boltzmann method to be a very promising computational method in different areas [3].

Therefore, in this paper, the Lattice Boltzmann method is proposed to solve the diffusion wave equation. In order to illustrate applications of it, two examples are introduced. The Lattice Boltzmann method has been verified by comparing to the results of the Muskingum method and analytical solution.

2. Theoretical basis for diffusion wave equation

Unsteady flow in an open channel can be represented by the one-dimensional Saint Venant equations of continuity and momentum, respectively

$$\frac{\partial Q}{\partial x}+\frac{\partial A}{\partial t}=0 \tag{1}$$

$$\frac{V}{g}\frac{\partial V}{\partial x}+\frac{1}{g}\frac{\partial V}{\partial t}+\frac{\partial y}{\partial x}=S_0-S_f \tag{2}$$

Where Q is the discharge, A is the cross-section area of flow, g is the gravitational acceleration, V is the average velocity of flow cross-section, y is the water flow depth, S_0 is the channel bed slope, S_f is the frictional slope, t is time and x is the distance along the flow direction.

For many flow equations, the inertia terms can be considered much smaller than friction slope and pressure gradient, therefore, be neglected. And if the assumption is made that the friction slope can be determined as in steady uniform flow, then it can be described by Chezy equation. The St Venant equations are reduced to diffusion wave equation

$$\frac{\partial Q}{\partial t}+C_d\frac{\partial Q}{\partial x}=\mu\frac{\partial^2 Q}{\partial x^2} \tag{3}$$

Where C_d is the diffusion wave celerity, μ is diffusion coefficient.

If C_d and μ is constant, Eq. (3) reduces to linear diffusion wave equation. The initial and boundary conditions for Eq. (3) can be expressed as

$$Q(x,0)=0 \quad (0\le x\le L) \qquad Q(0,t)=I(t) \quad (t\ge 0) \qquad \lim_{x\to\infty} Q(x,t)=0 \quad (t\ge 0) \tag{4}$$

Where L is the length of channel, $I(t)$ is the inflow of up boundary.

3. Lattice Boltzmann model

3.1. *Lattice Boltzmann equation*

The Lattice Boltzmann method can be considered as a finite difference scheme for the kinetic equation of the discrete-velocity distribution function [3]. It divides the space into uniform mesh and the particle distribution function $f_\alpha(\vec{x},\vec{e}_\alpha,t)$ is laid on each node. Based on the physical characteristics, and with the

BGK approximation [8], the lattice Boltzmann equation can be obtained as

$$f_\alpha(\vec{x}+\Delta t\cdot\vec{e}_\alpha,\vec{e}_\alpha,t+\Delta t)-f_\alpha(\vec{x},\vec{e}_\alpha,t)=-\frac{1}{\tau}(f_\alpha(\vec{x},\vec{e}_\alpha,t)-f_\alpha^{eq}(\vec{x},\vec{e}_\alpha,t)) \tag{5}$$

Where $f_\alpha(\vec{x},\vec{e}_\alpha,t)$ is the particle velocity distribution function, $\vec{e}_\alpha$ is the particle velocity, τ is the single relaxation time, $f_\alpha^{eq}(\vec{x},\vec{e}_\alpha,t)$ is the local equilibrium distribution function.

According to the theory of the lattice Boltzmann method, it consists of collision step and streaming step. In the collision step, particles at the points interact one another and change their velocity directions according to scattering rules, which can be expressed as

$$f_\alpha^{new}(\vec{x},\vec{e}_\alpha,t)=f_\alpha(\vec{x},\vec{e}_\alpha,t)-\frac{1}{\tau}(f_\alpha(\vec{x},\vec{e}_\alpha,t)-f_\alpha^{eq}(\vec{x},\vec{e}_\alpha,t)) \tag{6}$$

In the streaming step, particles move to the neighboring lattice points which is governed by

$$f_\alpha^{new}(\vec{x}+\Delta t\cdot\vec{e}_\alpha,\vec{e}_\alpha,t+\Delta t)=f_\alpha^{new}(\vec{x},\vec{e}_\alpha,t) \tag{7}$$

3.2. Lattice Boltzmann model of diffusion wave equation

In this paper, the D1Q5 five-bit model of Lattice Boltzmann method is developed to solve the diffusion wave equation. For the D1Q5 model, the discrete velocities are: $e_2=-2C_0$, $e_1=-C_0$, $e_0=0$, $e_3=C_0$, $e_4=2C_0$, where $C_0=\Delta x/\Delta t$, C_0 is the magnitude of velocity. The diagrammatic sketch of D1Q5 model is shown as Fig.1.

