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Spline collocation method for integro-differential equations with weakly singular kernels☆

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Abstract

In the first part of this paper we study the regularity properties of solutions to initial or boundary-value problems of Fredholm integro-differential equations with weakly singular or other nonsmooth kernels. We then use these results in the analysis of a piecewise polynomial collocation method for solving such problems numerically. Presented numerical examples display that theoretical results are in good accordance with actual convergence rates of proposed algorithms.

MSC: 65R20; 45J05

Keywords: Weakly singular integro-differential equation; Piecewise polynomial collocation method; Graded grid

1. Introduction

In this paper we study the convergence behavior of a piecewise polynomial collocation method for the numerical solution of initial or boundary-value problems of the form

$$u^{(n)}(t) = \sum_{i=0}^{n-1} a_i(t)u^{(i)}(t) + f(t) + \int_0^b \sum_{i=0}^{n-1} K_i(t,s)u^{(i)}(s)\,\mathrm{d}s, \tag{1.1}$$

$$\sum_{i=0}^{n-1} [\alpha_{ij}u^{(i)}(0) + \beta_{ij}u^{(i)}(b)] = 0, \qquad j = 1, \ldots, n, \tag{1.2}$$

where $0 \leqslant t \leqslant b$, $b \in \mathbb{R} = (-\infty, \infty)$, $b > 0$, $n \in \mathbb{N} = \{1, 2, \ldots\}$, $\alpha_{ij}, \beta_{ij} \in \mathbb{R}$, $i = 0, 1, \ldots, n-1$; $j = 1, \ldots, n$. We assume that $a_0, \ldots, a_{n-1}, f \in C^{m,\nu}[0, b]$, $K_0, \ldots, K_{n-1} \in W^{m,\nu}(\Delta)$, $m \in \mathbb{N}$, $\nu \in \mathbb{R}$, $-\infty < \nu < 1$.

The set $C^{m,\nu}[0, b]$, with $m \in \mathbb{N}$, $-\infty < \nu < 1$, is defined as the collection of all continuous functions $u : [0, b] \to \mathbb{R}$ which are m times continuously differentiable in $(0, b)$ and such that for all $t \in (0, b)$ and $i = 1, \ldots, m$ the following

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estimate holds:

$$|u^{(i)}(t)| \leqslant c \begin{cases} 1 & \text{if } i < 1-v, \\ 1 + |\log \varrho(t)| & \text{if } i = 1-v, \\ \varrho(t)^{1-v-i} & \text{if } i > 1-v. \end{cases} \tag{1.3}$$

Here $c = c(u)$ is a positive constant and

$$\varrho(t) = \min\{t, b-t\} \quad (0 < t < b)$$

is the distance from $t \in (0, b)$ to the boundary of the interval $[0, b]$. Equipped with the norm

$$\|u\|_{m,v} = \max_{0 \leqslant t \leqslant b} |u(t)| + \sum_{i=1}^{m} \sup_{0<t<b} (w_{i+v-1}(t)|u^{(i)}(t)|), \quad u \in C^{m,v}[0, b], \tag{1.4}$$

$C^{m,v}[0, b]$ is a Banach space. Here

$$w_\lambda(t) = \begin{cases} 1 & \text{for } \lambda < 0, \\ (1 + |\log \varrho(t)|)^{-1} & \text{for } \lambda = 0, \\ \varrho(t)^\lambda & \text{for } \lambda > 0, \end{cases}$$

with $t \in (0, b)$. Note that $C^m[0, b]$, the set of m times $(m \geqslant 1)$ continuously differentiable functions $u : [0, b] \to \mathbb{R}$, belongs to $C^{m,v}[0, b]$ for arbitrary $v < 1$. We define

$$C^{0,v}[0, b] = C[0, b], \quad -\infty < v < 1,$$

where $C[0, b]$ is the Banach space of continuous functions $u : [0, b] \to \mathbb{R}$ equipped with the usual norm $\|u\|_{C[0,b]} = \max_{0 \leqslant x \leqslant b} |u(x)|$.

