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A new method of solving nonlinear mathematical programming problems involving r-invex functions

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Abstract

A new approach to a solution of a nonlinear constrained mathematical programming problem involving r-invex functions with respect to the same function η is introduced. An η-approximated problem associated with an original nonlinear mathematical programming problem is presented that involves η-approximated functions constituting the original problem. The equivalence between optima points for the original mathematical programming problem and its η-approximated optimization problem is established under r-invexity assumption.

Keywords: Mathematical programming; η-approximated optimization problem; r-invex function; Optimality

1. Introduction

We consider the nonlinear constrained mathematical programming problem

$$(\mathrm{P}) \quad \begin{cases} f(x) \to \min \\ \text{subject to} \quad g_j(x) \leqslant 0, \quad j = 1, \ldots, m, \end{cases}$$

where $f : X \to R$ and $g : X \to R^m$ are differentiable functions on a nonempty open set $X \subset R^n$.

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We denote the feasible set in (P) by

$$D := \{x \in X\colon\ g_j(x) \leqslant 0,\ \ j = 1, \ldots, m\}$$

and consider a point $\bar{x} \in D$. The basic problem in optimization is to find conditions under which $\bar{x}$ locally or globally optimizes f on D. The idea is to use properties of the objective function, constraint functions and the feasible set. Thus, optimality conditions for $\bar{x}$ in constrained mathematical programming problems can be formulated in several different ways. Among the most used are those with convexity assumption imposed on all functions involved in a constrained mathematical programming problem.

Convex programming is the most thoroughly studied area of nonlinear optimization (see, for example, [5,8,11]). The assumption of convexity imposed on functions involved in constrained mathematical programming problems is important, because local and global optima coincide, so one talks only about an 'optimization solution.' Moreover, it is well known that, for constrained minimization program, saddle point of the Lagrangian is always a global minimum of the problem and they are also equivalent under convexity assumption and constraint qualification. Various optimality conditions can be derived for convex programming problems also from theorems of the alternative (for example, [8]).

However, in the recent years to relax convexity assumptions imposed in theorems on sufficient optimality conditions, various generalized convexity notions have been proposed. One of such generalizations of convexity notion is r-convexity introduced by Avriel [4] and Martos [10]. r-convex functions include the class of convex functions, whereas they themselves, as a class of functions, are contained in the class of quasi-convex functions. Also the class of invex functions introduced by Hanson [7] is one of such generalizations of convex functions. He considered differentiable functions $f : R^n \to R$ for which there exists vector-valued function $\eta : R^n \times R^n \to R^n$ such that, for all $x, u \in R^n$, the inequality

$$f(x) - f(u) \geqslant \nabla f(u)\eta(x, u) \tag{1}$$

holds. Hanson proved that if, in a mathematical programming problem with inequality constraints, instead of the convexity assumption, the objective function and each of the constraints function satisfy the inequality (1) with respect to the same function η, then the Karush–Kuhn–Tucker conditions (being necessary conditions for optimality) are also sufficient conditions for optimality. Hanson's work inspired others to further investigations concerning invexity. Craven [6] was the first to introduce the term "*invariant convex*."

Later, Antczak [3] introduced a new class of (nonconvex) differentiable functions and called them r-invex with respect to η. The class of r-invex functions with respect to η is an extension of the class of r-convex functions introduced by Avriel [4] and Martos [10] and invex functions with respect to η introduced by Hanson [7]. Antczak generalized Hanson's results for mathematical programming problems using r-invexity and he proved sufficient optimality conditions and Wolfe duality for constrained optimization problems involving r-invex functions with respect to the same function η. Further, Antczak [3] extended Martin's results [9] and he used the so-called alternative approach to prove sufficient optimality conditions and duality (Wolfe type). In this way, he gave the conditions of the r-invexity with respect to η of another type.

Considerable attention has been given recently to devising new methods which solve the original mathematical programming problem and its duals by the help of some associated optimization problem. But in almost of these approaches the notion of convexity plays a dominant role.

However, some new approaches of this type have been introduced lately, in which the convexity assumption has been relaxed to various invexity notions.

