Global Attractivity of Periodic Solution in a Model of Hematopoiesis

PEI-XUAN WENG*
Department of Mathematics
South China Normal University
Guangzhou 510631, P.R. China

(Received October 2000; revised and accepted February 2002)

Abstract—Sufficient conditions are obtained for the global attractivity of a positive periodic solution for the periodic equation

\[
\frac{dP(t)}{dt} = -\delta(t)P(t) - \frac{\beta(t)P(t)}{1 + P^n(t)} + \alpha(t) \int_0^\infty K(s) \frac{P(t-s)}{1 + P^n(t-s)} \, ds,
\]

for \( n > 0 \), where \( \delta(t), \beta(t), \alpha(t) \) are continuous positive periodic function on \([0, \infty)\) with a period \( \omega > 0 \) and \( K(s) \) is a delay kernel. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—Hematopoiesis, Periodic solution, Global attractivity.

1. INTRODUCTION

Mackey et al., [1-5] studied the dynamics of blood cells production and proposed some autonomous delay differential equations to describe the models. One of the equation is

\[
\frac{dN(t)}{dt} = -\delta N(t) - \frac{\beta \theta^n N(t)}{\theta^n + N^n(t)} + \frac{2\beta \theta^n N(t - \tau)}{\theta^n + N^n(t - \tau)} e^{-r\tau},
\]

where \( \delta, \beta, \theta, \tau, \alpha, n \) are positive constants. By a transition of variable \( N(t) = \theta x(t) \), (1.1) can be simplified to

\[
\frac{dx(t)}{dt} = -\delta x(t) - \frac{\beta x(t)}{1 + x^n(t)} + \frac{2\beta e^{-r\tau} x(t - \tau)}{1 + x^n(t - \tau)}.
\]  

If we assume that

\[ 2\beta e^{-r\tau} > \delta + \beta, \]

then (1.2) has a positive equilibrium \( x = M \) and it is proved by the author that this positive equilibrium could be global attractive under some assumptions (please see [6] for the case of \( n \in (0, 1) \)).
If one considers the periodic factors and the continuously distributed delay, one could obtain the following integro-differential equation:

\[
\frac{dP(t)}{dt} = -\delta(t)P(t) - \beta(t)\frac{P(t)}{1 + P_n(t)} + \alpha(t) \int_0^\infty K(s) \frac{P(t-s)}{1 + P_n(t-s)} \, ds, \tag{1.3}
\]

where \(\delta(t), \beta(t), \alpha(t)\) are continuous positive periodic function on \([0, \infty)\) with a period \(\omega > 0; K(s) : [0, +\infty) \to [0, \infty)\) is assumed to be piecewise continuous, nonincreasing eventually and normalized such that

\[
\int_0^\infty K(s) \, ds = 1. \tag{1.4}
\]

In this paper, we shall investigate the existence of periodic solution \(P^*(t)\) and its stability characteristics for equation (1.3).

For a positive periodic function \(g(t)\), let

\[
g_0 = \min_{0 \leq t \leq \omega} \{ g(t) \} \quad \text{and} \quad g^0 = \max_{0 \leq t \leq \omega} \{ g(t) \}, \tag{1.5}
\]

then we have \(0 < g_0 \leq g^0\).

Let \(f(x) = x/(1 + x^n), x \geq 0, n > 0,\) then \(f(x)\) has the following properties:

\(\text{(P}_1\text{)}\) if \(n > 1,\) then

\[\max_{x \geq 0} f(x) = f(x^*) = \frac{1}{n} (n-1)^{1-1/n}, \quad x^* = \left( \frac{1}{n-1} \right)^{1/n},\]

and \(f(x)\) is increasing for \(x \in (0, x^*)\), and decreasing for \(x \in (x^*, +\infty)\);

\(\text{(P}_2\text{)}\) if \(n = 1,\) then \(f(x)\) is increasing for \(x \in (0, +\infty),\) and \(\lim_{x \to +\infty} f(x) = 1;\)

\(\text{(P}_3\text{)}\) if \(0 < n < 1,\) then \(f(x)\) is increasing for \(x \in (0, +\infty),\) and \(\lim_{x \to +\infty} f(x) = 1;\)

\(\text{(P}_4\text{)}\) if \(n > 1,\) then for any \(x^0 > 0,\) there exist a \(x_0 < x^0\) such that

\[f(x_0) < f(x^0) \quad \text{and} \quad f(x) > f(x_0), \quad \text{for} \ x \in (x_0, x^0).\]

Together with (1.3), we usually consider the following initial condition:

\[P(s) = \varphi(s) \in C((-\infty, 0], R_+), \quad \varphi(0) > 0. \tag{1.6}\]

We shall show that the solution of (1.3) and (1.6) remains positive and exists for all \(t > 0.\) For instance, if \(t^* > 0\) is the first time while \(P(t^*)\) becomes zero, then

\[P(t^*) = 0, \quad \frac{dP(t^*)}{dt} \leq 0. \tag{1.7}\]

