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A pedagogical remark on the main theorem of perturbative renormalization theory ☆

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Dedicated to A.S. Wightman

Abstract

In this note, it is proven that, given two perturbative constructions of time-ordered products via the Bogoliubov-Epstein-Glaser recursion, both sets of coupling functions are related by a local formal power series, recursively determined by causality.

☆ The "Popineau-Stora" preprint has remained in the status of "unpublished material" until today. It circulated through Raymond Stora's collaborators and even beyond this inner circle.

For this Special Issue in Memoriam of Raymond Stora, and for the benefit of present and future colleagues working on *Mathematical Foundations of Quantum Field Theory*, it seems appropriate indeed to bring the "Popineau-Stora" preprint out of the shadows.

This idea was mooted to us by J.C. Varilly (CIMM) who typed out in June 2012 the transcribed LaTeX version from the CPT-1982 preprint. Checking the LaTeX version for accuracy and completeness was done by F. Thuillier (LAPTH) and S. Lazzarini (CPT).

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1. Introduction

Perturbative renormalization theory is by now a well understood subject, after the intense streamlining which took place about ten years ago [1]. It is amusing to recall that the Wightman axioms for local fields [2], abstracted from those features of perturbation theory which were mathematically sound, and complemented by a nonperturbative part, were not used until some fifteen years later to clean up the perturbative theory. If many renormalization schemes of good mathematical standards were proposed ten years ago [1], it was the merit of H. Epstein and V. Glaser [3] to streamline the construction proposed by N.N. Bogoliubov and co-workers [1], which is closest to the spirit of Wightman's axioms. It is in this construction that the role of locality and spectrum is most conspicuous, whereas the main combinatorial theorems were first proved in the momentum space scheme of W. Zimmermann and co-workers [1], or/and in the analytic renormalization schemes finally formulated by E.R. Speer, which culminated in the dimensional scheme proposed by G. 't Hooft and M. Veltman, now one of the most popular, for computational reasons [1].

The main theorems we referred to are the quantum action principles which describe the changes of Green functions under reparametrization and field changes of variables. We will say very little about the second of these (the Lam action principle), because there is very little to add to the version given by R. Flume [4], taking into account the technical remarks which will be found here. On the other hand the first theorem follows from the well known fact that, given two renormalization prescriptions, one can pass from the first to the second by a suitable change in the Lagrangian.

This is well known, but in our knowledge, this has not been proved either in all generality or directly. All that seems to exist in the published literature is a collection of comparisons of pairs of useful renormalization schemes. To our great surprise, such a proof does exist and is extremely simple if it were not for fixing the notation. The Bogoliubov–Epstein–Glaser construction is ideal for this purpose, and we propose to review it here in a slightly modified form which, whereas it is—fortunately—equivalent to the initial one, is a bit more symmetrical and easier to work with in dealing with formal problems.

In section 2 we fix the notation, hopefully precisely enough and briefly enough so that it will be clear that the arguments given in the following are really proofs!

In section 3 we review the Epstein–Glaser recursive construction in time-ordered rather than retarded-advanced form as it was originally performed.

In section 4, we state and prove the first main theorem.

Section 5 is devoted to a few concluding remarks.

2. Notations

The building block of the construction is a finite set of N free—generalized free—fields, which we shall assume to be massive, defined in d-dimensional Minkowski space time M_d, with signature $(+, -, \dots, -.)$.

The set of all Wick monomials of those fields and their derivatives can be and will be lexicographically ordered. Given a word $\mathbf{w}$ of this dictionary D, there corresponds to it a Wick monomial $W(x)$ which is an operatored-valued distribution defined within the Fock space of the problem. To each Wick monomial $W(x)$ we associate a space-time dependent coupling constant in $\mathcal{D}$ or $\mathcal{S}$, $g_W(x)$. The problem is to define

$$S(\mathbf{g}) = \mathbf{1} + \sum_n \frac{i^n}{n!} \sum_{\substack{w_i \in D \\ i=1,\dots,n}} \int \mathrm{T}\big(W_1(x_1)\dots W_n(x_n)\big) g_{W_1}(x_1)\dots g_{W_n}(x_n)\, dx_1 \dots dx_n \tag{1}$$

as a formal power series in $\mathbf{g} = \{ g_{w_i} \mid w_i \in D \}$ where T denotes time ordering, in such a way that causal factorization holds:

$$S(\mathbf{g}_1 + \mathbf{g}_2) = S(\mathbf{g}_1) S(\mathbf{g}_2) \quad \text{if} \quad \operatorname{supp} \mathbf{g}_1 \gtrsim \operatorname{supp} \mathbf{g}_2. \tag{2}$$

The symbol $\gtrsim$ means the following: $(\operatorname{supp} \mathbf{g}_1) \cap (\operatorname{supp} \mathbf{g}_2 + \overline{V}_-)$, where $\overline{V}_-$ is the closed past light cone in M_d.

