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Localization in dimension theory

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Abstract

Sullivan (1970, 1974) pointed out the availability and applicability of localization methods in homotopy theory. We shall apply the method to dimension theory and analyze covering dimension and cohomological dimension from the viewpoint. The notion of localized dimension with respect to prime numbers shall be introduced as follows: the *P-localized dimension of a space X is at most n* (denoted by $\dim_P X \leqslant n$) provided that every map $f: A \to S^n_P$ of a closed subset A of X into a P-localized n-dimensional sphere S^n_P admits a continuous extension over X.

The main results are:

(1) Let $P_1 \subseteq P_2 \subseteq \mathcal{P}$. Then $\dim_{P_1} X \leqslant \dim_{P_2} X$ (Theorem 1.1).

(2) Let X be a compactum. Then the following conditions are equivalent: (a) $\dim X < \infty$; (b) for some partition $P_1, \ldots, P_s$ of $\mathcal{P}$, $\max\{\dim_{P_i} X\colon i = 1, \ldots, s\} < \infty$; (c) for any partition $P_1, \ldots, P_s$ of $\mathcal{P}$, $\max\{\dim_{P_i} X\colon i = 1, \ldots, s\} < \infty$ (Theorem 1.2).

(3) Let X be a compactum, G an Abelian group. We have that $\sup\{c\text{-}\dim_{G_p} X\colon p \in \mathcal{P}\} = c\text{-}\dim_G X$ (Theorem 1.4). © 1998 Elsevier Science B.V.

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1. Introduction and preliminaries

Bockstein [2] showed that the integral cohomological dimension of a compactum coincides with the supremum of cohomological dimensions with respect to the rings of integers localized at each prime numbers. Dranishnikov [3] proved the existence of an infinite dimensional compactum whose integral cohomological dimension three (also, see [12]). Thus covering dimension cannot be approximated by cohomological dimension in

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the above-mentioned sense. A natural question is to find a new kind of dimension which does approximate the covering dimension.

Sullivan [24,25] pointed out the availability and applicability of localization methods in homotopy theory. In the sense, $K(\mathbb{Z}_{(p)}, n)$ is a localization of $K(\mathbb{Z}, n)$ at p. Thus we may consider that cohomological dimension with respect to $\mathbb{Z}_{(p)}$ is a localization of integral one at p. We shall apply the method to dimension theory and analyze covering dimension and cohomological dimension from the viewpoint.

Throughout this paper, we shall denote by $\mathcal{P}$ the set of all prime numbers. The full subcategory of the category $\mathcal{G}$ of all groups consisting of all nilpotent groups is denoted by $\mathcal{N}$. Let $P \subseteq \mathcal{P}$. A homomorphism $e : G \to G_P$ in $\mathcal{N}$ is said to be a *P-localizing map* if G_P is P-local (i.e., $x \mapsto x^n$, $x \in G_P$, is bijective for all $n \in P'$, where $n \in P'$ means that n is a product of primes in the complementary collection P' of primes with respect to P) and if $e^* : \mathrm{Hom}(G_P, K) \approx \mathrm{Hom}(G, K)$ provided $K \in \mathcal{N}$, with K P-local. We know that there exists the P-localization theory on the category $\mathcal{N}$ [15].

Definition. A connected CW-complex X is *nilpotent* if $\pi_1(X)$ is nilpotent and operates nilpotently on $\pi_n(X)$ for every $n \geqslant 2$.

Let $\mathcal{NH}$ be the homotopy category of nilpotent CW-complexes. $\mathcal{NH}$ contains the homotopy category of simply connected CW-complexes. Moreover, the *simple* CW-complexes are plainly in $\mathcal{NH}$; in particular, $\mathcal{NH}$ contains all connected Hopf spaces.

Definition. Let $X \in \mathcal{NH}$ and $P \subseteq \mathcal{P}$. Then X is *P-local* if $\pi_n(X)$ is P-local for all $n \geqslant 1$. A map $f : X \to Y$ in $\mathcal{NH}$ *P-localizes* if Y is P-local and

$$f^* : [Y, Z]_* \approx [X, Z]_*$$

for all P-local Z in $\mathcal{NH}$, where $[A, B]_*$ means the set of pointed homotopy classes of maps from A to B.

We note that if a P-localization theory exists on $\mathcal{NH}$, it is essentially unique. In fact, the following results [15,24] are very useful for us.

Theorem A. *Every X in $\mathcal{NH}$ admits a P-localization.*

Theorem B. *Let $f : X \to Y$ in $\mathcal{NH}$. Then the following statements are equivalent:*
(i) *f P-localizes,*
(ii) *$\pi_n f : \pi_n X \to \pi_n Y$ P-localizes for all $n \geqslant 1$, and*
(iii) *$H_n f : H_n X \to H_n Y$ P-localizes for all $n \geqslant 1$.*

In this paper, for $P \subseteq \mathcal{P}$ we define P-localized dimension as follows: the *P-localized dimension of a space X is at most n* (denoted by $\dim_P X \leqslant n$) provided that every map $f : A \to S^n_P$ of a closed subset A of X into a P-localized n-dimensional sphere S^n_P admits a continuous extension over X.