In [1], Antczak introduced a new approach with a modified objective function for solving a differentiable multiobjective optimization problems involving invex functions. He obtained optimality conditions for Pareto optimality by constructing for a considered multiobjective programming problem an equivalent vector minimization problem and then using an invexity concept in mathematical programming. Moreover, a definition of the so-called η-Lagrange function in such vector optimization problem was given, for which modified vector valued saddle points results were presented.

Recently, Antczak [2] proposed a new approach for solving a scalar nonlinear constrained mathematical programming problem involving invex and/or generalized invex functions. He showed one can obtain optimality conditions and duality results for a nonlinear constrained mathematical programming problem involving invex functions with respect to the same function η by constructing for it an equivalent minimization problem. Furthermore, Antczak applied the introduced approach to solve original dual problems in the sense of Mond–Weir.

Our aim in the present paper is to further develop the introduced earlier by Antczak approaches [2] for solving a nonlinear mathematical programming problem involving r-invex functions with respect to the same functions η. The key technique for solving a nonlinear mathematical programming problem which is used here is a construction for it an equivalent optimization problem. This associated η-approximated optimization problem is obtained by a modification both the objective function and the constraint functions in the given mathematical programming problem at an arbitrary but fixed feasible point $\bar{x}$. Then the notion of r-invexity is used to establish the equivalence between the original mathematical programming problem and its associated optimization problem. The key requirement we impose here is that the function η should satisfy some restrictions weaker than the so-called Condition (A) introduced in [1]. It turns out that the equivalent associated optimization problem obtained in this approach is, in general, less complicated and its optimal solutions are connected to the optimal points of the original minimization problem.

2. Preliminaries

In this section, we recall some definitions and preliminary results about r-invexity that will be used throughout the paper. The concept of an r-invex function was given by Antczak [3] as follows:

Definition 1. Let $f : X \to R$ be a differentiable function on a nonempty open set $X \subset R^n$. Then f is r-invex at $u \in X$ on X with respect to η if, for all $x \in X$,

$$\begin{aligned} &\frac{1}{r} e^{rf(x)} \geqslant \frac{1}{r} e^{rf(u)} \big(1 + r \nabla f(u) \eta(x, u)\big) && \text{if } r \neq 0, \\ &f(x) - f(u) \geqslant \nabla f(u) \eta(x, u) && \text{if } r = 0. \end{aligned} \tag{2}$$

If the inequality (2) holds for any $u \in X$ then f is r-invex on X with respect to η.

Lemma 2. *If f is an r-invex function with respect to η on X, and if k is any positive real number, then the function kf is $\frac{r}{k}$-invex with respect to η on X.*

More properties and characterizations of r-invexity were studied by Antczak in [3].

It is well known (see, for example, [5,8]) that the Karush–Kuhn–Tucker conditions are necessary conditions for optimality in a nonlinear mathematical programming problem (P).

Theorem 3. *Let $\bar{x}$ be an optimal solution in* (P) *and let some suitable constraint qualification* (CQ) [5] *be satisfied at $\bar{x}$. Then there exists $\bar{\xi} \in R^m_+$, $\bar{\xi} \geqslant 0$, such that*

$$\nabla f(\bar{x}) + \bar{\xi} \nabla g(\bar{x}) = 0, \tag{3}$$

$$\bar{\xi}_j g_j(\bar{x}) = 0, \quad j = 1, \ldots, m. \tag{4}$$

We denote by $J(\bar{x})$ the set

$$J(\bar{x}) := \{j = 1, \ldots, m \colon \bar{\xi}_j \neq 0\}.$$

3. An η-approximated optimization problem and optimality conditions

Let $\bar{x}$ be a feasible solution in (P). We consider the following η-approximated optimization problem $(\mathrm{P}^r_\eta(\bar{x}))$ given by

$$\left(\mathrm{P}^r_\eta(\bar{x})\right) \quad \begin{cases} \dfrac{1}{r} e^{rf(\bar{x})} + \nabla f(\bar{x}) \eta(x, \bar{x}) \to \min, \\ \text{subject to} \quad \dfrac{1}{r} e^{rg_j(\bar{x})} \big[1 + r \nabla g_j(\bar{x}) \eta(x, \bar{x})\big] \leqslant \dfrac{1}{r}, \quad j = 1, \ldots, m, \\ \quad \text{if } r \neq 0, \\ f(\bar{x}) + \nabla f(\bar{x}) \eta(x, \bar{x}) \to \min, \\ \text{subject to} \quad g_j(\bar{x}) + \nabla g_j(\bar{x}) \eta(x, \bar{x}) \leqslant 0, \quad j = 1, \ldots, m, \quad \text{if } r = 0, \end{cases}$$

where f, g, X are defined as in problem (P).