But we have from (1.3) that

\[\frac{dP}{dt} \bigg|_{t=t^*} = \alpha(t^*) \int_0^\infty K(s) \frac{P(t^*-s)}{1 + P_n(t^*-s)} \, ds > 0,\]

which contradicts to (1.7). Thus, \(P(s)\) remains positive on its existing interval \([0, T)\). We claim that \(T = +\infty.\) Otherwise, there exists a sequence \(\{t_k\}, t_k \uparrow T -\) as \(k \to \infty,\) for which

\[P(t) \leq P(t_k), \quad \text{for} \ t \leq t_k, \quad \text{and} \quad \lim_{k \to \infty} P(t_k) = +\infty. \tag{1.8}\]

If \(n \geq 1,\) we obtain from (1.3) that

\[
\frac{dP(t)}{dt} \leq -\delta_0 P(t) + \alpha^0 \int_0^\infty K(s) \frac{P(t-s)}{1 + P_n(t-s)} \, ds \\
\leq -\delta_0 P(t) + \alpha^0 F, \quad \text{for} \ t \in [0, T),
\]
where
\[ F = \begin{cases} f(x^n), & n > 1, \\ 1, & n = 1, \end{cases} \]
and this leads to
\[ \frac{d(e^{\delta t}P(t))}{dt} \leq \alpha^0 F e^{\delta t}, \quad \text{for } t \in [0, T). \]

By integrating the above inequality from 0 to \( t \), we deduce that
\[ P(t) \leq P(0)e^{-\delta t} + \frac{\alpha^0 F}{\delta_0} (1 - e^{-\delta t}) < +\infty, \quad \text{(1.9)} \]
which is contradictory to (1.8).

If \( n \in (0, 1) \), we have from (P_3) that
\[ f(P(t_k)) \geq f(P(t - r)). \quad t \in [0, t_k]. \quad \text{(1.10)} \]

We obtain from (1.3) that
\[ \frac{dP(t)}{dt} \leq -\delta_0 P(t) + \alpha^0 \int_0^\infty K(s)f(P(t - s)) \, ds, \quad \text{(1.11)} \]
which implies that
\[ \frac{d(e^{\delta t}P(t))}{dt} \leq \alpha^0 e^{\delta t} \int_0^\infty K(s)f(P(t - s)) \, ds, \quad t \in [0, T). \]

Integrating the above inequality from 0 to \( t_k \), one obtains from (1.10) that
\[
\begin{align*}
e^{\delta t_k} P(t_k) & \leq P(0) + \alpha^0 \int_0^{t_k} e^{\delta u} \left( \int_0^{\infty} K(s)f(P(u - s)) \, ds \right) \, du \\
& \leq P(0) + \alpha^0 f(P(t_k)) \int_0^{t_k} e^{\delta u} \, du \\
& = P(0) + \frac{\alpha^0}{\delta_0} f(P(t_k)) (e^{\delta t_k} - 1),
\end{align*}
\]
which leads to
\[ P(t_k) \leq P(0)e^{-\delta t_k} + \frac{\alpha^0}{\delta_0} f(P(t_k)) (1 - e^{-\delta t_k}). \]

Thus, we obtain
\[ 1 \leq \frac{P(0)e^{-\delta t_k}}{P(t_k)} + \frac{\alpha^0}{\delta_0} \frac{1 - e^{-\delta t_k}}{1 + P(t_k)} \to 0, \quad (k \to \infty), \]
which is a contradiction. Therefore, \( T = +\infty \). In this paper, we shall only consider the positive solutions of (1.3).

2. ESTIMATES FOR SOLUTIONS OF (1.3)

In this section, we mainly give the estimates of bounds for solutions of (1.3). It is noted that it is possible to choose the delay kernel \( K(s) \) to be one of the functions \( K_m \) where
\[ K_m(t) = \frac{\lambda^m t^m e^{-\lambda t}}{(m+1)!}, \quad m = 1, 2, \ldots, t > 0, \]
or an appropriate linear combination of \( K_m \). Thus,

(H) one may choose a positive integer \( k \) large enough such that
\[ \mu \triangleq \int_0^\infty e^{\delta_0 s/k} K(s) \, ds < +\infty. \quad \text{(2.1)} \]
Let
\[ p^0 = \begin{cases} \frac{2\alpha^0 f(x^*)}{\delta_0}, & n > 1, \\ \frac{2\alpha^0}{\delta_0}, & n = 1, \\ \left(\frac{2k\alpha^0 \mu}{\delta_0}\right)^{1/n}, & 0 < n < 1. \end{cases} \]

Then we have the following lemma.