Here are the main results of the paper:

Theorem 1.1. *Let $P_1 \subseteq P_2 \subseteq \mathcal{P}$. Then we have the following inequality*:

$$\dim_{P_1} X \leqslant \dim_{P_2} X.$$

Theorem 1.2. *Let X be a compactum. Then the following conditions are equivalent*:
(1) $\dim X < \infty$,
(2) *for some partition* $P_1, \ldots, P_s$ *of* $\mathcal{P}$, $\max\{\dim_{P_i} X\colon i = 1, \ldots, s\} < \infty$,
(3) *for any partition* $P_1, \ldots, P_s$ *of* $\mathcal{P}$, $\max\{\dim_{P_i} X\colon i = 1, \ldots, s\} < \infty$.

Corollary 1.3. *Let X be a compactum and $P_1, \ldots, P_s$ a partition of $\mathcal{P}$. Then if* $\dim_{P_i} X = c\text{-}\dim_{\mathbb{Z}_{(P_i)}} X$ *for* $i \in \{1, \ldots, s\}$, $\dim X = c\text{-}\dim_{\mathbb{Z}} X$.

Theorem 1.4. *Let X be a compactum, G an Abelian group. We have the following equality*:

$$c\text{-}\dim_G X = \sup\{c\text{-}\dim_{G_p} X\colon p \in \mathcal{P}\}.$$

In this paper, we use the following definition of cohomological dimension: the *cohomological dimension of a space X with respect to a coefficient group G is less than or equal to n* (denoted by $\dim_G X \leqslant n$) provided that every map $f : A \to K(G, n)$ of a closed subset A of X into an Eilenberg–MacLane space $K(G, n)$ of type (G, n) admits a continuous extension over X. By the dimension of a space X (denoted by $\dim X$) we mean the *covering dimension* of X. $\mathbb{Z}$ is the additive group of all integers and $\mathbb{Q}$ is the additive group of all rational numbers. $\mathbb{Z}_{(P)}$ is the ring of integers localized at P, that is, the subring of $\mathbb{Q}$ consisting of rationals expressible as fractions k/l with $l \in P'$. We denote by $\mathbb{Z}_p$ and $\mathbb{Z}_{p^\infty}$ the cyclic group of order p and the quasicyclic group of type p^∞, respectively. Recall that $K \in AE(X)$ means that any map $f : A \to K$, A closed in X, extends over X.

2. Localized dimension with respect to prime numbers

In this paper, for $P \subseteq \mathcal{P}$ we define P-localized dimension as follows: the *P-localized dimension of a space X is at most n* (denoted by $\dim_P X \leqslant n$) provided that every map $f : A \to S^n_P$ of a closed subset A of X into a P-localized n-dimensional sphere S^n_P admits a continuous extension over X. Note that since a P-localization of the n-dimensional sphere is unique up to homotopy type, the definition above is well-defined. We use $\dim_{\mathbb{Q}}$ instead of $\dim_{\{\emptyset\}}$.

The first half of the section is devoted to developing the basic properties.

Proposition 2.1. *If* $\dim_P X \leqslant n$, *then* $\dim_P X \leqslant n + 1$.

Proof. We shall give a direct proof by using a classical argument (also see the remark below). Let $\dim_P X \leqslant n$. Select a map $f : A \to S^{n+1}_P$ of a closed subset A of X into a P-localized $(n + 1)$-dimensional sphere. Since we have a homotopy equivalence (see Appendix A1(1)) h: $S^{n+1}_P \cong (S^1 \wedge S^n)_P \approx S^1 \wedge S^n_P \approx \Sigma S^n_P$, where ΣZ means the suspension of Z, it suffices to show that $h \circ f$ can be extended over X. We can represent ΣS^n_P as $\mathrm{Con}^+ S^n_P \cup \mathrm{Con}^- S^n_P$ with $\mathrm{Con}^+ S^n_P \cap \mathrm{Con}^- S^n_P = S^n_P$, where $\mathrm{Con}\, Z$ means the cone on Z. Select open sets U and V of X such that $(h \circ f)^{-1}(\mathrm{Con}^+ S^n_P \setminus S^n_P) \subseteq U$, $(h \circ f)^{-1}(\mathrm{Con}^- S^n_P \setminus S^n_P) \subseteq V$ and $U \cap V = \emptyset$. Because of $\dim_P X \leqslant n$, we have an extension $\tilde{f}: X \setminus (U \cup V) \cup (h \circ f)^{-1}(S^n_P) \to S^n_P$ of $h \circ f|_{(h \circ f)^{-1}(S^n_P)}$. Then by using $\mathrm{Con}^+ S^n_P, \mathrm{Con}^- S^n_P \in AR$ we can get an extension $F: X \to \Sigma S^n_P$ of $h \circ f$. □

Remark. The proposition above follows from Dranishnikov's theorem [5, Lemma 1] and the closed subset theorem of P-localized dimension.