We introduce a more compact notation:

$$\begin{aligned} &\mathrm{T}\big(W_1(x_1)\dots W_n(x_n)\big) = \big[\mathrm{T}_{\mathbf{W}}(X)\big]_{W_1 \dots W_n} \\ &X = (x_1, \dots, x_n), \qquad |X| = n, \end{aligned} \tag{3}$$

so that $\mathrm{T}_{\mathbf{W}}(X)$ denotes the set of time ordered products of n Wick monomials, and we may rewrite

$$S(\mathbf{g}) = \mathbf{1} + \sum_n \frac{i^n}{n!} \int \mathrm{T}_{\mathbf{W}}(X)\, \mathbf{g}_{\mathbf{W}}(X)\, dX \tag{4}$$

where a summation over $\mathbf{W}$ is implied. Summation over repeated $\mathbf{W}$ indices will be used throughout.

It is well known that the heuristic formula for T, in terms of Heaviside step functions, leads to the cursed ultraviolet divergences due to mishandling of the distribution character of products of Wick monomials [2,3].

Besides, T-products will be defined in such a way that they fulfil Wick's theorem (which holds for ordinary products): let us say that for words $\mathbf{w}$ in D syllables are made with vowels (fields) and consonants (derivatives).

Two words in D differ if they do not have the same syllables—independent of order (if fermion fields are involved some games with Grassmann algebra valued source functions have to be played in order to get the right signs). Then Wick's theorem asserts:

$$\frac{\mathrm{T}_{\mathbf{W}}(X)}{|X|!} = \sum_{\mathbf{W}_1 \cup \mathbf{W}_2 = \mathbf{W}} \frac{\langle \mathrm{T}_{\mathbf{W}_1}(X) \rangle}{|W_1|!} \frac{:W_2(X):}{|W_2|!} \tag{5}$$

where $|W_i|$ is the product over elements of X of the number of repeated syllables in each subword involved.

Among the other properties required for $\mathrm{T}_{\mathbf{W}}(X)$, for which we refer to the original article, note

$$\partial_{\mu_1} \mathrm{T}\big(W_1(x_1)\dots W_n(x_n)\big) = \mathrm{T}\big(\partial_{\mu_1} W_1(x_1) \dots W_n(x_n)\big) \tag{6}$$

which we shall need later on.

3. The recursive construction of Bogoliubov, Epstein, Glaser: the time ordered version

The recursion hypothesis formulated by Epstein and Glaser is that $\mathrm{T}_{\mathbf{W}}(X)$ has been constructed, fulfilling all required properties for $|X| < n$, such that, in particular:

$$\text{(i)}\quad \mathrm{T}_{\mathbf{W}}(X) = \mathrm{T}_{\mathbf{W}_1}(X_1)\,\mathrm{T}_{\mathbf{W}_2}(X_2) \quad \text{if} \quad X_1 \gtrsim X_2,\ |X_1| < n,\ |X_2| < n,\ \mathbf{W}_1 \cup \mathbf{W}_2 = \mathbf{W}. \tag{7}$$

$$\text{(ii)}\quad [\mathrm{T}_{\mathbf{W}_1}(X_1), \mathrm{T}_{\mathbf{W}_2}(X_2)] = 0 \quad \text{if} \quad X_1 \sim X_2,\ |X_1| < n,\ |X_2| < n. \tag{8}$$

The question is now to construct $\mathrm{T}_{\mathbf{W}}(X)$ for $|X| = n$. We are going to use causal factorization in the form (i), (ii) as much as possible.