**Lemma 2.1.** The following conclusions hold:

(i) if \( n \geq 1 \), then for \( \forall \alpha > 0 \) and initial function \( \varphi : \max_{-\infty < s \leq 0} |\varphi(s)| \leq \alpha \), there exists a \( T_0 = T_0(\alpha) > 0 \) such that

\[ P(t) \leq P^0, \quad \text{for } t \geq T_0 \tag{2.2} \]

for \( P(t) \) which satisfies (1.3) and (1.6);

(ii) if \( 0 < n < 1 \), then under Assumption (H), the conclusion of (i) is true.

Suppose that \( P(t) \) is any positive solution of (1.3) satisfying (1.6), then if \( n \geq 1 \), we have from (1.9) that the conclusion of (i) is true. For the aim to show (ii), we introduce a lemma which is a simplified form of Lemma 1 in [7] without proof.

**Lemma 2.2.** Assume that

\[ x(t) \leq D + \int_0^t f(t, s)x(s)^{\gamma} \, ds, \quad t \in [0, b), \]

where \( D > 0, \gamma \in (0, 1) \) are constants; \( f(t, s) \) is nondecreasing for \( t \) if \( s \) is fixed, and \( x(t) \geq 0 \) for \( t \in [0, b) \). Then one has

\[ x(t) \leq D \left[ 1 + (1 - \gamma)D^{\gamma-1} \int_0^t f(t, s) \, ds \right]^{1/(1-\gamma)}. \]

**Proof of Lemma 2.1 (ii).** Define \( \gamma = 1 - n \in (0, 1) \) and choose a positive integer \( k \) as in (H). We have from (1.11) that

\[ \frac{dP(t)}{dt} \leq -\delta_0 P(t) + \alpha^0 \int_{-\infty}^\infty K(s) P^\gamma(t-s) \, ds \]

\[ = -\delta_0 P(t) + \alpha^0 \int_{-\infty}^0 K(t-u)P^\gamma(u) \, du + \alpha^0 \int_0^t K(t-u)P^\gamma(u) \, du \]

\[ \leq -\delta_0 \frac{P(t)}{k} + \alpha^0 \int_{-\infty}^0 K(t-u)P^\gamma(u) \, du + \alpha^0 \int_0^t K(t-u)P^\gamma(u) \, du. \]

Note that \( \max_{-\infty < s \leq 0} P(s) \leq \alpha \). We have

\[ \frac{d}{dt} \left[ P(t)e^{(\delta_0/k)t} \right] \leq \alpha^0 d^r e^{(\delta_0/k)t} \int_{t}^\infty K(u) \, du + \alpha^0 e^{(\delta_0/k)t} \int_0^t K(t-u)P^\gamma(u) \, du. \]
Integrating the above inequality from 0 to $t$ and changing the integrating order, we obtain

$$P(t) e^{(\delta_0/k)t} \leq P(0) + \alpha^0 \int_0^t \int_0^t K(u) e^{(\delta_0/k)u} du ds + \alpha^0 \int_0^t K(\gamma(u)) e^{(\delta_0/k)\gamma(u)} du ds$$

$$= P(0) + \alpha^0 \int_0^t K(u) du \int_0^t e^{(\delta_0/k)s} ds + \int_t^\infty K(u) du \int_0^t e^{(\delta_0/k)s} ds$$

$$+ \alpha^0 \int_0^t P^\gamma(u) du \int_0^t K(s) e^{(\delta_0/k)(s+u)} ds du$$

$$= P(0) + \frac{k \gamma u^2}{\delta_0} \left[ \int_0^t K(u) \left( e^{(\delta_0/k)u} - 1 \right) du + \int_t^\infty K(u) \left( e^{(\delta_0/k)u} - 1 \right) du \right]$$

$$+ \alpha^0 \int_0^t P^\gamma(u) du \int_0^t K(s) e^{(\delta_0/k)(s+u)} ds du$$

$$\leq d + \frac{k \mu \alpha^0 d^\gamma}{\delta_0} + \alpha^0 \mu \int_0^t e^{(\delta_0/k)w} P^\gamma(u) du.$$

Let $D = d + k \mu \alpha^0 d^\gamma/\delta_0$, then we deduce that

$$P(t) e^{(\delta_0/k)t} \leq D + \alpha^0 \mu \int_0^t e^{(n\delta_0/k)u} \left[ P(u) e^{(\delta_0/k)u} \right]^n du.$$

By the use of Lemma 2.2, we have

$$P(t) e^{(\delta_0/k)t} \leq D \left[ 1 + (1 - \gamma) D^{\gamma-1} \int_0^t \alpha^0 \mu e^{(n\delta_0/k)u} du \right]^{1/(1-\gamma)}$$

$$= D \left[ 1 + D^{-n} \alpha^0 \mu \frac{k}{\delta_0} \left( e^{(n\delta_0/k)t} - 1 \right) \right]^{1/n}.$$

Thus, we get

$$P(t) \leq \left[ D^n e^{-(n\delta_0/k)t} + \frac{k \alpha^0 \mu}{\delta_0} \right]^{1/n},$$

and the conclusion of (ii) in Lemma 2.1 is true.