The set $W^{m,v}(\Delta)$, with $m \in \mathbb{N}$, $-\infty < v < 1$,

$$\Delta = \{(t, s) : 0 \leqslant t \leqslant b, 0 \leqslant s \leqslant b, t \neq s\},$$

consists of all m times continuously differentiable functions $K : \Delta \to \mathbb{R}$ satisfying for all $(t, s) \in \Delta$ and all non-negative integers i and j such that $i + j \leqslant m$ the condition

$$\left| \left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j K(t, s) \right| \leqslant c \begin{cases} 1 & \text{if } v + i < 0, \\ 1 + |\log |t-s|| & \text{if } v + i = 0, \\ |t-s|^{-v-i} & \text{if } v + i > 0, \end{cases} \tag{1.5}$$

where $c = c(K)$ is a positive constant. For $i = j = 0$, condition (1.5) yields

$$|K(t, s)| \leqslant c \left\{ \begin{matrix} 1 & \text{if } v < 0 \\ 1 + |\log |t-s|| & \text{if } v = 0 \\ |t-s|^{-v} & \text{if } v > 0 \end{matrix} \right\}, \quad (t, s) \in \Delta.$$

Thus, a kernel $K \in W^{m,v}(\Delta)$ $(m \in \mathbb{N}, v < 1)$ is at most weakly singular for $0 \leqslant v < 1$; if $v < 0$, then $K(t, s)$ is bounded on Δ but its derivatives may have singularities as $s \to t$. Most important examples of weakly singular kernels are given by

$$K_0(t, s) = \kappa(t, s) \log |t-s|, \quad K_v(t, s) = \kappa(t, s)|t-s|^{-v}, \quad 0 < v < 1,$$

where $\kappa \in C^m([0, b] \times [0, b])$. Clearly, $K_0 \in W^{m,0}(\Delta)$ and $K_v \in W^{m,v}(\Delta)$, $m \in \mathbb{N}$, $0 < v < 1$.

A special case of $\{(1.1), (1.2)\}$ with $\beta_{ij} = 0$, $i = 0, \ldots, n-1$; $j = 1, \ldots, n$, and $K_i(t, s) = 0$ for $s > t$, $i = 0, \ldots, n-1$, is a initial value problem for a Volterra integro-differential equation. Collocation methods for the numerical solution of Volterra integro-differential equations are studied, for example, in [1–3,9,13–15].

The numerical solution of Fredholm integro-differential equations with help of collocation methods in case of smooth kernels is investigated in [4–6,8,10,12,17]. The solution of such equations by Galerkin method is studied in [18], by Tau method in [7], by Taylor method in [11] and by the modified Adomian decomposition method in [19].

There is very little literature on the numerical solution of Fredholm integro-differential equations with weakly singular kernels [17,18]. This is in remarkable contrast to the number of works on weakly singular Fredholm integral equations, see, for example, [16] and the literature given therein. We refer also to [4,5] where the numerical solution of Fredholm integro-differential equations with discontinuous kernels (but smooth solutions) is considered.

The main purpose of the present paper is to generalize the results obtained in [2,3,9] for first order Volterra integro-differential equations to a wide class of arbitrary order Fredholm or Volterra integro-differential equations with weakly singular or other nonsmooth kernels.

In the first part of this paper (Section 2) we study the regularity properties of the solution to $\{(1.1), (1.2)\}$ in case when the kernels $K_0(t,s), \ldots, K_{n-1}(t,s)$ may be weakly singular at $t=s$ and the derivatives of the functions $a_0, \ldots, a_{n-1}$ and f may be unbounded on the interval $[0,b]$. Then we use these results in the analysis of a piecewise polynomial collocation method for solving such equations numerically. Using special graded grids, we derive optimal global convergence estimates and analyze the attainable order of global and local convergence of numerical solutions for all values of the grading exponent of the underlying grid (Sections 4 and 5). In Section 3 we introduce some auxiliary results which we need in the convergence analysis of proposed algorithms. In Section 6 we present some numerical illustrations showing a good accordance with theoretical results. The main results of the paper are formulated in Theorems 2.1, 4.1 and 5.1.