Note first that the regions

$$C_I = \langle X = I \cup I' \mid I \gtrsim I' \rangle, \qquad I, I' \neq \emptyset, \tag{9}$$

cover almost all of $M_d^{|X|}$. In fact, it is a simple geometrical Lemma that

$$\bigcup_{\substack{I, I' \neq \emptyset \\ I \cup I' = X}} C_I = M_d^{|X|} \setminus \operatorname{diag} X, \tag{10}$$

where

$$\operatorname{diag} X = \left(X \big|_{x_1 = \cdots = x_n} \right) \tag{11}$$

for $|X| = n$.

Let us now consider

$$\mathrm{T}_{\mathbf{W}}^{(I)}(X) = \mathrm{T}_{\mathbf{W}_I}(I)\,\mathrm{T}_{\mathbf{W}_{I'}}(I'), \qquad \mathbf{W}_I \cup \mathbf{W}_{I'} = \mathbf{W} \tag{12}$$

as a distribution, this can be restricted to C_I. It is a simple matter to show that

$$\mathrm{T}_{\mathbf{W}}^{(I)}(X)\Big|_{C_I \cap C_J} = \mathrm{T}_{\mathbf{W}}^{(J)}(X)\Big|_{C_I \cap C_J}. \tag{13}$$

Indeed, using (i), this can be written

$$\begin{aligned} &\mathrm{T}_{\mathbf{W}_{I\cap J}}(I \cap J)\,\mathrm{T}_{\mathbf{W}_{I\cap J'}}(I \cap J')\,\mathrm{T}_{\mathbf{W}_{I'\cap J}}(I' \cap J)\,\mathrm{T}_{\mathbf{W}_{I'\cap J'}}(I' \cap J') \\ &\quad = \mathrm{T}_{\mathbf{W}_{J\cap I}}(J \cap I)\,\mathrm{T}_{\mathbf{W}_{J\cap I'}}(J \cap I')\,\mathrm{T}_{\mathbf{W}_{J'\cap I}}(J' \cap I)\,\mathrm{T}_{\mathbf{W}_{J'\cap I'}}(J' \cap I'), \end{aligned} \tag{14}$$

which is true by (ii), because in $C_I \cap C_J$, $J \cap I' \sim I \cap J'$.

If the various $\mathrm{T}_{\mathbf{W}}$'s were scalar distributions instead of operator valued distributions they would thus define a unique distribution $\overset{\circ}{\mathrm{T}}_{\mathbf{W}}(X)$ in all of $M_d^{|X|} \setminus \operatorname{diag} X$. The situation is reduced to a problem on scalar distributions by means of Wick's theorem applied to $\mathrm{T}_{\mathbf{W}_I}(I)\,\mathrm{T}_{\mathbf{W}_{I'}}(I')$. The theory of power counting elaborated by Epstein and Glaser insures that all the coefficients of the Wick expansion of $\overset{\circ}{\mathrm{T}}_{\mathbf{W}}(X)$ are indeed continuable to all of M_d with a "degree of singularity on diag X" which is the same as theirs. Since we have nothing to add here we refer the reader to the original article. It is clear at this point that $\mathrm{T}_{\mathbf{W}}(X)$ obtained through Wick's expansion does fulfil Wick's theorem. (i) is trivial, by construction; so is (ii) because if $I \sim I'$ ($I \cup I' = X$), then

$$\mathrm{T}_{\mathbf{W}}(X)\big|_{I\sim I'} = \mathrm{T}_{\mathbf{W}_I}(I)\,\mathrm{T}_{\mathbf{W}_{I'}}(I')\big|_{I\sim I'} = \mathrm{T}_{\mathbf{W}_{I'}}(I')\,\mathrm{T}_{\mathbf{W}_I}(I)\big|_{I\sim I'},$$

the first equality holding in $C_I = \{I \gtrsim I'\}$, the second in $C_{I'} = \{I' \gtrsim I\}$.

In their initial construction, Epstein and Glaser defined a certain commutator $\mathcal{C}_{\mathbf{W}}(X)$ with support the union of two opposite closed cones intersecting on diag X, by virtue of (i) and (ii), and they decomposed it as $\mathcal{C}_{\mathbf{W}}(X) = \mathcal{R}_{\mathbf{W}}(X) - \mathcal{A}_{\mathbf{W}}(X)$, the difference of a retarded and an advanced function, decomposition whose ambiguity, with support on diag X is the same as that of the continuation $\mathrm{T}_{\mathbf{W}}(X)$ of $\overset{\circ}{\mathrm{T}}_{\mathbf{W}}(X)$. $\mathrm{T}_{\mathbf{W}}(X)$ was then deduced from $\mathcal{R}_{\mathbf{W}}(X)$ through the known formulae relating time ordered products to retarded products [3]. In time, the present version was a by-product of Ref. [5]. The problem of cutting a distribution into two pieces with given supports is really the same problem as that of continuing a distribution through the boundary of the open set where it is defined, both analytically and geometrically [6]. We have just chosen the second route here.