Proposition 2.2. *Let X be a metrizable space. We have the following inequality:*

$$c\text{-}\dim_{\mathbb{Z}_{(P)}} X \leqslant \dim_P X.$$

In particular, if X is finite dimensional or an ANR, the equality holds.

Proof. The first part of proposition follows from Dranishnikov's theorem of noncompact version (see the proof of [6, Theorems 9 and 6], or [11]).

Assume that $\dim X < \infty$ or X is ANR. Let $c\text{-}\dim_{\mathbb{Z}_{(P)}} X = n$. Since the homotopy group of S^n_P contains only a combination of $\mathbb{Z}_{(P)}$, $\mathbb{Z}_{p_1^{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p_l^{k_l}}$ ($p_i \in P$, $k_i \in \mathbb{N}$) and trivial, we see $c\text{-}\dim_{\pi_m(S^n_P)} X \leqslant m$ for $m \in \mathbb{N}$. Thus we have, by using the Postnikov decomposition of S^n_P, that $\dim_P X \leqslant n = c\text{-}\dim_{\mathbb{Z}_{(P)}} X$ (cf. [10]). □

Proposition 2.3 (Dydak [10]). *Suppose $p: E \to B$ is a map, B is a regular cell complex and X is a metrizable space such that $p^{-1}(\sigma) \in AE(X)$ for each cell $\sigma \in B$. Then $E \in AE(X)$ if $B \in AE(X)$.*

Proposition 2.4. *Let X be a metrizable space. We have the following equality:*

$$c\text{-}\dim_{\mathbb{Q}} X = \dim_{\mathbb{Q}} X.$$

Proof. We shall show the inequality $c\text{-}\dim_{\mathbb{Q}} X \geqslant \dim_{\mathbb{Q}} X$.

Since $S^{2k+1}_{\mathbb{Q}} \approx K(\mathbb{Q}, 2k+1)$ and

$$\pi_i\big(S^{2k}_{\mathbb{Q}}\big) = \begin{cases} \mathbb{Q}, & i = 2k,\ 4k-1, \\ 0, & i \neq 2k,\ 4k-1, \end{cases}$$

it suffices to show the inequality with respect to rational-localized even dimensional one.

Let $K(\mathbb{Q}, 2k) \in AE(X)$. Factorize the natural inclusion map $i: S^{2k}_{\mathbb{Q}} \hookrightarrow K(\mathbb{Q}, 2k)$ by a homotopy equivalence $i: S^{2k}_{\mathbb{Q}} \approx S$ and a fibration $p: S \to K(\mathbb{Q}, 2k)$. Then we have that a fibre F of p is $K(\mathbb{Q}, 4k-1)$ (by the Serre's homotopy exact sequence). Thus we see $p^{-1}(\sigma) \approx F \times \sigma \in AE(X)$ by a cell structure of $K(\mathbb{Q}, 2k)$ (note $4k - 1 \geqslant 2k$ for $k \in \mathbb{N}$). Therefore $S^{2k}_{\mathbb{Q}} \approx S \in AE(X)$ follows from Proposition 2.3. □

Lemma 2.5. *Let $P \subsetneqq \mathcal{P}$, $p \in \mathcal{P} \setminus P$ and $p: S^n \to S^n$ be a map of degree p. Then any map $f: S^n \to S^n_P$ can be extended over the mapping cylinder $M(p)$ of p, where S^n identifies the top of $M(p)$.*

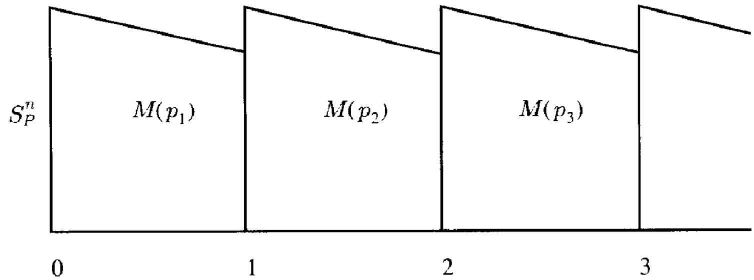

Fig. 1.

Proof. Let $\{p_1, p_2, \ldots\}$ be an enumeration of $\mathcal{P} \setminus P$ with $|\{i \in \mathbb{N}\colon p_i = q\}| = \aleph_0$ for $q \in \mathcal{P} \setminus P$. We can assume S^n_P the Milnor's infinite telescope $M(p_1) \cup M(p_2) \cup \cdots$, where $M(p_i)$ is the mapping cylinder of a map $p_i : S^n \to S^n$ of degree p_i and $M(p_{i-1}) \cap M(p_i) = S^n \times \{i-1\}$ (see Fig. 1).