Remark 1.1. For simplicity we confine ourselves to problem with homogenous conditions (1.2). But similar results can be derived also in case of a equation (1.1) with nonhomogenous conditions

$$\sum_{i=0}^{n-1} [\alpha_{ij} u^{(i)}(0) + \beta_{ij} u^{(i)}(b)] = \gamma_j, \quad j = 1, \ldots, n, \tag{1.6}$$

where $\alpha_{ij}, \beta_{ij}, \gamma_j \in \mathbb{R}$, $i = 0, \ldots, n-1$; $j = 1, \ldots, n$. This follows from the observation that if u is the solution of problem $\{(1.1), (1.6)\}$, then $\tilde{u} = u - p$ is the solution of $\{(1.1), (1.2)\}$ in which only the forcing function $f(t)$ is replaced by

$$\tilde{f}(t) = f(t) - p^{(n)}(t) + \sum_{i=0}^{n-1} a_i(t) p^{(i)}(t) + \int_0^b \sum_{i=0}^{n-1} K_i(t,s) p^{(i)}(s)\, ds, \quad 0 \leqslant t \leqslant b,$$

with a polynomial $p(t)$ satisfying (1.6).

2. Smoothness of the solution

Consider the following two homogenous equations, corresponding to the initial Eq. (1.1):

$$u^{(n)}(t) = \sum_{i=0}^{n-1} a_i(t) u^{(i)}(t) + \int_0^b \sum_{i=0}^{n-1} K_i(t,s) u^{(i)}(s)\, ds, \quad 0 \leqslant t \leqslant b, \tag{2.1}$$

$$u^{(n)}(t) = 0, \quad 0 \leqslant t \leqslant b. \tag{2.2}$$

The existence and regularity of the solution of problem $\{(1.1), (1.2)\}$ is described in the following theorem.

Theorem 2.1. *Let $n \in \mathbb{N}$, $\alpha_{ij}, \beta_{ij} \in \mathbb{R}$, $i = 0, \ldots, n-1$; $j = 1, \ldots, n$. Assume that f, $a_i \in C^{m,\nu}[0,b]$, $K_i \in W^{m,\nu}(\Delta)$, $i = 0, \ldots, n-1$, $m \in \mathbb{N}$, $\nu \in \mathbb{R}$, $-\infty < \nu < 1$. Moreover, assume that both the homogenous problem $\{(2.1), (1.2)\}$ and the homogenous problem $\{(2.2), (1.2)\}$ have only the trivial solution $u = 0$. Then problem $\{(1.1), (1.2)\}$ possesses a unique solution $u \in C^{m+n,\nu-n}[0,b]$ and for its derivatives $u', u'', \ldots, u^{(n)}$ we have that $u^{(i)} \in C^{m+n-i,\nu-n+i}[0,b]$, $i = 1, \ldots, n$.*

For the proof of Theorem 2.1 we need the following auxiliary results.

Lemma 2.1. *If $m \geqslant k \geqslant 0$ and $\mu \leqslant \nu < 1$, then $C^{m,\mu}[0,b] \subset C^{k,\nu}[0,b]$ and*

$$\|v\|_{k,\nu} \leqslant c\|v\|_{m,\mu}$$

for $v \in C^{m,\mu}[0,b]$ with a constant c.

Lemma 2.2. *If $a, v \in C^{k,\nu}[0,b]$, $k \in \mathbb{N}$, $\nu < 1$, then*

$$\|av\|_{k,\nu} \leqslant c\|a\|_{k,\nu}\|v\|_{k,\nu}$$

where c is a constant.

The statements of Lemmas 2.1 and 2.2 follow from the definition of the space $C^{k,\nu}[0,b]$ and its norm (1.4).

Consider now an equation

$$u^{(n)}(t) = v(t), \quad 0 \leqslant t \leqslant b, \quad v \in L^\infty(0,b). \tag{2.3}$$

If problem {(2.2), (1.2)} has only the trivial solution $u=0$, then problem {(2.3), (1.2)} has a unique solution

$$u(t) = \int_0^b G(t,s)v(s)\,ds, \quad t \in [0,b], \tag{2.4}$$

where $G(t,s)$ is the Green's function of problem {(2.3), (1.2)}. The derivatives of the function u given by (2.4) can be expressed in the form

$$u^{(i)}(t) = (J_i v)(t), \quad t \in [0,b], \quad i = 0, \ldots, n-1, \tag{2.5}$$

where

$$(J_i v)(t) = \int_0^b \frac{\partial^i G(t,s)}{\partial t^i} v(s)\,ds, \quad t \in [0,b], \quad i = 0, \ldots, n-1. \tag{2.6}$$

Actually, by (2.6) is defined n linear integral operators $J_i : L^\infty(0,b) \to C[0,b]$, $i = 0, \ldots, n-1$, associated with the Green's function $G(t,s)$ of problem {(2.3), (1.2)}.