**Remark 2.3.** If in the above proving procedure, instead of integrating the inequality from 0 to $t$, we integrating the inequality from $t_0$ to $t$, then (2.2) could be changed into

$$P(t) \leq P_0, \quad \text{for } t \geq T_0 + t_0.$$

Define a function

$$g(x) = - \left( \delta^0 + \beta^0 \right) + \frac{\alpha_0}{1 + x^n}, \quad x \geq 0, \quad n > 0.$$

Under the assumption: $\alpha_0 > \delta^0 + \beta^0$, we obtain

$$g(A) = 0, \quad A = \left( \frac{\alpha_0}{\delta^0 + \beta^0} - 1 \right)^{1/n}, \quad g(x) > 0 \text{ for } x \in (0, A). \quad (2.3)$$

In view of (P2)-(P4) of $f(x)$, one can choose a constant $b \in (0, 1)$ such that

$$P_0 = bA < \begin{cases} \min \{ P_0, x^* \}, & \text{as } n > 1, \\ P_0, & \text{as } 0 < n \leq 1, \end{cases}$$

and

$$f(P_0) < f(P_0^0), \quad f(x) > f(P_0), \quad \text{for } x \in (P_0, P_0^0). \quad (2.4)$$
LEMMA 2.4. Assume that $P(t)$ be any positive solution of (1.3) and $\alpha_0 > \delta^0 + \beta^0$, then there exists a $T_1 > 0$ such that

$$P(t) \geq P_0, \quad \text{for } t \geq T_1. \quad (2.5)$$

PROOF. Let $P(t)$ be any positive solution of (1.3). If (2.5) is not true, then there are two possibilities:

(I) there is a $T_2 > 0$ such that $P(t) < P_0$ for $t \geq T_2$;

(II) $P(t)$ is oscillatory about $P_0$.

Note that $P_0 \in (0, A)$, so we have from (2.3) that

$$g(P_0) = -(\delta^0 + \beta^0) + \frac{\alpha_0}{1 + P_0} > 0. \quad (2.6)$$

Suppose that (I) holds. By using (1.4) and (2.6), one can select a $\sigma > 0$ such that

$$0 < \eta \triangleq \int_0^{\sigma} K(s) \, ds \leq 1, \quad -(\delta^0 + \beta^0) + \frac{\alpha_0 \eta}{1 + P_0} > 0. \quad (2.7)$$

We derive from (1.3) and the positivity of $P(t)$ that

$$\frac{dP(t)}{dt} \geq -(\delta^0 + \beta^0) P(t) + \alpha_0 \int_0^{\sigma} K(s) f(P(t - s)) \, ds. \quad (2.8)$$

Under (I), there are three subpossibilities:

(1') $P(t)$ is decreasing for $t \geq T_3 \geq T_2$;

(2') there is a sequence $\{t_k\}$ such that $t_k > 0$, $\lim_{k \to \infty} t_k = +\infty$ and $P(t_k)(k = 1, 2, \ldots)$ are local minimums;

(3') $P(t)$ is increasing for $t \geq T_4 \geq T_2$.

First, we claim that (1') is impossible. In fact, due to (2.8) and (2.7), we obtain

$$\frac{dP(t)}{dt} \geq -(\delta^0 + \beta^0) P(t) + \alpha_0 \int_0^{\sigma} K(s) f(P(t - s)) \, ds > 0, \quad t \geq T_3 + \sigma,$$

which is a contradiction to (1').

Second, we investigate case (2'). Let $B = \lim_{t \to \infty} \inf P(t)$. We can show that $B > 0$. Otherwise $B = 0$. According to the definition of lower limit and the positivity of $P(t)$, one can find a subsequence $\{t_{k_n}\} \subset \{t_k\}$ such that

$$P(t_{k_n}) = \min \{P(t) \mid 0 \leq t \leq t_{k_n}\}, \quad \lim_{n \to \infty} P(t_{k_n}) = 0. \quad (2.9)$$

Choose some $t_{k_{nm}} \in \{t_{k_n}\}$ so that $t_{k_{nm}} > T_2 + \sigma$, then we have from (1), (2.8), (2.9), and (2.7) that

$$0 = \frac{d}{dt} P(t_{k_{nm}}) \geq -(\delta^0 + \beta^0) P(t_{k_{nm}}) + \alpha_0 \int_0^{\sigma} K(s) f(P(t_{k_{nm}} - s)) \, ds$$

$$\geq -(\delta^0 + \beta^0) P(t_{k_{nm}}) + \alpha_0 \int_0^{\sigma} K(s) \frac{P(t_{k_{nm}} - s)}{1 + P_0} \, ds$$

$$\geq P(t_{k_{nm}}) \left[ -(\delta^0 + \beta^0) + \frac{\alpha_0 \eta}{1 + P_0} \right] > 0, \quad (2.10)$$

which is impossible. Thus, we obtain

$$P_0 \geq B = \lim_{t \to \infty} \inf P(t) = \lim_{k \to \infty} \inf P(t_k) > 0. \quad (2.11)$$
Note
\[-(\delta^0 + \beta^0) B + \alpha_0 \eta f(B) > B \left[-(\delta^0 + \beta^0) + \frac{\alpha_0 \eta}{1 + P_0^\nu}\right] > 0.\]  
(2.12)