Now we can select $i_0 \in \mathbb{N}$ such that $p_{i_0} = p$ and $\operatorname{Im} f \subseteq M(p_1) \cup \cdots \cup M(p_{i_0-1})$. Then we have homotopies $H : S^n \times [0,1] \to M(p_1) \cup \cdots \cup M(p_{i_0-1})$ with $H_0 = f$ and $H_1 = j_1 \circ r \circ f$ and $H' : S^n \times [0,1] \to M(p_{i_0})$ with $H'_0 = j_1 \circ r \circ f$ and $H'_1 = j_2 \circ r' \circ j_1 \circ r \circ f$, where $r : M(p_1) \cup \cdots \cup M(p_{i_0-1}) \to S^n$ and $r' : M(p_{i_0}) \to S^n$ are the natural retractions to the bottoms of $M(p_{i_0-1})$ and $M(p_{i_0})$, respectively and the natural homeomorphisms $j_1 : S^n \to S^n \times \{i_0 - 1\}$ and $j_2 : S^n \to S^n \times \{i_0\}$, respectively. Since we see that

$$r' \circ j_1 \circ (r \circ f) = p_{i_0} \circ (r \circ f) \simeq (r \circ f) \circ p \quad \text{in } S^n,$$

we have a homotopy $H'' : S^n \times [0,1] \to S^n$ with $H''_0 = r' \circ j_1 \circ (r \circ f)$ and $H''_1 = (r \circ f) \circ p$.

Then we can get an extension $F : M(p) \to S^n_P$ defined by

$$F(x,t) = \begin{cases} H(x,3t), & 0 \leqslant t \leqslant 1/3, \\ H'(x,3t-1), & 1/3 \leqslant t \leqslant 2/3, \\ (H''(x,3t-2), i_0), & 2/3 \leqslant t < 1, \\ (r \circ f(x), i_0), & t = 1. \end{cases} \qquad \square$$

Theorem 2.6. *Let $P_1 \subseteq P_2 \subseteq \mathcal{P}$. Then we have the following inequality:*

$$\dim_{P_1} X \leqslant \dim_{P_2} X.$$

Proof. We use the notations above. Let $\dim_{P_2} X = n$. Select a map $f : A \to S^n_{P_1}$ from a closed set A of X into a P_1-localized n-dimensional sphere $S^n_{P_1}$. Pick $i \in \mathbb{N}$ with $\operatorname{Im} f \subseteq M(p_1) \cup \cdots \cup M(p_i)$ and the natural retraction $r : M(p_1) \cup \cdots \cup M(p_i) \to S^n \times \{i\}$. Then the composition $h^{-1} \circ r \circ f : A \to S^n \times \{0\} \subseteq S^n_{P_2}$, where $h : S^n \times \{0\} \to S^n \times \{i\}$ is a homeomorphism, has an extension F over $S^n_{P_2}$, as $\dim_{P_2} X = n$. Consider an extension $e : S^n_{P_2} \to S^n_{P_1}$ of h by using Lemma 2.5. Since the composition $e \circ F$ is an extension of $e \circ F|_A = r \circ f \approx f$ in $S^n_{P_1}$. Thus the existence of an extension of f over X follows from the homotopy extension theorem. $\square$

Proposition 2.7. *If $\dim_P X \leqslant n$, then $\dim_P X \times [0,1] \leqslant n+1$.*

Proof. We can easily see the inequality by [6] and Appendix A1(1). □

Proposition 2.8. *Let* $f, g : X \to S_P^n$ *be maps with* $\dim_P Z \leqslant n-1$, *where* $Z = \{x \in X\colon\ f(x) \neq g(x)\}$. *Then* $f \simeq g$ *in* S_P^n.

Proof. We can easily see it in a similar fashion of the covering dimension by using Proposition 2.7, see [17] quoted in the references. □

Lemma 2.9. *Let* $\dim_P X \leqslant n-1$. *Then any map* $f : X \to S_P^n$ *is inessential.*

Proof. It follows from Proposition 2.8. □

We shall illustrate by the following example, which is essentially constructed from an idea of Dydak and Walsh [12], the essential differences between the theory of cohomological dimension and the theory of P-localized dimension.

Example 2.10. There exists a compactum X such that $\dim_2 X \geqslant 3$ and $c\text{-}\dim_{\mathbb{Z}_{(2)}} X = 2$.

For the construction, we need first a preliminary proposition.

Proposition 2.11. *Let* p *be a prime. Then we have for* n *odd* $[K(\mathbb{Z}^t, s), \Omega^k S_p^n] = 0$ *for* $k \geqslant n$, $s \geqslant 2$ *and* $t \geqslant 1$, *and for* n *even* $[K(\mathbb{Z}^t, s), \Omega^k S_p^n] = 0$ *for* $k \geqslant 2n-1$, $s \geqslant 2$ *and* $t \geqslant 1$.