In the sequel for given Banach spaces E and F, we denote by $\mathscr{L}(E,F)$ the Banach space of bounded linear operators $A : E \to F$ with the norm $\|A\|_{\mathscr{L}(E,F)} = \sup\{\|Az\|_F : z \in E, \|z\|_E \leqslant 1\}$.

Lemma 2.3. *Let $G(t,s)$ be the Green's function of problem* {(2.3), (1.2)} *and let J_i $(i=0,1,\ldots,n-1)$ be defined by the formula* (2.6). *Assume that problem* {(2.2), (1.2)} *has only the trivial solution $u=0$. Then J_i is linear and compact as an operator from $L^\infty(0,b)$ into $C[0,b]$. Moreover, J_i is a bounded operator from $C^{k,\nu}[0,b]$ into $C^{k+n-i,\nu-n+i}[0,b]$ for every $k \in \mathbb{N}_0 = \{0,1,\ldots\}$ and $\nu \in \mathbb{R}$, $\nu < 1$.*

Proof. Since the general solution of Eq. (2.2) is an arbitrary polynomial of degree $n-1$, the Green's function $G(t,s)$ for {(2.3), (1.2)} can be expressed both for $t < s$ and for $t > s$ as the polynomial of degree $n-1$ with respect to t and s. Moreover, $\partial^i G(t,s)/\partial t^i$, $i=0,\ldots,n-2$, the derivatives of $G(t,s)$ with respect to t, are continuous in $\overline{\Delta} = [0,b] \times [0,b]$ and $\partial^{n-1}G(t,s)/\partial t^{n-1}$ is continuous and bounded in Δ. From this it follows that J_i, $i = 0, \ldots, n-1$, are (see [16]) linear and compact as operators from $L^\infty(0,b)$ to $C[0,b]$.

In order to prove the second assertion of Lemma we show that J_i $(i = 0, \ldots, n-1)$ maps every bounded set of $C^{k,\nu}[0,b]$ into bounded set of $C^{k+n-i,\nu-n+i}[0,b]$, $k \in \mathbb{N}_0$, $\nu < 1$. Let $v \in C^{k,\nu}[0,b]$ be a function for which $\|v\|_{k,\nu} \leqslant c_0$, where c_0 is a constant. Then

$$|v^{(i)}(t)| \leqslant c_0 \begin{cases} 1 & \text{if } i < 1-\nu, \\ 1 + |\log \varrho(t)| & \text{if } i = 1-\nu, \\ \varrho(t)^{1-\nu-i} & \text{if } i > 1-\nu, \end{cases} \tag{2.7}$$

for $t \in (0, b)$ and $i = 0, \ldots, k$. From (2.5)–(2.7) we obtain that

$$|(J_i v)^{(j)}(t)| = |u^{(i+j)}(t)| \leqslant c_0 \int_0^b \left| \frac{\partial^{i+j} G(t,s)}{\partial t^{i+j}} \right| \mathrm{d}s \leqslant c_0 c_1, \tag{2.8}$$

where $t \in [0, b]$, $j = 0, \ldots, n - i - 1$, and c_1 is a constant. For $j \geqslant n - i$ we have

$$(J_i v)^{(j)}(t) = u^{(i+j)}(t) = v^{(i+j-n)}(t), \quad t \in [0, b],$$

and it follows from (2.7) and (2.8) that

$$|(J_i v)^{(j)}(t)| \leqslant c \begin{cases} 1 & \text{if } j < 1 - (v - n + i), \\ 1 + |\log \varrho(t)| & \text{if } j = 1 - (v - n + i), \\ \varrho(t)^{1-(v-n+i)-j} & \text{if } j > 1 - (v - n + i), \end{cases}$$

with a constant $c = c_0 \max\{1, c_1\}$ for all $t \in (0, b)$ and all $j = 0, \ldots, k + n - i$. This yields that $J_i v$ belongs to a bounded set of $C^{k+n-i, v-n+i}[0, b]$. $\square$

Proof of Theorem 2.1. Using $a_i \in C^{m,v}[0, b]$ and $K_i \in W^{m,v}(\Delta)$ we define the operators A_i and T_i by settings

$$(A_i w)(t) = a_i(t) w(t), \quad t \in [0, b], \quad i = 0, \ldots, n - 1, \tag{2.9}$$

$$(T_i w)(t) = \int_0^b K_i(t, s) w(s)\, \mathrm{d}s, \quad t \in [0, b], \quad i = 0, \ldots, n - 1. \tag{2.10}$$