There is another small technical alternative we may offer to the Epstein–Glaser construction. In the course of verifying all the recursion hypotheses which we have not bothered to list one may, at one point worry about Lorentz invariance. If not so important by far as translation invariance, this is important in practice. The question is whether one can find $\mathrm{T}_{\mathbf{W}}(X)$ with the same Lorentz covariance properties as those of $\overset{\circ}{\mathrm{T}}_{\mathbf{W}}(X)$. Epstein and Glaser solve this question by using momentum space analyticity properties of the constructed kernels and integrating over the compact O(4) subgroup of the complex Lorentz group that leaves the analyticity domain invariant—another tribute to A. S. Wightman [2]. The same question can be solved in the reals and boils down to a trivial cohomology problem for $\mathrm{SL}(2,\mathbb{C})$—trivial because the cohomology of $\mathrm{SL}(2,\mathbb{C})$ with values in a finite dimensional representation space is trivial. This was indicated to us long ago by B. Malgrange and P. Cartier in a similar context [7].

Consider a set of scalar coefficients occurring in the Wick expansion of $\overset{\circ}{\mathrm{T}}_{\mathbf{W}}(X)$, $\overset{\circ}{\boldsymbol{\tau}}(X)$, transforming covariantly:

$$\overset{\circ}{\boldsymbol{\tau}}(\Lambda X) = \mathcal{D}(\Lambda)\overset{\circ}{\boldsymbol{\tau}}(X), \qquad \Lambda \in \mathrm{SL}(2,\mathbb{C}) \tag{15}$$

(in the sense of distributions), according to the covariance properties of the involved fields and derivatives. Here, $\mathcal{D}(\Lambda)$ is a finite dimensional representation of $\mathrm{SL}(2,\mathbb{C})$. Among the continuations of $\overset{\circ}{\boldsymbol{\tau}}(X)$:

$$\boldsymbol{\tau}(X) = \underset{\sim}{\boldsymbol{\tau}}(X) + \sum \mathbf{c}_i\, P_i(\partial)\,\delta(X), \tag{16}$$

where $\underset{\sim}{\boldsymbol{\tau}}(X)$ is a given continuation, $P_i(\partial)$ a monomial of derivatives, $\mathbf{c}_i$ complex coefficients, with values in the representation space of $\mathcal{D}$, and the summation ranges over all polynomials of a given degree, in agreement with power counting theory [3], is there one such that

$$\boldsymbol{\tau}(\Lambda X) = \mathcal{D}(\Lambda)\boldsymbol{\tau}(X)\ ? \tag{17}$$

Now,

$$\boldsymbol{\tau}(\Lambda X) - \mathcal{D}(\Lambda)\boldsymbol{\tau}(X) = \underset{\sim}{\boldsymbol{\tau}}(\Lambda X) - \mathcal{D}(\Lambda)\underset{\sim}{\boldsymbol{\tau}}(X) + \sum\big(\Delta^i_j(\Lambda) - \delta^i_j\mathcal{D}(\Lambda)\big)\mathbf{c}_i\, P_j(\partial)\,\delta(X), \tag{18}$$

where $\Delta(\Lambda)$ is the representation acting on $\{P_i(\partial)\}$. Now

$$\underset{\sim}{\boldsymbol{\tau}}(\Lambda X) - \mathcal{D}(\Lambda)\underset{\sim}{\boldsymbol{\tau}}(X)\Big|_{M_d^{|X|}\setminus \mathrm{diag}\ X} = \overset{\circ}{\boldsymbol{\tau}}(\Lambda X) - \mathcal{D}(\Lambda)\overset{\circ}{\boldsymbol{\tau}}(X) = 0. \tag{19}$$

Thus,

$$\underset{\sim}{\boldsymbol{\tau}}(\Lambda\, X) - \mathcal{D}(\Lambda)\underset{\sim}{\boldsymbol{\tau}}(X) = \sum_i \mathbf{c}_i(\Lambda)\, P_i(\partial)\, \delta(X) \tag{20}$$

for some set of coefficients $\mathbf{c}_i(\Lambda)$.