This together with the continuity of the function 
\[-(\delta^0 + \beta^0)x + \alpha_0 \eta f(x)\]
leads to a conclusion that we can find a small enough \(\varepsilon_0 > 0\) such that \(B - \varepsilon_0 > 0\) and
\[-(\delta^0 + \beta^0)(B + \varepsilon_0) + \alpha_0 \eta f(B - \varepsilon_0) > 0.\]  
(2.13)

For such a \(\varepsilon_0\), we know from the definition of lower limit and (2.11) that there exists a \(T_5 > T_2\) and a subsequence \(\{t_{k_j}\} \subset \{t_k\}\) such that
\[
P_0 \geq P(t) > B - \varepsilon_0, \quad \text{for } t \geq T_5, \quad P(t_{k_j}) < B + \varepsilon_0, \quad j = 1, 2, \ldots.\]  
(2.14)

Note that \(f(x)\) is increasing in \((0, P_0)\). We derive from (2.8), (2.14), and (2.13) that
\[
0 = \frac{d}{dt} P(t_{k_j}) \geq -(\delta^0 + \beta^0) P(t_{k_j}) + \alpha_0 \int_0^\sigma K(s) f(P(t_{k_j} - s)) \, ds \\
\geq -(\delta^0 + \beta^0)(B + \varepsilon_0) + \alpha_0 \eta f(B - \varepsilon_0) > 0
\]  
(2.15)

if \(t_{k_j} > T_5 + \sigma\). Equation (2.15) says that (2.0) is false.

Third, for case (3.0), we know that \(\lim_{t \to \infty} P(t)\) exists and
\[
0 < C = \lim_{t \to \infty} P(t) \leq P_0.
\]  
(2.16)

By a similar deriving procedure as in (2.13), one can select a small \(\varepsilon_1 > 0\) satisfying \(C - \varepsilon_1 > 0\) and
\[-(\delta^0 + \beta^0) C + \frac{\alpha_0 \eta (C - \varepsilon_1)}{1 + P_0^\nu} > 0.\]  
(2.17)

Due to the monotonicity of \(P(t)\), there exists a \(T_6 \geq T_4\) such that
\[
C - \varepsilon_1 < P(t) \leq C \leq P_0, \quad \text{for } t \geq T_6.
\]  
(2.18)

Thus, we get from (2.8) and (2.17) that
\[
\frac{dP(t)}{dt} \geq -(\delta^0 + \beta^0) P(t) + \alpha_0 \int_0^\sigma K(s) \frac{P(t - s)}{1 + P_0^n} \, ds \\
\geq -(\delta^0 + \beta^0) C + \frac{\alpha_0 \eta (C - \varepsilon_1)}{1 + P_0^n} > 0, \quad \text{for } t \geq T_6 + \sigma,
\]
which leads to \(\lim_{t \to \infty} P(t) = \infty\). This is impossible. In summary, we conclude from the above discussion that (1) is not true.

Now, we study Case (II). Suppose that \(\{t_k\}\) is a sequence such that
\[
\lim_{k \to \infty} t_k = +\infty, \quad P(t_k) \leq P_0 \quad (k = 1, 2, \ldots),
\]
and \(\{P(t_k)\}\) are the local minimums of \(P(t)\). Furthermore, we have
\[
E = \lim \inf_{t \to \infty} P(t) = \lim \inf_{k \to \infty} P(t_k) \leq P_0.
\]

We can show that \(E > 0\). Otherwise, \(E = 0\), then by a similar deriving procedure as in (2.9), we can find some \(t_{k_m} \in \{t_k\}\) satisfying \(t_{k_m} > T_6 + \sigma\) (\(T_6\) is defined in Lemma 2.1) and
\[
P(t_{k_m}) = \min \{P(t) \mid t_{k_m} \leq t \leq t_{k_m}\} \leq P_0.
\]  
(2.19)
For any \( s \in [0, \sigma] \), if \( P_0 \leq P(t_{k_m} - s) \leq P_0^0 \), we have from (2.4) and (2.19) that
\[
f(P(t_{k_m} - s)) \geq f(P_0) \geq \frac{P(t_{k_m})}{1 + P_0^0},
\]
if \( P(t_{k_m} - s) < P_0 \), we have from (2.19) that
\[
f(P(t_{k_m} - s)) \geq \frac{P(t_{k_m} - s)}{1 + P_0^0} \geq \frac{P(t_{k_m})}{1 + P_0^0}.
\]
However, we always have
\[
f(P(t_{k_m} - s)) > \frac{P(t_{k_m})}{1 + P_0^0}, \quad \text{for} \quad s \in [0, \sigma]. \tag{2.20}
\]
Use (2.20) to obtain (2.10) and we conclude that \( E > 0 \).