Let

$$D^r_\eta(\bar{x}) = \begin{cases} x \in X \colon \frac{1}{r} e^{rg_j(\bar{x})} [1 + r \nabla g_j(\bar{x}) \eta(x, \bar{x})] \leqslant \frac{1}{r}, \quad j = 1, \ldots, m, & \text{if } r \neq 0, \\ x \in X \colon g_j(\bar{x}) + \nabla g_j(\bar{x}) \eta(x, \bar{x}) \leqslant 0, \quad j = 1, \ldots, m, & \text{if } r = 0, \end{cases}$$

denote the set of all feasible solutions in $(\mathrm{P}^r_\eta(\bar{x}))$.

To prove some results in this paper we need some restrictions imposed on the function η. Antczak [1] introduced the following condition:

Condition (A). We denote by $\eta(\cdot, \bar{x})$ the function $x \to \eta(x, \bar{x})$. It will be said that η satisfies Condition (A) (at the point $\bar{x}$), when $\eta(\cdot, \bar{x})$ is a differentiable function at the point $x = \bar{x}$ with respect to the first component and satisfies the following conditions: $\eta(\bar{x}, \bar{x}) = 0$ and $\eta_x(\bar{x}, \bar{x}) = \alpha \cdot \mathbf{1}$, where $\eta_x(\bar{x}, \bar{x})$ denotes the derivative of $\eta(\cdot, \bar{x})$ at the point $x = \bar{x}$, and α is some positive real number.

Now, we show that the Karush–Kuhn–Tucker optimality conditions for the original mathematical programming problem (P) and its associated η-approximated problem $(\mathrm{P}^r_\eta(\bar{x}))$ have the same form if, the function η is assumed to satisfy Condition (A).

Indeed, the Karush–Kuhn–Tucker necessary optimality conditions for the η-approximated problem $(\mathrm{P}^r_\eta(\bar{x}))$ have the following form:

Theorem 4. *Let $\bar{x}$ be an optimal solution in $(\mathrm{P}^r_\eta(\bar{x}))$ and let some suitable constraint qualification* (CQ) [5] *be satisfied at $\bar{x}$. Then there exists $\bar{\xi} \in R^m_+$, $\bar{\xi} \geqslant 0$, such that*

$$\nabla f(\bar{x}) + \bar{\xi} \nabla g(\bar{x}) = 0, \tag{5}$$

$$\bar{\xi}_j g_j(\bar{x}) = 0, \quad j = 1, \ldots, m. \tag{6}$$

Proof. The Karush–Kuhn–Tucker optimality conditions [8] for the η-approximated problem $(\mathrm{P}^r_\eta(\bar{x}))$ have the following form:

$$\big(\nabla f(\bar{x}) + \bar{\xi} \nabla g(\bar{x})\big)\eta_x(\bar{x}, \bar{x}) = 0, \tag{7}$$

$$\bar{\xi}\big(g_j(\bar{x}) + \nabla g_j(\bar{x})\eta(\bar{x}, \bar{x})\big) = 0. \tag{8}$$

The function η is assumed to satisfy Condition (A). Therefore, $\eta(\bar{x}, \bar{x}) = 0$ and $\eta_x(\bar{x}, \bar{x}) = \alpha \cdot \mathbf{1}$. Thus, from (7) and (8) we obtain (5) and (6), respectively. This means that the necessary optimality conditions (5)–(6) in problem $(\mathrm{P}^r_\eta(\bar{x}))$ are the same form as the necessary optimality conditions (3)–(4) in problem (P).