Proof. We assume S_P^n the Milnor's infinite telescope $M(p_1) \cup M(p_2) \cup \cdots$ like in Lemma 2.5. Then we have, by using [12,28], the following:

$$\begin{aligned}
\big[K(\mathbb{Z}^t, s), \Omega^k S_p^n\big] &\approx \big[S^k \wedge K(\mathbb{Z}^t, s), S_p^n\big] \\
&\approx \varinjlim\big\{\big[S^k \wedge K(\mathbb{Z}^t, s), S^n\big] \xrightarrow{p_1} \big[S^k \wedge K(\mathbb{Z}^t, s), S^n\big] \xrightarrow{p_2} \cdots\big\} \\
&\approx \varinjlim\big\{\big[K(\mathbb{Z}^t, s), \Omega^k S^n\big] \xrightarrow{p_1} \big[K(\mathbb{Z}^t, s), \Omega^k S^n\big] \xrightarrow{p_2} \cdots\big\} \\
&= 0. \qquad \square
\end{aligned}$$

Construction of Example 2.10. Let $f : S^3 \to \Omega^3 S_2^3$ be an essential map (note that $\pi_3(\Omega^3 S_2^3) \approx \pi_6(S_2^3) \approx \mathbb{Z}_{2^2}$, for $\pi_6(S^3) \approx \mathbb{Z}_{12}$ and Appendix A2(1)). Construct an inverse sequence $\{(P_i, \tau_i), p_{i+1}\}$ of compact polyhedra and maps, by using Proposition 2.11, satisfying (cf. [12]):

(i) $P_1 = S^3$,

(ii) for $m \in \mathbb{N}$, given a map $g : L \to K(\mathbb{Z}, 2)$ of a subcomplex L of P_m with respect to the triangulation τ_m, the composition $g \circ p_{m+1}|_{p_{m+1}^{-1}(L)}$ extends to a map $G : P_{m+1} \to K(\mathbb{Z}, 2)$,

(iii) for each $i \in \mathbb{N}$ and each $\varepsilon > 0$, there is an $m_0 \geqslant i$ such that for $m \geqslant m_0$ the diameter of $p_{i+1} \circ \cdots \circ p_m(\sigma) \subseteq P_i$ is less than ε for each $\sigma \in \tau_m$, and

(iv) each of the compositions $f \circ p_2 \circ \cdots \circ p_i : P_i \to \Omega^3 S_2^3$ is essential.

Put

$$X = \varprojlim\{(P_i, \tau_i), p_{i+1}\}.$$

Then we have $c\text{-}\dim_{\mathbb{Z}} X \leqslant 2$ (i.e., $c\text{-}\dim_{\mathbb{Z}_{(p)}} X \leqslant 2$ for $p \in \mathcal{P}$) and essential map $f \circ p_{\infty,1} : X \to S^3 \to \Omega^3 S^3_2$. Then by the universality of localizations, we have a map $f_2 : (S^3_2, *) \to ((\overline{\Omega}^3 S^3)_2, *)$ with $f_2 \circ e_2 \simeq f$. It follows that $e_2 \circ p_{\infty,1} : X \to S^3 \to S^3_2$ is essential:

$$\begin{array}{ccccc} X & \xrightarrow{p_{\infty,1}} & (S^3, *) & \xrightarrow{f} & (\overline{\Omega}^3 S^3_2, *) \\ & & {\scriptstyle e_2}\downarrow & & \downarrow{\scriptstyle \approx} \\ & & (S^3_2, *) & \xrightarrow{f_2} & ((\overline{\Omega}^3 S^3)_2, *) \end{array}$$

where $\overline{\Omega} Y$ means the component of the loop space of Y containing the constant map.

Thus by Lemma 2.9 we have $\dim_2 X \geqslant 3$.

Remark 2.12. From the example above it follows that $\mathrm{Tor}(\mathbb{Z}_2, \pi_q(S^2)) \neq *$ for infinitely many q [22].

The remainder of the section is devoted to developing the main results.

A finite collection $P_1, \ldots, P_s$ of subsets of $\mathcal{P}$ is called a *partition of* $\mathcal{P}$ if $P_1 \cup \cdots \cup P_s = \mathcal{P}$ (we do not assume that P_i are pairwise disjoint).

Theorem 2.13. *Let X be a compactum. Then the following conditions are equivalent*:

(1) $\dim X < \infty$,

(2) *for some partition* $P_1, \cdots, P_s$ *of* $\mathcal{P}$, $\max\{\dim_{P_i} X\colon i = 1, \ldots, s\} < \infty$,

(3) *for any partition* $P_1, \cdots, P_s$ *of* $\mathcal{P}$, $\max\{\dim_{P_i} X\colon i = 1, \ldots, s\} < \infty$.

Proof. (1) $\Rightarrow$ (3) follows from $\dim X \geqslant \dim_P X$ for $P \subseteq \mathcal{P}$. (3) $\Rightarrow$ (2) are trivial. We shall show (2) $\Rightarrow$ (1).[2]

Let $\max\{\dim_{P_i} X\colon i = 1, \ldots, s\} < m$ for some partition $P_1, \ldots, P_s$ of $\mathcal{P}$. We shall show that $\dim X \leqslant m + 1$. We may assume that P_i are pairwise disjoint, because of Theorem 2.6.

According to Sullivan [24, 2.13] or [25, p. 23] a sphere S^n is homotopy equivalent to the pullback $Z_n \equiv \prod_{i=1}^{s}\{g_i\colon S^n_{P_i} \to S^n_{\mathbb{Q}}\}$, where each g_i is a Serre fibration.