Now

$$\begin{aligned}
&\underset{\sim}{\boldsymbol{\tau}}(\Lambda\Lambda'\, X) - \mathcal{D}(\Lambda\Lambda')\underset{\sim}{\boldsymbol{\tau}}(X)\\
&\quad = \underset{\sim}{\boldsymbol{\tau}}(\Lambda\Lambda'\, X) - \mathcal{D}(\Lambda)\underset{\sim}{\boldsymbol{\tau}}(\Lambda' X) + \mathcal{D}(\Lambda)\big[\underset{\sim}{\boldsymbol{\tau}}(\Lambda' X) - \mathcal{D}(\Lambda')\underset{\sim}{\boldsymbol{\tau}}(X)\big]\\
&\quad = \sum_i \mathbf{c}_i(\Lambda\Lambda')\, P_i(\partial)\, \delta(X)\\
&\quad = \sum_i \mathbf{c}_i(\Lambda)\, P_i(\partial)\, \delta(\Lambda' X) + \mathcal{D}(\Lambda)\sum_i \mathbf{c}_i(\Lambda')\, P_i(\partial)\, \delta(X)\\
&\quad = \sum_i \mathbf{c}_i(\Lambda)\, \Delta^i_j(\Lambda')\, P_j(\partial)\, \delta(X) + \mathcal{D}(\Lambda)\sum_i \mathbf{c}_i(\Lambda')\, P_i(\partial)\, \delta(X).
\end{aligned} \tag{21}$$

Thus, the $\mathbf{c}_i(\Lambda)$'s fulfil the consistency condition:

$$\mathbf{c}_j(\Lambda\Lambda') = \mathbf{c}_i(\Lambda)\, \Delta^i_j(\Lambda') + \mathcal{D}(\Lambda)\mathbf{c}_j(\Lambda'), \tag{22}$$

or, putting

$$\mathbf{c}_j(\Lambda) = \mathbf{d}_i(\Lambda)\, \Delta^i_j(\Lambda'), \tag{23}$$

$$\mathbf{d}_i(\Lambda\Lambda') = \mathbf{d}_i(\Lambda) + \mathcal{D}(\Lambda)\big[\widetilde{\Delta}\big]^j_i\, \mathbf{d}_j(\Lambda'), \tag{24}$$

where $\widetilde{\Delta}$ is contragredient to Δ. Thus $\{\mathbf{d}_i(\Lambda)\}$ is a one cocycle of $\mathrm{SL}(2, \mathbb{C})$ with values in the representation space carrying $\mathcal{D} \otimes \widetilde{\Delta}$. By the vanishing of the corresponding cohomology [7],

$$\mathbf{d}_i(\Lambda) = \big[\mathcal{D}(\Lambda)\widetilde{\Delta}^j_i(\Lambda) - \delta^j_i\big]\, \mathbf{d}_j \tag{25}$$

for some constant $\mathbf{d}_j$.

Thus

$$\mathbf{c}_i(\Lambda) = \big[\mathcal{D}(\Lambda) - \Delta^j_i(\Lambda)\big]\, \mathbf{d}_j \tag{26}$$

and

$$\mathbf{c}_i = \mathbf{d}_i \tag{27}$$

does the job.

4. The first main theorem of renormalization theory

Let us now assume we have constructed two solutions $S_T(\mathbf{g})$, $S_{T'}(\mathbf{G})$ of the Bogoliubov–Epstein–Glaser recursion, involving two sequences of time ordered products $\mathrm{T}(X)$, $\mathrm{T}'(X)$. We want to prove that there exists a local formal power series

$$G_{\mathbf{W}}(x) = g_{\mathbf{W}}(x) + \sum_1^{\infty} G_{\mathbf{W},n}(\mathbf{g}, D\mathbf{g})(x) \tag{28}$$

of $\mathbf{g}(x)$ and its derivatives, such that

$$S_T(\mathbf{g}) = S_{T'}(\mathbf{G}). \tag{29}$$

The term $G_{\mathbf{W},n}$ involves n factors $\mathbf{g}$ or $D\mathbf{g}$. Let us denote by $S^{(n)}$, $\mathbf{G}^{(n)}$ the n-th order approximation to S, $\mathbf{G}$. Let us assume we have found $\mathbf{G}^{(n-1)}$ such that