Remark 5. Condition (A) was introduced by Antczak in [1]. However, the modified Condition (A) given above is weaker than Condition (A) in [1]. This follows from the fact that the second relation in it is weaker than in [1]. However, it turns out that the equivalence between the original mathematical programming problem (P) involving r-invex functions at $\bar{x}$ on D with respect to the same function η and its associated η-approximated optimization problem $(\mathrm{P}^r_\eta(\bar{x}))$ can be proved without using Condition (A). In order to prove this equivalence it should be assumed only the first relation from Condition (A), that is, the relation $\eta(\bar{x}, \bar{x}) = 0$. Note that this restriction imposed on the function η, which is weaker than Condition (A), extends the class of functions η with respect to which all functions involved in (P) are r-invex at $\bar{x}$ on D. It is useful, of course, from the practical point of view.

Now, we establish the equivalence between the optimization problems (P) and $(\mathrm{P}^r_\eta(\bar{x}))$, that is, we prove that if, $\bar{x}$ is optimal solution in the original mathematical programming problem (P), then it is also optimal in its associated η-approximated optimization problem $(\mathrm{P}^r_\eta(\bar{x}))$.

Theorem 6. *Let $\bar{x}$ be an optimal solution in problem* (P). *Moreover, we assume that some suitable constraint qualification* (CQ) [5] *is satisfied at $\bar{x}$. If η satisfies the condition $\eta(\bar{x}, \bar{x}) = 0$ then $\bar{x}$ is also optimal in problem $(\mathrm{P}^r_\eta(\bar{x}))$.*

Proof. By assumption, $\bar{x}$ is optimal (P) and some suitable constraint qualification (CQ) is satisfied at $\bar{x}$. Then there exists $\bar{\xi} \geqslant 0$ such that the Karush–Kuhn–Tucker conditions (3)–(4) are fulfilled.

We proceed by contradiction. Let $\bar{x}$ be not optimal in $(\mathrm{P}^r_\eta(\bar{x}))$. This implies that there exists $\tilde{x}$ feasible for $(\mathrm{P}^r_\eta(\bar{x}))$ such that

$$\frac{1}{r}e^{rf(\bar{x})} + \nabla f(\bar{x})\eta(\tilde{x},\bar{x}) < \frac{1}{r}e^{rf(\bar{x})} + \nabla f(\bar{x})\eta(\bar{x},\bar{x}).$$

Thus

$$\nabla f(\bar{x})\eta(\tilde{x},\bar{x}) < 0. \tag{9}$$

Using the feasibility of $\tilde{x}$ in $(\mathrm{P}^r_\eta(\bar{x}))$ we have

$$\frac{1}{r}e^{rg_j(\bar{x})}\bigl[1 + r\nabla g_j(\bar{x})\eta(\tilde{x},\bar{x})\bigr] \leqslant \frac{1}{r}, \quad j = 1,\ldots,m,$$

and so

$$\frac{1}{r}e^{\frac{r}{\bar{\xi}_j}\bar{\xi}_j g_j(\bar{x})}\left[1 + \frac{r}{\bar{\xi}_j}\bar{\xi}_j\nabla g(\bar{x})\eta(\tilde{x},\bar{x})\right] \leqslant \frac{1}{r}, \quad j \in J(\bar{x}).$$

Using the Karush–Kuhn–Tucker condition (4) together with $\bar{\xi}_j \geqslant 0$, $j = 1,\ldots,m$, we obtain

$$\bar{\xi}\nabla g(\bar{x})\eta(\tilde{x},\bar{x}) \leqslant 0. \tag{10}$$

By (9) and (10), we get the inequality

$$\bigl[\nabla f(\bar{x}) + \bar{\xi}\nabla g(\bar{x})\bigr]\eta(\tilde{x},\bar{x}) < 0,$$

which contradicts (3). Hence, $\bar{x}$ is optimal in $(\mathrm{P}^r_\eta(\bar{x}))$. □

Remark 7. Note that we have established Theorem 6 without any assumption to which class of functions all functions involved in problems (P) and $(\mathrm{P}^r_\eta(\bar{x}))$ belong. However, we assume that a suitable constraint qualification (CQ) is fulfilled at the optimal solution $\bar{x}$ in problem (P). It turns out that this assumption is essential to establish Theorem 6 and it will not be omitted. To illustrate this fact we give the following example.