Assume that $\dim X > m+1$. Then there exists a closed subset A of X and an essential map $f : A \to Z_{m+1}$. Theorem 2.6 implies that $\dim_{\mathbb{Q}} X < m$. Thus, $S^{m+1}_{\mathbb{Q}} \in AE(X \times I^2)$ by Proposition 2.7. Therefore, $\Omega S^{m+1}_{\mathbb{Q}} \in AE(X \times I)$ by an argument of evaluation maps. Also, note that $S^{m+1}_{P_i} \in AE(X \times I)$. Consider the fibration $\Omega S^{m+1}_{\mathbb{Q}} \to F_i \to S^{m+1}_{P_i}$ (in the homotopy category) generated by the fibration $g_i : F_i \hookrightarrow S^{m+1}_{P_i} \to S^{m+1}_{\mathbb{Q}}$. Then it follows from a Ferry's argument [26, Appendix A] (or Proposition 2.3) that $F_i \in AE(X \times$

[2] This proof, which is shorter than the original one, was suggested by the referee. The author is grateful to him for kind suggestion.

I). Let $g: Z_{m+1} \to S^{m+1}_{\mathbb{Q}}$ denote the natural projection. Since $S^{m+1}_{\mathbb{Q}} \in AE(A \times I)$, $g \circ f$ is homotopic to a constant map via some homotopy Φ_t with $\Phi_0 = g \circ f$, $\Phi_1 = *$. Let $\overline{\Phi}_t : A \to Z_{m+1}$ be a lift of Φ_t with $\overline{\Phi}_0 = f$ (use Proposition 2.3). Note that $\overline{\Phi}_1(A) \subseteq g^{-1}(*) = \prod F_i$. Since $F_i \in AE(X \times I)$ for all i, we have $\prod F_i \in AE(\mathrm{Con}(A))$, where $\mathrm{Con}(Z)$ means the cone of Z. Hence $\overline{\Phi}_1$ is an inessential map. This means that f is inessential. This is a contradiction. □

Remark 2.14. There exists an infinite dimensional compactum X such that for any partition $P_1, \ldots, P_s$ of $\mathcal{P}$, $\max\{c\text{-}\dim_{\mathbb{Z}_{(P_i)}} X \colon i = 1, \ldots, s\} < \infty$ (use the Dranishnikov's example).

Remark 2.15. Let $\mathcal{P} = \{p_1, p_2, \ldots\}$. There is an infinite dimensional compactum Y such that $\dim_{p_i} Y = i$ for $i \in \mathbb{N}$ (use the fundamental compacta [3] and the countable sum theorem for P-localized dimension).

Remark 2.16. By Theorem 2.13 and an argument of cohomological dimension, we have

$$\dim X = \sup\{\dim_{P_i} X \colon i = 1, \ldots, s\}$$

for any partition $P_1, \ldots, P_s$ of $\mathcal{P}$.

We note that the above does not always hold for noncompact spaces [8].

Corollary 2.17. *Let X be a compactum and $P_1, \ldots, P_s$ a partition of $\mathcal{P}$. Then if $\dim_{P_i} X = c\text{-}\dim_{\mathbb{Z}_{(P_i)}} X$ for $i \in \{1, \ldots, s\}$, $\dim X = c\text{-}\dim_{\mathbb{Z}} X$.*

Remark 2.18. There is a compactum X such that $\dim X = c\text{-}\dim_{\mathbb{Z}} X = \infty$, $\dim_2 X \geqslant 3$ and $c\text{-}\dim_{\mathbb{Z}_{(2)}} X = 2$ (use Example 2.10 and the fundamental compacta).

From Dydak's theorem, we obtain the following Menger–Urysohn's type sum formula.

Theorem 2.19. *Let $X = A \cup B$ be a metrizable space. Then we have the following inequality:*

$$\dim_P X \leqslant \dim_P A + \dim_P B + 1.$$

Proof. We have the following homotopy equivalence:

$$\begin{aligned} S^m_P * S^n_P &\approx S^1 \wedge (S^m_P \wedge S^n_P) \approx \left(S^1 \wedge S^m\right)_P \wedge S^n_P \approx \left(S^1 \wedge S^m \wedge S^n\right)_P \\ &\cong S^{m+n+1}_P \end{aligned}$$

(see Appendix A1(1) and (2)). Thus the inequality follows from [11]. □

Corollary 2.20. *Let $X = A \cup B$ be a metrizable space. Then we have the following inequality:*

$$c\text{-}\dim_{\mathbb{Q}} X \leqslant c\text{-}\dim_{\mathbb{Q}} A + c\text{-}\dim_{\mathbb{Q}} B + 1.$$

In particular, if X is finite dimensional, then the inequality with respect to $\mathbb{Z}_{(p)}$ holds.

Proof. It follows from Theorem 2.19, Proposition 2.4 and Proposition 2.2. □

Remark. For more general cases, which contain the result above, see Dydak [11].

3. Localization and cohomological dimension

The object of this section is to develop the relation between localization and cohomological dimension.