Finally, by replacing \( B, T_5 \) by \( E, T_7 \ (T_7 \geq T_6) \) from (2.11)–(2.15), we obtain a contradiction which says that (II) is impossible. Thus, the proof is complete.

### 3. EXISTENCE AND GLOBAL ATTRACTIVITY OF PERIODIC SOLUTIONS OF (1.3)

The following Lemmas 3.1 and 3.2 change the existence problem of periodic solution of (1.3) into a same problem of another equation with finite delay. The proof could be found in [8], so we omit the details.

**Lemma 3.1.** Any \( \omega \)-periodic solution \( P(t) \) of (1.3) is also an \( \omega \)-periodic solution of the following equation:
\[
\frac{dP(t)}{dt} = -\delta(t)P(t) - \frac{\beta(t)P(t)}{1 + P^n(t)} + \alpha(t)\int_0^\omega H(s)\frac{P^n(t - s)}{1 + P^n(t - s)}\,ds, \quad t \geq 0, \tag{3.1}
\]
where \( H(s) = \sum_{j=0}^\infty K(s + j\omega), s \in [0, \omega] \) and vice versa.

The following lemma gives sufficient condition for the existence of positive periodic solution of (3.1).

**Lemma 3.2.** If \( n \in (0, 1] \), we assume that (H) holds. For \( n > 0 \), assume that \( \alpha_0 > \delta^0 + \beta^0 \), then there exists a positive \( \omega \)-periodic solution \( P^*(t) \) of (3.1) such that
\[
P_0 \leq P^*(t) \leq P_0^0, \quad \text{for} \quad t \geq 0. \tag{3.2}
\]

**Proof.** Note from (1.4) that \( \int_0^\omega H(s)\,ds = 1 \) suppose that \( P(t) \) is any positive solution of (1.3). We have from (3.1) that
\[
\frac{dP(t)}{dt} \geq -\left(\delta^0 + \beta^0\right)P(t) + \alpha_0\int_0^\omega H(s)f(P(t - s))\,ds, \quad t \geq 0. \tag{3.3}
\]
Replacing \( \eta, \sigma \) and \( K(s) \) with \( 1, \omega \) and \( H(s) \), respectively, in Lemma 2.4, we obtain that there exists a \( T_1 > 0 \) such that
\[
P(t) \geq P_0, \quad \text{for} \quad t \geq T_1. \tag{3.4}
\]
On the other hand, we could show by a similar way as for Lemma 2.1 that there exists \( T_0 > 0 \) such that
\[
P(t) \leq P_0^0, \quad \text{for} \quad t \geq T_0, \tag{3.5}
\]
where \( T_0 \) is dependent on \( d \ (d = \max_{-\omega \leq s \leq 0} P(s)) \). Anyway, we claim from (3.4) and (3.5) that solutions of (3.1) are uniform-bounded and uniform-ultimately bounded (see the definition in [9, p. 120]). In view of Theorem 37.1 in [10], we obtain the conclusion of Lemma 3.2.
THEOREM 3.3. If \( n \in (0,1] \), we assume that \((H)\) holds. For \( n > 0 \), assume that \( \alpha_0 > \delta^0 + \beta^0 \), then there exists a positive \( \omega \)-periodic solution \( P^*(t) \) of \((1.3)\) such that
\[
P_0 \leq P^*(t) \leq P^0, \quad \text{for } t \geq 0.
\]

Next, we investigate the global attractivity of \( P^*(t) \).

THEOREM 3.4. Assume that \( \delta_0 + \beta_0 > \alpha_0 \), and
\[
\int_0^\infty sK(s)\,ds < +\infty,
\]
then every positive solution \( P(t) \) of \((1.3)\) satisfying
\[
\lim_{t \to \infty} P(t) = 0.
\]

PROOF. Define a Lyapunov functional \( V(t) = V(P(t)) \) as follows:
\[
V(t) = P(t) + \int_0^\infty K(s) \left[ \int_{t-s}^t \alpha(u+s)f(P(u))\,du \right] \,ds.
\]