Example 8. We consider the following mathematical programming problem:

$$f(x) = \log(x^2 + x + 1) \to \min,$$
$$g(x) = \begin{cases} \log(x^2+1) & \text{if } x \leqslant 0, \\ 0 & \text{if } x \geqslant 0. \end{cases}$$

Note that the set of all feasible solutions $D = \{x \in R\colon\ x \geqslant 0\}$, and, moreover, f and g are differentiable on R. Further, $\bar{x} = 0$ is optimal in the considered optimization problem (P). If, for example, we set that

$$\eta(x,\bar{x}) = x^2 + x - \bar{x}^2 - \bar{x},$$

then it is not difficult to prove by Definition 1 that both the objective function f and the constraint function g are 1-invex at $\bar{x}$ on D with respect to η. Using the η-approximation approach to solve this problem we have that the associated η-approximated optimization problem $(\mathrm{P}^1_\eta(\bar{x}))$ has the following form:

$$(\mathrm{P}^1_\eta(\bar{x}))\qquad \begin{cases} x^2+x+1\to \min,\\ \quad x\in R. \end{cases}$$

It is not difficult to see that the η-approximation approach introduced in this paper enlarges the set of all feasible solutions from D to $D^1_\eta(\bar{x})=R$. Thus, the approximated problem $(\mathrm{P}^1_\eta(\bar{x}))$ has the unbounded set of all feasible solutions and, therefore, $\bar{x}$ is not optimal in this problem (it is not difficult to see that the feasible solution $x=-\frac{1}{2}$ is optimal in $(\mathrm{P}^1_\eta(\bar{x}))$). This results follows from the fact that a suitable constraint qualification (for example, linear independence constraint qualification (LICQ) [5]) is not fulfilled at the optimal point $\bar{x}=0$ in the considered mathematical programming problem (P).

Now, if we assume that the objective function f and the constraint function g are r-invex at $\bar{x}$ on D with respect to the same function η satisfying the condition $\eta(\bar{x},\bar{x})=0$, then we establish that the optimal solution $\bar{x}$ in the η-approximated optimization problem $(\mathrm{P}^r_\eta(\bar{x}))$ is also optimal in the original mathematical programming problem (P).

Theorem 9. *Let $\bar{x}$ be an optimal solution in problem $(\mathrm{P}^r_\eta(\bar{x}))$ and let some suitable constraint qualification* [5] *be satisfied at $\bar{x}$. We assume that f and g are r-invex at $\bar{x}$ on D with respect to the same function η satisfying $\eta(\bar{x},\bar{x})=0$. Then $\bar{x}$ is also optimal in problem* (P)*.*

Proof. Since $\bar{x}$ is optimal in $(\mathrm{P}^r_\eta(\bar{x}))$ then the following inequality:

$$\frac{1}{r}e^{rf(\bar{x})}+\nabla f(\bar{x})\eta(x,\bar{x})\geqslant \frac{1}{r}e^{rf(\bar{x})}+\nabla f(\bar{x})\eta(\bar{x},\bar{x}) \tag{11}$$

holds for all $x\in D^r(\bar{x})$. Hence, the relation $\eta(\bar{x},\bar{x})=0$ implies that the inequality

$$\nabla f(\bar{x})\eta(x,\bar{x})\geqslant 0 \tag{12}$$

holds for all $x\in D^r(\bar{x})$.

We now show that $D\subset D^r(\bar{x})$. By assumption, g is r-invex at $\bar{x}$ on D with respect to η. Then, by Definition 1, the inequality

$$\frac{1}{r}e^{rg(x)}\geqslant \frac{1}{r}e^{rg(\bar{x})}\big[1+r\nabla g(\bar{x})\eta(x,\bar{x})\big]$$

is fulfilled for all $x\in D$. Since $x\in D$ then $g(x)\leqslant 0$. Thus,

$$\frac{1}{r}\geqslant \frac{1}{r}e^{rg(\bar{x})}\big[1+r\nabla g(\bar{x})\eta(x,\bar{x})\big].$$

From the inequality above follows that $x\in D^r(\bar{x})$. Hence, $D\subset D^r(\bar{x})$.