We shall use the following result of Bockstein [2]:

Bockstein Theorem. *For any Abelian group G and a compactum X, we have the following*:

$$c\text{-}\dim_G X = \sup\{c\text{-}\dim_H X\colon\ H \in \sigma(G)\}.$$

For an Abelian group G its Bockstein basis $\sigma(G)$ is a subset of

$$\{\mathbb{Q}\} \cup \bigcup_{p\in\mathcal{P}} \{\mathbb{Z}_{(p)}, \mathbb{Z}_p, \mathbb{Z}_{p^\infty}\}$$

defined as follows:

(i) $\mathbb{Q} \in \sigma(G)$ iff $G \otimes \mathbb{Q} \neq 0$,
(ii) $\mathbb{Z}_{(p)} \in \sigma(G)$ iff $G \otimes \mathbb{Z}_{p^\infty} \neq 0$,
(iii) $\mathbb{Z}_p \in \sigma(G)$ iff $p\text{-Tor}\, G \otimes \mathbb{Z}_p \neq 0$,
(iv) $\mathbb{Z}_{p^\infty} \in \sigma(G)$ iff $G \otimes \mathbb{Z}_p = 0$ and $\mathrm{Tor}(G, \mathbb{Z}_p) \neq 0$.

Theorem 3.1. *Let X be a compactum, G an Abelian group. We have the following equality*:

$$c\text{-}\dim_G X = \sup\{c\text{-}\dim_{G_p} X\colon\ p \in \mathcal{P}\}.$$

Proof. Let $c\text{-}\dim_G X = n$. Note that

$$G_p \approx \varinjlim \{G \xrightarrow{p_1} G \xrightarrow{p_2} \cdots\},$$

where $p_i \in \mathcal{P}\setminus\{p\}$ and a prime of $\mathcal{P}\setminus\{p\}$ rises for infinitely many p_i. Then the Milnor's infinite telescope $K = M(\tilde{p}_1) \cup M(\tilde{p}_2) \cup \cdots$, where $\tilde{p}_i : K(G,n) \to K(G,n)$ is a map induced by p_i, is an Eilenberg–MacLane complex of type (G_p, n). By using the complex, we shall see that $c\text{-}\dim_{G_p} X \leqslant n$ for $p \in \mathcal{P}$.

Let $f : A \to K$ be a map from a closed set A of X to K. Select $i \in \mathbb{N}$ such that $f(A) \subseteq M(\tilde{p}_1) \cup \cdots \cup M(\tilde{p}_i)$. Then we have the following:

$$f \simeq r \circ f : A \to M(\tilde{p}_1) \cup \cdots \cup M(\tilde{p}_i),$$

where $r : M(\tilde{p}_1) \cup \cdots \cup M(\tilde{p}_i) \to K(G,n)$ is the natural retraction to the bottom of $M(\tilde{p}_i)$. Thus we have, by using the homotopy extension theorem, an extension

$$F : X \to M(\tilde{p}_1) \cup \cdots \cup M(\tilde{p}_i) \subseteq K$$

of f.

We shall show the reverse inequality. By the Bockstein Theorem, it suffices to show that if $H \in \sigma(G)$, then there is a prime q such that $H \in \sigma(G_q)$.

- $H = \mathbb{Q}$: for any $q \in \mathcal{P}$, $G_q \otimes \mathbb{Q} \approx G \otimes \mathbb{Z}_{(q)} \otimes \mathbb{Q} \approx G \otimes \mathbb{Q} \neq 0$.
- $H = \mathbb{Z}_{(p)}$: put $q = p$. Then $G_q \otimes \mathbb{Z}_{p^\infty} \approx G \otimes \mathbb{Z}_{(q)} \otimes \mathbb{Z}_{p^\infty} \approx G \otimes \mathbb{Z}_{p^\infty} \neq 0$.
- $H = \mathbb{Z}_p$: put $q = p$. Then

$$p\text{-Tor}\, G_q \otimes \mathbb{Z}_p = p\text{-Tor}(G \otimes \mathbb{Z}_{(q)}) \otimes \mathbb{Z}_p \underset{\text{subgroup}}{\supset} p\text{-Tor}\, G \otimes \mathbb{Z}_{(q)} \otimes \mathbb{Z}_p$$
$$\approx p\text{-Tor}\, G \otimes \mathbb{Z}_p \neq 0.$$

- $H = \mathbb{Z}_{p^\infty}$: put $q = p$. Then $G_q \otimes \mathbb{Z}_p \approx G \otimes \mathbb{Z}_{(q)} \otimes \mathbb{Z}_p \approx G \otimes \mathbb{Z}_p = 0$.

We shall show that $\text{Tor}(G_q, \mathbb{Z}_p) \neq 0$.