By using \((3.6)\), one can verify that the improper integral in \((3.8)\) converges uniformly, and hence, is differentiable in \( t \). Calculating the upper right derivative of \( V \) along the positive solutions of \((1.3)\), we obtain
\[
D^+V(t) = -\delta(t)P(t) - \beta(t)f(P(t)) + \int_0^\infty K(s)\alpha(t+s)f(P(t))\,ds
\]
\[
\leq - [\delta_0 + \beta_0 - \alpha_0] f(P(t)).
\]
Integrating \((3.9)\) from \( 0 \) to \( t \), we have
\[
V(t) - V(0) \leq \left[ \alpha_0 - (\delta_0 + \beta_0) \right] \int_0^t f(P(s))\,ds, \quad t \geq 0.
\]
which leads to \( f(P(t)) \in L_1[0,\infty) \). The boundedness of \( P(t) \) on \([0, +\infty)\) follows from
\[
0 \leq P(t) \leq V(t) \leq V(0), \quad \text{for } t \geq 0.
\]
This together with the boundedness of \( \alpha(t), \delta(t), \beta(t), f(P(t)) \), and \((1.3)\) implies the boundedness of \( \frac{dP}{dt} \) on \([0, +\infty)\). Since
\[
\frac{d}{dt} f(P(t)) = \frac{1 + (1-n)P^n}{(1 + P^n)^2} \frac{dP}{dt}
\]
is bounded for \( t \in [0, +\infty) \), \( f(P(t)) \) is uniformly continuous on \([0, +\infty)\). By Barbalat’s lemma \([11,12]\), we conclude that \( \lim_{t \to \infty} f(P(t)) = 0 \) which implies \((3.8)\).

THEOREM 3.5. Assume that \((3.6)\) holds and
\[
(\alpha^0 + \beta^0) H < \delta_0 \leq \delta^0 + \beta^0 < \alpha_0, \quad n > 1,
\]
where
\[
H = \max \left\{ \frac{(n-1)^2}{4^n}, \left| \frac{1 + (1-n)P^n_0}{(1 + P^n_0)^2} \right| \right\}.
\]
Then there exists an unique positive \( \omega \)-periodic solution \( P^*(t) \) of \((1.3)\) such that any positive solution \( P(t) \) of \((1.3)\) satisfies
\[
\lim_{t \to \infty} |P(t) - P^*(t)| = 0.
\]
PROOF. The existence of $P^*(t)$ is ensured by Theorem 3.3. The uniqueness is obvious if we have showed the global attractivity. Let $P(t)$ be any positive solution of (1.3). Make a transformation of variables

$$z(t) = P(t) - P^*(t), \quad t \in R.$$ 

Then $z(t)$ satisfies

$$\frac{dz(t)}{dt} = -\delta(t)z(t) - \beta(t) [f(P(t)) - f(P^*(t))] + \alpha(t) \int_0^\infty K(s) [f(P(t-s)) - f(P^*(t-s))] \, ds. \tag{3.11}$$

Define a Lyapunov functional

$$V(t) = |z(t)| + \int_0^\infty K(s) \left[ \int_{t-s}^t \alpha(u+s) |f(P(u)) - f(P^*(u))| \, du \right] \, ds.$$ 

Calculating the upper right derivative of $V$ along the solutions of (3.11), we get

$$D^+ V(t) \leq -\delta_0 |z(t)| + \left( \beta^0 + \alpha^0 \right) |f(P(t)) - f(P^*(t))|$$

$$\leq -\delta_0 |z(t)| + \left( \beta^0 + \alpha^0 \right) \frac{1 + (1-n)\xi^n(t)}{(1 + \xi^n(t))^2} |z(t)|, \tag{3.12}$$

where $\xi(t)$ lies between $P(t)$ and $P^*(t)$ for $t \geq 0$. Note that

$$|P(t)| \geq P_0, \quad \text{for } t \geq T_1. \tag{3.13}$$

Define a function

$$h(x) = \frac{1 + (1-n)x^n}{(1 + x^n)^2}, \quad x \geq 0, \quad n > 1.$$ 

Then we have

1. $h(x^*) = 0, \quad x^* = \left( \frac{1}{n-1} \right)^{1/n}, \quad h(x^{**}) = \min_{x \geq 0} h(x) = -\frac{(n-1)^2}{4n}, \quad x^{**} = \left( \frac{n + 1}{n-1} \right)^{1/n}, \quad h(0) = \max_{x \geq 0} h(x) = 1, \quad \lim_{x \to \infty} h(x) = 0.$$

2. $h(x)$ is decreasing for $x \in (0, x^{**})$ and increasing for $x \in (x^{**}, +\infty).$

Therefore, we have

$$|h(x)| \leq \max \left\{ \frac{(n-1)^2}{4n}, |h(P_0)| \right\} \triangleq H, \quad \text{for } x \geq P_0. \tag{3.14}$$

Now we derive from (3.12)-(3.14) that

$$D^+ V(t) \leq -\left[ \delta_0 - (\alpha^0 + \beta^0) H \right] |z(t)|, \quad \text{for } t \geq T_1.$$ 

By a similar argument as in the proof of Theorem 3.4, we obtain that

$$\lim_{t \to \infty} z(t) = 0, \quad \text{i.e., } \lim_{t \to \infty} [P(t) - P^*(t)] = 0.$$ 