Suppose that $\bar{x}$ is not optimal in (P). Then there exists $\tilde{x}$ feasible in (P) such that

$$f(\tilde{x})<f(\bar{x}). \tag{13}$$

By $\tilde{x}\in D$ and from $D\subset D(\bar{x})$ follows that $\tilde{x}$ is also feasible in $(\mathrm{P}^r_\eta(\bar{x}))$. By assumption, f is r-invex at $\bar{x}$ on D. Hence, using Definition 1 together with (13), we get the inequality

$$\nabla f(\bar{x})\eta(\tilde{x},\bar{x}) < 0$$

which contradicts (12). This means that $\bar{x}$ is optimal in (P). □

We denote by

$$f_{\text{opt}} := f(\bar{x}),$$

that is, the optimal value in the original mathematical programming problem (P), and also by f^r_{opt} we denote the optimal value in its associated η-approximated optimization problem $(\mathrm{P}^r_\eta(\bar{x}))$.

In view of Theorems 6 and 9, if we assume that both the objective and the constraint functions involved in problem (P) are r-invex at $\bar{x}$ on the set of all feasible solutions D with respect to the same function η, and, moreover, some suitable constraint qualification at $\bar{x}$ and the relation $\eta(\bar{x},\bar{x}) = 0$ are satisfied, then problems (P) and $(\mathrm{P}^r_\eta(\bar{x}))$ are equivalent in the sense discussed above. Further, the optimal value f^r_{opt} in the η-approximated optimization problem $(\mathrm{P}^r_\eta(\bar{x}))$ is equal to

$$\begin{aligned} f^r_{\text{opt}} &= \frac{1}{r}e^{rf_{\text{opt}}} \quad \text{if } r \neq 0,\\ f^r_{\text{opt}} &= f_{\text{opt}} \quad \text{if } r = 0. \end{aligned} \tag{14}$$

Hence, the optimal value in the original mathematical programming problem (P) is equal to

$$\begin{aligned} f_{\text{opt}} &= \frac{1}{r}\log\bigl(rf^r_{\text{opt}}\bigr) \quad \text{if } r \neq 0,\\ f_{\text{opt}} &= f^r_{\text{opt}} \quad \text{if } r = 0. \end{aligned} \tag{15}$$

Now, we give an example of a mathematical programming problem (P), which, by using the approach discussed in this paper, is transformed to an equivalent linear optimization problem $(\mathrm{P}^r_\eta(\bar{x}))$.

Example 10. We consider the following nonlinear mathematical programming problem:

$$(\mathrm{P}) \quad \begin{cases} f(x_1,x_2) = \log\Bigl(e^{x_1} + x_1^2 - \frac{1}{2}\sin x_1 + e^{x_2} - \frac{1}{2}\arctan(\sin x_2) + x_2^2 - 1\Bigr) \\ \qquad \rightarrow \min, \\ g_1(x_1,x_2) = \log\bigl(x_1^2 e^{x_1^2} - x_2 + 1\bigr) \leqslant 0, \\ g_2(x_1,x_2) = \log\bigl(x_2^2 e^{x_2^2} - x_1 + 1\bigr) \leqslant 0. \end{cases}$$

Note that $D = \{(x_1,x_2) \in R \times R\colon\ 0 \leqslant x_2^2 e^{x_2^2} \leqslant x_1 \wedge 0 \leqslant x_1^2 e^{x_1^2} \leqslant x_2\}$, and $\bar{x} = (0,0)$ is optimal in the considered nonlinear mathematical programming problem (P). We set

$$\eta(x,\bar{x}) = \begin{bmatrix} \eta_1(x,\bar{x}) \\ \eta_2(x,\bar{x}) \end{bmatrix} = \begin{bmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \end{bmatrix}. \tag{16}$$