Since $G \otimes \mathbb{Z}_p = 0$ and $\text{Tor}(G, \mathbb{Z}_p) \neq 0$, we have, by the structure theorem of divisible groups [14], that $0 \neq p\text{-Tor}\, G = \bigoplus \mathbb{Z}_{p^\infty}$. Consider the following short exact sequence [13]:

$$\begin{aligned}
0 &\to p\text{-Tor}\, G \otimes p\text{-Tor}\, \mathbb{Z}_{(q)} \\
&\to p\text{-Tor}(G \otimes \mathbb{Z}_{(q)}) \\
&\to \left(G/\, p\text{-Tor}\, G \otimes p\text{-Tor}\, \mathbb{Z}_{(q)}\right) \oplus \left(p\text{-Tor}\, G \otimes \mathbb{Z}_{(q)}/\, p\text{-Tor}\, \mathbb{Z}_{(q)}\right) \to 0.
\end{aligned}$$

Then we have

$$\begin{aligned}
p\text{-Tor}(G_q) &\approx p\text{-Tor}(G \otimes \mathbb{Z}_{(q)}) \approx p\text{-Tor}\, G \otimes \mathbb{Z}_{(q)}/\, p\text{-Tor}\, \mathbb{Z}_{(q)} \\
&\approx \bigoplus \mathbb{Z}_{p^\infty} \otimes \mathbb{Z}_{(q)} \approx \bigoplus \mathbb{Z}_{p^\infty} \neq 0.
\end{aligned}$$

It follows that $\text{Tor}(G_q, \mathbb{Z}_p) \neq 0$. □

Remark 3.2. Note that there are infinitely many mutually nonisomorphic Abelian groups with p-localizations isomorphic to $\mathbb{Z}_{(p)}$ for every prime p (for example,

$$G(n) = \{k/l \in \mathbb{Q}\colon\ l \text{ is } n\text{th power-free}\}$$

for $n \geqslant 2$).

Corollary 3.3. *Let X be a finite dimensional compactum and K a simply connected CW-complex. The following are equivalent:*

(1) $K \in AE(X)$,

(2) $K_p \in AE(X)$ *for each prime* $p \in \mathcal{P}$.

Proof. Theorem 3.1 follows that

$$\begin{aligned}
c\text{-dim}_{H_i(K)} X &= \sup\left\{c\text{-dim}_{H_i(K)_p} X\colon\ p \in \mathcal{P}\right\} \\
&= \sup\left\{c\text{-dim}_{H_i(K_p)} X\colon\ p \in \mathcal{P}\right\},
\end{aligned}$$

see Appendix. We can see the equivalence by using [6]. □

The author is interested in the following problems:

Problem.

(1) Suppose $K \in AE(X)$ is a simply connected CW-complex and P is a set of primes. Is $K_P \in AE(X)$?

(2) Suppose K is a simply connected CW-complex and $K_p \in AE(X)$ for all primes p. Is $K \in AE(X)$?

(3) Suppose $K \in AE(X)$ is a nilpotent CW-complex and P is a set of primes. Is $K_P \in AE(X)$?

Appendix A

A1. *We have the following pointed homotopy equivalences:*

(1) $S^1 \wedge S^m_P \approx (S^1 \wedge S^m)_P$,

(2) $S^m_P \wedge S^n_P \approx (S^m \wedge S^n)_P$,

(3) $\overline{\Omega}^n(X_P) \approx (\overline{\Omega}^n X)_P$, *where* $\overline{\Omega}X$ *means the component of the loop space of* X *containing the constant map.*

Proof. (1) By the universality of localization, it suffices to show that

$$\mathrm{id} \wedge e_P : S^1 \wedge S^m \to S^1 \wedge S^m_P$$

P-localizes ($m \geqslant 1$).

Because of the following homotopy equivalences:

$$S^1 \wedge S^m_P \cong S^1 \wedge S^0 \wedge S^m_P \approx S^0 * S^m_P,$$

we see that $S^1 \wedge S^m_P$ is simply connected.

That $H_n(S^1 \wedge S^m_P)$ is a P-local group for $n \geqslant 1$ follows from

$$\begin{aligned} H_n(S^1 \wedge S^m_P) &\approx H_n(S^1 \times S^m_P, S^1 \vee S^m_P) = H_n\big((S^1, *) \times (S^m_P, *)\big) \\ &\approx \bigoplus_{i+j=n} H_i(S^1, *) \otimes H_j(S^m_P, *) \oplus \bigoplus_{i+j=n-1} H_i(S^1, *) * H_j(S^m_P, *) \\ &\approx \begin{cases} \mathbb{Z}_{(P)}, & n = m+1, \\ 0, & n \neq m+1,\ n \geqslant 1. \end{cases} \end{aligned}$$

Finally, we must show that

$$(\mathrm{id} \wedge e_P)_* : H_n(S^1 \wedge S^m) \to H_n(S^1 \wedge S^m_P)$$

is P-isomorphic for $n \geqslant 1$. But it can easily see that by using the homology exact sequences and the five lemma modulo the Serre's class.

By a similar way, we can see (2) and (3). □

A2. *We have the following isomorphisms:*

(1) $\pi_n(X_P) \approx \pi_n(X)_P$,

(2) $H_n(X_P) \approx H_n(X)_P$.

Proof. It follows from Theorem B in the introduction and the universality of localization. □

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