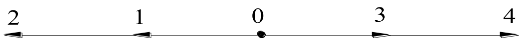

Fig.1 D1Q5 five-bit model of Lattice Boltzmann method

The Knudsen number ε which is defined as $\varepsilon=l/L$, where l is the mean free path, and L is the characteristic length, is introduced to be taken as the time step Δt, Eq. (5) thus changes into

$$f_\alpha(\vec{x}+\varepsilon\cdot\vec{e}_\alpha,\vec{e}_\alpha,t+\varepsilon)-f_\alpha(\vec{x},\vec{e}_\alpha,t)=-\frac{1}{\tau}(f_\alpha(\vec{x},\vec{e}_\alpha,t)-f_\alpha^{eq}(\vec{x},\vec{e}_\alpha,t)) \tag{8}$$

The Chapman-Enskog expansion which is a formal multiscaling expansion [9] is applied to the distribution function $f_\alpha(\vec{x},\vec{e}_\alpha,t)$

$$f_\alpha(\vec{x},\vec{e}_\alpha,t)=f_\alpha^{eq}(\vec{x},\vec{e}_\alpha,t)+\varepsilon f_\alpha^{(1)}(\vec{x},\vec{e}_\alpha,t)+\varepsilon^2 f_\alpha^{(2)}(\vec{x},\vec{e}_\alpha,t)+\varepsilon^3 f_\alpha^{(3)}(\vec{x},\vec{e}_\alpha,t)+\varepsilon^4 f_\alpha^{(4)}(\vec{x},\vec{e}_\alpha,t)+o(\varepsilon^5) \tag{9}$$

Due to the Knudsen number ε is very small, so

$$\sum_\alpha f_\alpha(\vec{x},\vec{e}_\alpha,t)=\sum_\alpha f_\alpha^{eq}(\vec{x},\vec{e}_\alpha,t) \qquad \sum_\alpha f_\alpha^{(k)}(\vec{x},\vec{e}_\alpha,t)=0 \qquad k=1,2,3,4 \tag{10}$$

Introduce different time and space scales, and their differential coefficient form are

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial t_0}+\varepsilon\frac{\partial}{\partial t_1}+\varepsilon^2\frac{\partial}{\partial t_2}+\varepsilon^3\frac{\partial}{\partial t_3}+\varepsilon^4\frac{\partial}{\partial t_4}+o(\varepsilon^5) \qquad \frac{\partial}{\partial x}=\frac{\partial}{\partial x}+o(\varepsilon) \tag{11}$$

The second-order Taylor expansion is used on Eq. (8), it is

$$\varepsilon(\frac{\partial}{\partial t}+\vec{e}_\alpha\frac{\partial}{\partial x})f_\alpha(\vec{x},\vec{e}_\alpha,t)+\frac{\varepsilon^2}{2}(\frac{\partial}{\partial t}+\vec{e}_\alpha\frac{\partial}{\partial x})^2 f_\alpha(\vec{x},\vec{e}_\alpha,t)=-\frac{1}{\tau}(f_\alpha(\vec{x},\vec{e}_\alpha,t)-f_\alpha^{eq}(\vec{x},\vec{e}_\alpha,t)) \tag{12}$$

Combining Eqs (9), (11) and (12), the series of Lattice Boltzmann equations in different time scales are obtained

$$o(\varepsilon^0):\quad f_\alpha^{(0)}=f_\alpha^{eq} \tag{13}$$

$$o(\varepsilon^1):\quad (\frac{\partial}{\partial t_0}+\vec{e}_\alpha\frac{\partial}{\partial x})f_\alpha^{(0)}=-\frac{1}{\tau}f_\alpha^{(1)} \tag{14}$$

$$o(\varepsilon^2):\quad \frac{\partial f_\alpha^{(0)}}{\partial t_1}+(\frac{1}{2}-\tau)(\frac{\partial}{\partial t_0}+\vec{e}_\alpha\frac{\partial}{\partial x})^2 f_\alpha^{(0)}=-\frac{1}{\tau}f_\alpha^{(2)} \tag{15}$$

$$o(\varepsilon^3):\quad \frac{\partial f_\alpha^{(0)}}{\partial t_2}+(2-\frac{1}{\tau})\frac{\partial f_\alpha^{(1)}}{\partial t_1}+(\tau^2-\tau+\frac{1}{6})(\frac{\partial}{\partial t_0}+\vec{e}_\alpha\frac{\partial}{\partial x})^3 f_\alpha^{(0)}=-\frac{1}{\tau}f_\alpha^{(3)} \tag{16}$$