Then, it is not difficult to show by Definition 1 that both the objective function f and the constraints functions g_1 and g_2 are 1-invex at $\bar{x}$ on D with respect to the same function η

defined above. Moreover, a suitable condition of regularity of constraints, for example, linear independence constraint qualification (LICQ) [5] is satisfied at $\bar{x}$. Note that the function η defined above satisfies the relation $\eta(\bar{x}, \bar{x}) = 0$. Now, using the approach discussed in the paper, we construct problem $(\mathrm{P}^1_\eta(\bar{x}))$ by the η-approximation both the objective function f and the constraint function g at $\bar{x}$. Thus, we obtain the following linear optimization problem:

$$\big(\mathrm{P}^1_\eta(\bar{x})\big) \qquad \begin{cases} 1 + \dfrac{1}{2}x_1 + \dfrac{1}{2}x_2 \to \min, \\ -x_2 \leqslant 0, \quad -x_1 \leqslant 0. \end{cases}$$

It is not difficult to see, that $\bar{x} = (0, 0)$ is also optimal in the above optimization problem $(\mathrm{P}^1_\eta(\bar{x}))$, that is, in the η-approximated optimization problem which is constructed by a modification of the objective function and the constraint function in the original problem. Since both the objective function f and the constraint function g are 1-invex at $\bar{x}$ on D with respect to η, then the assumptions of Theorem 9 are fulfilled. Thus, by Theorems 6 and 9, $\bar{x}$ is optimal in both optimization problems. Then, the optimal value in the η-approximated optimization problem $(\mathrm{P}^1_\eta(\bar{x}))$ is equal to $f^1_{\mathrm{opt}} = 1$. Since all functions involved in the considered mathematical programming problem (P) are 1-invex at $\bar{x}$ on D then using (15) we are in position to calculate the optimal value f_{opt} in the original optimization problem (P). Thus, by (15), $f_{\mathrm{opt}} = \log(1) = 0$.

Remark 11. Note that there exists more than one a function η satisfying all conditions of Theorems 6 and 9. In other words, there exists more than one associated η-approximated optimization problem $(\mathrm{P}_\eta(\bar{x}))$, which is equivalent to the original mathematical programming problem (P). This property is useful from the practical point of view. Indeed, for example, for the considered mathematical programming problem (P) from Example 10, we set

$$\eta(x, \bar{x}) = \begin{bmatrix} \eta_1(x, \bar{x}) \\ \eta_2(x, \bar{x}) \end{bmatrix} = \begin{bmatrix} e^{x_1} - e^{\bar{x}_1} \\ e^{x_2} - e^{\bar{x}_2} \end{bmatrix}.$$

Note that both the objective function f and the constraints functions g_1 and g_2 are 1-invex at $\bar{x}$ on D with respect to η given above. Then we construct the following η-approximated optimization problem:

$$\big(\mathrm{P}^1_\eta(\bar{x})\big) \qquad \begin{cases} 1 + \dfrac{1}{2}e^{x_1} + \dfrac{1}{2}e^{x_2} \to \min, \\ 1 - e^{x_2} \leqslant 0, \quad 1 - e^{x_1} \leqslant 0. \end{cases}$$

Hence, in fact, we constructed for the original mathematical programming problem (P) more than one an associated η-approximated optimization problem $(\mathrm{P}^1_\eta(\bar{x}))$. Moreover, any constructed associated η-approximated optimization problem $(\mathrm{P}^1_\eta(\bar{x}))$ is equivalent in the sense discussed in the paper. This property is, of course, important from the practical point of view.

In the following example, we show that the assumption of r-invexity imposed on the objective function f and the constraint function g is consistent and it will not be omitted to prove that the optimal solution $\bar{x}$ in problem $(\mathrm{P}^r_\eta(\bar{x}))$ is also optimal in (P).

Example 12. We consider the following optimization problem (P):

$$(\mathrm{P}) \quad \begin{cases} \log\bigl(\arctan(x^2+1)+2\bigr) \to \min, \\ \log(x^4-x^2+1) \leqslant 0. \end{cases}$$

Note that $D = \{x \in R\colon\ -1 \leqslant x \leqslant 1\}$ and, moreover, $\bar{x} = 0$ is optimal in the considered nonlinear mathematical programming problem. It is not difficult to show that there is no a function η with respect to which both the objective function and the constraint function are r-invex at $\bar{x}$ on D. Indeed, the constraint function g is not r-invex at $\bar{x}$ on D with respect to any function η and any real number r. It follows from the fact that its stationary point $\bar{x} = 0$ is not a minimum point (see [3, Theorem 34]). Therefore, there does not exist an η-approximated optimization problem associated with the mathematical programming problem considered in this example.