We complete the proof.
Theorem 3.6. Assume that (H) and (3.6) hold. Furthermore, assume that

\[(\alpha^0 - \beta_0) H < \delta_0 \leq \delta^0 + \beta^0 < \alpha_0, \quad n \in (0, 1),\]

where

\[H = \frac{1 + (1 - n)P_0^\alpha}{(1 + P_0^\alpha)^2}.\]

Then there exists an unique positive \(\omega\)-periodic solution \(P^*(t)\) of (1.3) such that any positive solution \(P(t)\) of (1.3) satisfies

\[\lim_{t \to \infty} |P(t) - P^*(t)| = 0.\]

Proof. By using the same Lyapunov function as in Theorem 3.5, and noting that \(f(P)\) is increasing, we obtain

\[D^+ V(t) \leq -\delta(t) |z(t)| - \beta(t) \langle \text{sgn } z(t) \rangle [f(P(t)) - f(P^*(t))]
\]

\[+ \int_0^\infty K(s)\alpha(t + s) [f(P(t)) - f(P^*(t))] \, ds\]

\[\leq -\delta_0 |z(t)| + (\alpha^0 - \beta_0) |f(P(t)) - f(P^*(t))|
\]

\[= -\delta_0 |z(t)| + (\alpha^0 - \beta_0) h(\xi(t)) |z(t)|,
\]

where \(\xi(t)\) lies between \(P(t)\) and \(P^*(t)\) for \(t \geq 0\), and

\[h(x) = \frac{1 + (1 - n)x^n}{(1 + x^n)^2}, \quad x \geq 0, \quad n \in (0, 1].\]

Since \(h(x)\) has the following properties:

1. \(h(0) = 1, h(x) > 0\) for \(x \in (0, +\infty)\);
2. \(h'(x) < 0, h(x)\) is decreasing for \(x \in (0, +\infty), \lim_{x \to \infty} h(x) = 0;\)

we get that

\[0 \leq h(x) \leq H = h(P_0), \quad \text{for } x \geq P_0.\]

Thus, one derive from (3.13) and (3.16) that

\[D^+ V(t) \leq - [\delta_0 - (\alpha^0 - \beta_0) H] |z(t)|, \quad \text{for } t \geq T_1,
\]

which implies that \(\lim_{t \to \infty} z(t) = 0.\) We complete the proof.

Remark 3.7. Condition (3.10) for \(n > 1\) is not as good as (3.15) for \(n \in (0, 1],\) since \(f(x)\) is not increasing on \([0, +\infty)\) for \(n > 1\), we are not able to estimate the sign of the term

\[-\beta(t) \langle \text{sgn } z(t) \rangle [f(P(t)) - f(P^*(t))].\]

But (3.10) could still be true if \(\alpha(t), \delta(t), \beta(t), n\) are selected suitably. For example, if

\[\delta(t) = |\sin t| + 8, \quad \beta(t) = \frac{1}{2} |\cos t| + \frac{1}{2}, \quad \alpha(t) = 20, \quad n = \frac{3}{2},\]

then we have

\[\delta_0 = 8, \quad \delta^0 = 9, \quad \beta_0 = \frac{1}{2}, \quad \beta^0 = 1, \quad \alpha_0 = \alpha^0 = 20,
\]

\[x^* = \left(\frac{1}{\sqrt[n]{2} - 1}\right)^{1/n} = 4^{1/3}, \quad f(x^*) = \frac{x^*}{1 + (x^*)^n} = \frac{4^{1/3}}{3} \approx 1.5874.\]
\[ A = \left( \frac{\alpha_0}{\delta^0 + \beta^0} - 1 \right)^{1/n} = \left( \frac{20}{9 + 1} - 1 \right)^{2/3} = 1, \]
\[ P_0 = \frac{2\alpha^0}{\delta_0} f(x^*) = \frac{2 \times 20 \times 4^{1/3}}{8 \times 3} = 4^{1/3} \times \frac{5}{3} \approx 2.6457. \]

Choose \( b = 0.8, P_0 = bA = 0.8, \) then
\[
\frac{(n - 1)^2}{4n} = \frac{1}{24} \quad \frac{P_0}{1 + (P_0)^n} = \frac{0.8}{1 + 0.8 \times \sqrt[3]{0.8}} \approx 0.8 \times \frac{1.715544}{1.715544} \approx 0.466325, \\
f(P_0) = \frac{5/3 \times 4^{1/3}}{1 + (5/3 \times 4^{1/3})^{3/2}} \approx 0.49887, \\
h(P_0) = \frac{1 + (1 - n)P_0^a}{(1 + P_0)^2} = \frac{1 - 1/2 \times 0.8^{3/2}}{1 + (0.8)^{3/2}} \approx 0.218207.
\]

Thus, we have
\[ f(P^0) > f(P_0), \quad H \approx 0.218207, \]
\[ (\alpha^0 + \beta^0) H = 21 \times 0.218207 < 8 = \delta_0 \leq 0 + 1 = \delta^0 + \beta^0 < \alpha_0 = 20. \]

REFERENCES