$$o(\varepsilon^4):\quad (1-2\tau)(\frac{\partial}{\partial t_0}+\vec{e}_\alpha\frac{\partial}{\partial x})\frac{\partial f_\alpha^{(0)}}{\partial t_2}+(\frac{3\tau^2}{2}-\tau^3-\frac{7\tau}{12}+\frac{1}{24})(\frac{\partial}{\partial t_0}+\vec{e}_\alpha\frac{\partial}{\partial x})^4 f_\alpha^{(0)}$$
$$+(1-\frac{1}{2\tau})\frac{\partial f_\alpha^{(2)}}{\partial t_1}+\frac{\partial f_\alpha^{(0)}}{\partial t_3}+(2\tau^2-2\tau+\frac{1}{4})(\frac{\partial}{\partial t_0}+\vec{e}_\alpha\frac{\partial}{\partial x})^2\frac{\partial f_\alpha^{(0)}}{\partial t_1}=-\frac{1}{\tau}f_\alpha^{(4)} \tag{17}$$

Zero-order moment of $f_\alpha^{eq}(\vec{x},\vec{e}_\alpha,t)$ is solved on Eqs (13) to (17), and according to Eq. (10), the macroscopic manifestation can be set

$$\sum_\alpha f_\alpha^{(0)}=Q\qquad \sum_\alpha \vec{e}_\alpha f_\alpha^{(0)}=C_d\cdot Q\qquad \sum_\alpha \vec{e}_\alpha^2 f_\alpha^{(0)}=C_d^2Q+k\cdot\mu\cdot Q$$
$$\sum_\alpha \vec{e}_\alpha^3 f_\alpha^{(0)}=3C_d\cdot\mu\cdot k\cdot Q+C_d^3Q\qquad \sum_\alpha \vec{e}_\alpha^4 f_\alpha^{(0)}=\frac{k(2\tau^2-2\tau+\frac{1}{4})\cdot\mu^2\cdot Q}{(\frac{3}{2}\tau^2-\tau^3-\frac{7}{12}\tau+\frac{1}{24})}+6C_d^2\mu\cdot k\cdot Q+C_d^4Q \tag{18}$$

In order to recover the diffusion wave equation, scale adhesion is needed (that is the time scale t_0,t_1,t_2,t_3,t_4 should be restored back to the time scale *t*), so let $(14)+\varepsilon\times(15)+\varepsilon^2\times(16)+\varepsilon^3\times(17)$ and sum the five directions, thus

$$\frac{\partial Q}{\partial t}+C_d\frac{\partial Q}{\partial x}+\varepsilon(\frac{1}{2}-\tau)\cdot k\cdot\mu\frac{\partial^2 Q}{\partial x^2}+o(\varepsilon^4)=0 \tag{19}$$

Compare to Eq. (3), when $k=1/\varepsilon(\tau-0.5)$, the diffusion wave equation with the fourth order accuracy of truncation error can be got. The equilibrium distribution function of D1Q5 five-bit model can be solved by Eq. (18), they are

$$f_0^{(0)}=A-\frac{5}{4}C+\frac{1}{4}E\qquad f_1^{(0)}=\frac{1}{6}(4C-4B+D-E)\qquad f_2^{(0)}=\frac{1}{24}(2B-2D+E-C)$$
$$f_3^{(0)}=\frac{1}{6}(4C+4B-D-E)\qquad f_4^{(0)}=\frac{1}{24}(2D-2B+E-C) \tag{20}$$

Where $A=Q$, $B=C_d\cdot Q/C_0$, $C=(C_d^2Q+k\cdot\mu\cdot Q)/C_0^2$, $D=(3C_d\cdot\mu\cdot k\cdot Q+C_d^3\cdot Q)/C_0^3$,

$$E=\frac{1}{C_0^4}(\frac{k(2\tau^2-2\tau+\frac{1}{4})\mu^2Q}{(\frac{3}{2}\tau^2-\tau^3-\frac{7}{12}\tau+\frac{1}{24})}+6C_d^2\mu\cdot k\cdot Q+C_d^4Q)$$

The initial moment of the distribution function on each node can be instead by the equilibrium distribution function calculated by Eq. (20). Through the collision step and streaming step of particles, updating the distribution function of each node. According to Eq. (18), the macroscopic discharge on each node can be got.

4. Applications

In this paper, two examples are taken to illustrate applications of the Lattice Boltzmann method. The Muskingum method and analytical solution of linear diffusion wave equation are also applied to solve these examples.