Remark 13. The assumption that a function η satisfies the relation $\eta(\bar{x}, \bar{x}) = 0$ is essential to confirm the equivalence between problems (P) and $(\mathrm{P}^r_\eta(\bar{x}))$ in the sense discussed in the paper.

In the next example, we consider the case when the relation $\eta(\bar{x}, \bar{x}) = 0$ is not satisfied. We show that in this case there is no equivalence between (P) and $(\mathrm{P}^r_\eta(\bar{x}))$, although the all functions involved in the considered mathematical programming problem (P) are r-invex with respect to the same function η at $\bar{x}$ on the set all feasible solutions.

Example 14. We consider the following mathematical programming problem:

$$f(x_1, x_2) = \log\bigl(4x_1^2 - x_2^2 + 2\bigr) \to \min,$$

$$g(x_1, x_2) = \log \frac{(x_1-1)^2 + x_2^2 + 1}{3} \leqslant 0.$$

Note that the set of all feasible solutions $D = \{(x_1, x_2) \in R^2\colon\ (x_1-1)^2 + x_2^2 \leqslant 2\}$, and $\bar{x}^1 = (0, 1)$ and $\bar{x}^2 = (0, -1)$ are optimal in the considered optimization problem. Further, it can be proved by Definition 1 that f and g are 1-invex at $\bar{x}$ on D with respect to the same function η defined by

$$\eta(x, \bar{x}^1) = \begin{bmatrix} \eta_1(x, \bar{x}^1) \\ \eta_2(x, \bar{x}^1) \end{bmatrix} = \begin{bmatrix} x_1^2 + \frac{1}{2}x_2^2 + 2 \\ -\frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \end{bmatrix}.$$

Note that the relation $\eta(\bar{x}, \bar{x}) = 0$ is not satisfied. For the considered mathematical programming problem we construct its associated η-approximated optimization problem $(\mathrm{P}^1_\eta(\bar{x}^1))$ in the form

$$\bigl(\mathrm{P}^1_\eta(\bar{x}^1)\bigr) \quad \begin{cases} 1 + x_1^2 - x_2^2 \to \min, \\ \quad (x_1, x_2) \in R^2. \end{cases}$$

It is not difficult to see that $\bar{x}^1 = (0, 1)$ is not optimal in this optimization problem (the optimization problem above has an unbounded set of all feasible solution). Thus, the considered optimization problems are no equivalent in the sense discussed in the paper. This follows from the fact that the function η, with respect to which both the objective function f and the constraint function g are 1-invex at $\bar{x}^1$ on D, does not satisfy the relation $\eta(\bar{x}, \bar{x}) = 0$.

Further, if we set

$$\eta(x, \bar{x}^2) = \begin{bmatrix} \eta_1(x, \bar{x}^2) \\ \eta_2(x, \bar{x}^2) \end{bmatrix} = \begin{bmatrix} x_1^2 + \frac{1}{2}x_2^2 + 2 \\ -\frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 \end{bmatrix},$$

then it is not difficult to show that f and g are 1-invex with respect to η at $\bar{x}^2$ on D. However, the function η defined above does not satisfy the relation $\eta(\bar{x}, \bar{x}) = 0$. We construct the associated η-approximated optimization problem $(\mathrm{P}_\eta^1(\bar{x}^2))$:

$$\left(\mathrm{P}_\eta^1(\bar{x}^2)\right) \quad \begin{cases} 1 - x_1^2 - x_2^2 \to \min, \\ \quad (x_1, x_2) \in R^2. \end{cases}$$

It is not difficult to see that $\bar{x}^2 = (0, -1)$ is not optimal in this optimization problem (the optimization problem above has an unbounded set of all feasible solution). Thus, the considered optimization problems (P) and $(\mathrm{P}_\eta^1(\bar{x}^2))$ are no equivalent in the sense discussed in the paper.

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