$$S_T^{(n-1)}(\mathbf{g}) = S_{T'}^{(n-1)}(\mathbf{G}^{(n-1)}), \tag{30}$$

and let us look for

$$\mathbf{G}^{(n)} = \mathbf{G}^{(n-1)} + \mathbf{G}_{\ ,n} \tag{31}$$

such that

$$S_T^{(n)}(\mathbf{g}) = S_{T'}^{(n)}(\mathbf{G}^{(n)}). \tag{32}$$

Equating the differences of both sides of Eq. (31) and (32), we get, for $\mathbf{W} = \mathbf{W}_1 \cup \cdots \cup \mathbf{W}_n$,

$$\begin{aligned} &\int T_{\mathbf{W}}(x_1, \ldots, x_n)\, g_{\mathbf{W}_1}(x_1) \ldots g_{\mathbf{W}_n}(x_n)\, dx_1 \ldots dx_n \\ &\quad = \int T'_{\mathbf{W}}(x_1, \ldots, x_n)\, g_{\mathbf{W}_1}(x_1) \ldots g_{\mathbf{W}_n}(x_n)\, dx_1 \ldots dx_n \\ &\qquad + \sum_{\substack{p=2 \\ n_1, \ldots, n_p \\ n_1 + \cdots + n_p = n}}^{p=n-1} \int T'_{\mathbf{W}}(x_1, \ldots, x_p)\, G_{\mathbf{W}_1}^{(n_1)}(x_1) \ldots G_{\mathbf{W}_p}^{(n_p)}(x_p)\, dx_1 \ldots dx_p \\ &\qquad + \int T'_{\mathbf{W}}(x)\, G_{\mathbf{W},n}(x)\, dx. \end{aligned} \tag{33}$$

Note that the middle sum only involves previously known terms $G_{\mathbf{W}}^{(k)}$ for $k \le n-1$, the only place where $G_{\mathbf{W}}^{(n)}$ is involved being the last term. In order to prove that this equation does determine $G_{\mathbf{W}}^{(n)}$, it is enough to prove that both sides admit the same causal factorizations outside diag$\,(x_1 = \cdots = x_n)$, and thus, the left hand side differs from the sum of the first two terms on the right hand side by a counter-term supported by diag$\,(x_1 = \cdots = x_n)$, which does indeed define $G_{\mathbf{W},n}$.

This is achieved by substituting $g_{\mathbf{W}} = {}^{(1)}g_{\mathbf{W}} + {}^{(2)}g_{\mathbf{W}}$ with supp$\,{}^{(1)}g_{\mathbf{W}} \gtrsim$ supp$\,{}^{(2)}g_{\mathbf{W}}$, which entails, by locality, $G_{\mathbf{W}}^{(k)} = {}^{(1)}G_{\mathbf{W}}^{(k)} + {}^{(2)}G_{\mathbf{W}}^{(k)}$ for $k < n$. Here ${}^{(i)}G$ is obtained from G by replacing g by ${}^{(i)}g$, $i = 1, 2$. By varying the supports of ${}^{(1)}g_{\mathbf{W}}$, ${}^{(2)}g_{\mathbf{W}}$ and identifying terms with different powers of ${}^{(1)}g$, ${}^{(2)}g$ we obtain the desired result: Eq. (33) can be rewritten

$$\begin{aligned} &S_T^{(n)}({}^{(1)}\mathbf{g} + {}^{(2)}\mathbf{g}) - S_T^{(n-1)}({}^{(1)}\mathbf{g} + {}^{(2)}\mathbf{g}) \\ &\quad = S_{T'}^{(n)}({}^{(1)}\mathbf{G}^{(n)} + {}^{(2)}\mathbf{G}^{(n)}) - S_{T'}^{(n-1)}({}^{(1)}\mathbf{G}^{(n-1)} + {}^{(2)}\mathbf{G}^{(n-1)}). \end{aligned} \tag{34}$$