The first example is Longjie-Qiaojia River reach which located upstream of the Yangtze River in China. The length of the reach is 247km and the mean bed slope is 0.00112. A flood event was measured on 18-20 June 1967, at two stations located at the upstream and the downstream ends of the reach. The diffusion wave celerity C_d =5.92m/s, diffusion coefficient μ =19456 m^2/s can be got by the hydraulic elements of two cross-sections. In this example, for the Lattice Boltzmann method, the single relax time τ =1.5, the space step Δx =1000m and the time step Δt =10s. The other reach is the Jiangnan-Fuxi River reach, which is 20.1km long. In 13-24 May 1967, a flood event was measured at two stations located at the upstream and the downstream ends of the reach. Based on the hydraulic elements of Jiangnan and Fuxi stations, the celerity of the diffusion wave C_d =2,792m/s and diffusion coefficient μ =8376 m^2/s can be got. In this example, for the Lattice Boltzmann method, the single relax time τ =1.5, the space step Δx =402m and the time step Δt =5s.

The computed results are compared to the observed data of two reaches in Table 1. The comparisons of observed and computed hydrograph are shown in Fig.2.

Table 1. The comparisons of observed and computed values of two river reaches

River	Method	Q_m observed (m^3/s)	Q_m computed (m^3/s)	Error of peak (%)	Deterministic Coefficient
Longjie-Qiaojia River reach	LB-D1Q5	11400.0	13111.8	-15.02	0.939
	Analytical solution		12878.0	-12.96	0.950
	Muskingum		13390.0	-17.46	0.953
Jiangnan-Fuxi River reach	LB-D1Q5	4660.0	4667.6	-0.16	0.987
	Analytical solution		4726.0	-1.42	0.992
	Muskingum		4668.4	-0.18	0.987

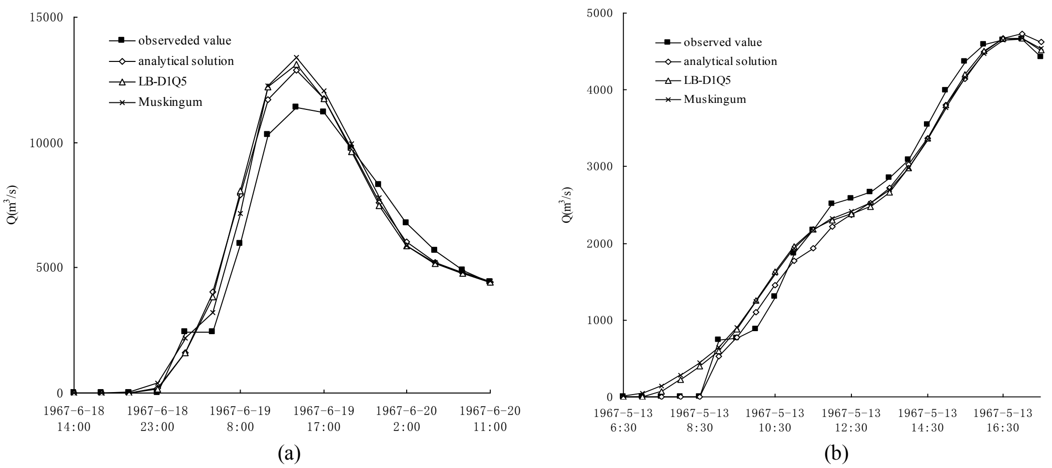

Fig.2. The comparisons of observed and computed hydrograph (a) Longjie-Qiaojia River reach (b) Jiangnan-Fuxi River reach

Fig.2 shows that the hydrographs calculated by the Lattice Boltzmann method, Muskingum method and analytical solution are in agreement with the observed flood hydrographs for two flood events. It also demonstrates that the Lattice Boltzmann method can be used to solve the linear diffusion wave equation and to predict the flood hydrograph. Table1 shows that the D1Q5 five-bit model of Lattice Boltzmann method is more accurate than the Muskingum method and less accurate than the analytical solution. It is because that the Lattice Boltzmann method is a numerical method, it can not match the observed flood hydrograph better than analytical solution.

5. Conclusion

In this paper, the diffusion wave equation is selected to describe the flood wave movement in the river and the D1Q5 five-bit model of Lattice Boltzmann method is presented to solve this partial differential equation. Two different river reaches are taken to illustrate applications of the Lattice Boltzmann method. The Muskingum method and analytical solution which can be got by Laplace transform are selected to verify the accuracy of the Lattice Boltzmann method. These three methods all compare well with the observed flood hydrographs for two flood events and the Lattice Boltzmann method is more accurate than the Muskingum method and less accurate than the analytical solution. It proves that the Lattice Boltzmann method can be used to solve the diffusion wave equation and predict the flood hydrograph well. Additionally, this method can not only save computation time but also easy to use and be programmed. These characteristics make the Lattice Boltzmann method have potential capability in solving flood routing problems.

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