By causal factorization for $S_T(\mathbf{g})$, $S_{T'}(\mathbf{G})$, the cross terms we are solely interested in, read

$$\left[S_T({}^{(1)}\mathbf{g})\, S_T({}^{(2)}\mathbf{g})\right]_n = \left[S_{T'}({}^{(1)}\mathbf{G}^{(n)})\, S_{T'}({}^{(2)}\mathbf{G}^{(n)})\right]_n, \tag{35}$$

$$\begin{aligned} S_T({}^{(1)}\mathbf{g})_p\, S_T({}^{(2)}\mathbf{g})_q = S_{T'}({}^{(1)}\mathbf{G}^{(n)})_p\, S_{T'}({}^{(2)}\mathbf{G}^{(n)})_q = S_{T'}({}^{(1)}\mathbf{G}^{(n-1)})_p\, S_{T'}({}^{(2)}\mathbf{G}^{(n-1)})_q \\ \text{for} \quad p + q = n, \quad 1 \le p \le n-1, \end{aligned} \tag{36}$$

where the last step uses the previous remark that, for the indicated terms, only ${}^{(1),(2)}G_{\mathbf{W}}^{(n-1)}$ is involved. This is the required coincidence of the causal factorizations of both sides of Eq. (33). It is true because for $p \le n-1$,

$$S_T^{(p)}({}^{(1)}\mathbf{g}) = S_{T'}^{(p)}({}^{(1)}\mathbf{G}^{(n-1)}), \qquad S_T^{(p)}({}^{(2)}\mathbf{g}) = S_{T'}^{(p)}({}^{(2)}\mathbf{G}^{(n-1)}), \tag{37}$$

by the recursion hypothesis.

5. Remarks

1. It is not exactly true that this construction determines $G_{\mathbf{W}}^{(n)}$ uniquely, as expected: only $\int T'_{\mathbf{W}}(x)\, G_{\mathbf{W},n}(x)\, dx$ is determined, i.e. the corresponding action integral: $G_{\mathbf{W},n}$ is only determined modulo the ambiguity which stems from the identities

$$\int \partial_\mu \big[T'_{\mathbf{W}}(x)\, G_{\mathbf{W}}(x) \big]\, dx = 0.$$

2. It would be of interest to classify all admissible sequences with respect to their behaviour with respect to power counting and, in particular, those for which some subsets $\mathcal{S}$ of the $G_{\mathbf{W}}$'s are expressible in terms of the corresponding subset of $g_{\mathbf{W}}$. This happens for many known renormalization schemes of renormalizable theories.

In practice, one has to study the adiabatic limit for some such subsets: $g_{\mathbf{W}}(x) \to \gamma_{\mathbf{W}}$, $\mathbf{W} \in \mathcal{S}$, where the $\gamma_{\mathbf{W}}$'s are coupling constants, the other $g_{\mathbf{W}}$'s being used to define local observables and Green functions thereof.

3. In many cases of interest, namely, in particular in the renormalizable case corresponding to a classical total Lagrangian density, one can reorganize the multiple series in $\gamma_{\mathbf{W}}$ into a single series in $\hbar$ involving no infinite resummation, at each order, for $\langle S_T(\gamma_{\mathcal{S}}/\hbar, g/\hbar)\rangle_0$ from which S_T can be recovered by use of the asymptotic LSZ formula guaranteed to hold true thanks to causal factorization, provided mass renormalization has been performed correctly.

6. Conclusion

Although the main interest of workers in field theory has by now shifted far ahead of formal perturbation theory, and although the main theorem referred to here has been well known for quite some time now, we have found it amusing to stress once more the role of locality in the structure of the formal perturbative expansions, still often used, before one knows better.

References

[1] It is difficult to give a complete bibliography. Some initials and names are by now familiar: BPH stands for Bogoliubov–Parasiuk–Hepp. BPHZ stands for Bogoliubov–Parasiuk–Hepp–Zimmermann. It is probably in this momentum-space scheme that most of the useful combinatoric properties of formal perturbation theory were first worked out by J.H. Lowenstein and Y.M.P. Lam, under the heading "quantum action principle". This was permitted by the precise normal ordered product algorithm presented by W. Zimmermann to describe composite operators. Let us also mention the analytic renormalization scheme of E.R. Speer, and the dimensional renormalization scheme of G. 't Hooft and M. Veltman, one of the most popular, rigorized by E.R. Speer and others. To say the least, A.S. Wightman was not foreign to a number of these advances. For obvious reasons, we reserve a special Ref. number to Ref. [3].

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[7] Through H. Epstein if our memories are correct.