Clearly,

$$A_i, T_i \in \mathscr{L}(C[0, b], C[0, b]), \quad i = 0, \ldots, n - 1. \tag{2.11}$$

Consider the equation

$$v = Tv + f, \tag{2.12}$$

where

$$T = \sum_{i=0}^{n-1} (A_i J_i + T_i J_i), \tag{2.13}$$

with J_i, $i = 0, \ldots, n - 1$, defined by (2.6). It follows from Lemma 2.3 that J_i, $i = 0, \ldots, n - 1$, are linear and compact as operators from $C[0, b] \subset L^\infty(0, b)$ into $C[0, b]$. This together with (2.11) and (2.13) yields that T is linear and compact as operator from $C[0, b]$ into $C[0, b]$.

Eq. (2.12) is equivalent to problem {(1.1), (1.2)} in the following sense: if $u \in C^n[0, b]$ is the solution of {(1.1), (1.2)} then $v = u^{(n)}$ is the solution of (2.12); conversely, if $v \in C[0, b]$ is the solution of (2.12) then $u = J_0 v$ is the solution of {(1.1), (1.2)}.

Further, since problem {(2.1), (1.2)} has only the trivial solution, equation $v = Tv$ has only the trivial solution $v = 0$, too. Thus, by the Fredholm alternative, the equation (2.12) has for every $f \in C^{m,v}[0, b] \subset C[0, b]$ the unique solution $v \in C[0, b]$.

It follows from Lemmas 2.1–2.3 that

$$J_i \in \mathscr{L}(C^{k-1,v}[0, b], C^{k,v}[0, b]), \quad A_i \in \mathscr{L}(C^{k,v}[0, b], C^{k,v}[0, b]) \tag{2.14}$$

for $i = 0, \ldots, n - 1$; $k = 1, \ldots, m$. Since $K_i \in W^{m,v}(\Delta) \subset W^{k,v}(\Delta)$, $i = 0, \ldots, n - 1$; $k = 1, \ldots, m$, then (see [16]) $T_i \in \mathscr{L}(C^{k,v}[0, b], C^{k,v}[0, b])$. This together with (2.13) and (2.14) yields that

$$T \in \mathscr{L}(C^{k-1,v}[0, b], C^{k,v}[0, b]), \quad k = 1, \ldots, m.$$

Further, since $f \in C^{m,\nu}[0,b] \subset C^{k,\nu}[0,b]$, $k=1,\ldots,m$, and v, the unique solution of equation (2.12), belongs to $C[0,b]=C^{0,\nu}[0,b]$, then by induction we get that

$$v = Tv + f \in C^{1,\nu}[0,b], \ldots, \quad v = Tv + f \in C^{m,\nu}[0,b].$$

Now we obtain from (2.4), (2.6) and Lemma 2.3 that

$$u = J_0 v, \tag{2.15}$$

the unique solution of {(1.1), (1.2)}, belongs to $C^{m+n,\nu-n}[0,b]$. Moreover,

$$u^{(i)} = J_i v \in C^{m+n-i,\nu-n+i}[0,b], \quad i=1,\ldots,n-1. \qquad \square$$

3. Piecewise polynomial interpolation

For $N \in \mathbb{N}$, let

$$\Pi_N = \{t_0,\ldots,t_{2N} : 0 = t_0 < t_1 < \cdots < t_{2N} = b\}$$

be a partion (a graded grid) of the interval $[0,b]$ with the grid points

$$t_j = \frac{b}{2}\left(\frac{j}{N}\right)^r, \quad j=0,1,\ldots,N,$$

$$t_{N+j} = b - t_{N-j}, \quad j=1,\ldots,N, \tag{3.1}$$

where the grading exponent $r \in \mathbb{R}$, $r \geqslant 1$. The number $r \in [1,\infty)$ characterizes the nonuniformity of the grid Π_N. If $r=1$, then the grid points (3.1) are distributed uniformly; for $r>1$ the points (3.1) are more densely clustered near the endpoints of the interval $[0,b]$. It is easy to see that

$$0 < t_j - t_{j-1} \leqslant \frac{rb}{2N}, \quad j=1,\ldots,2N.$$

For given integers $m \geqslant 0$ and $-1 \leqslant d \leqslant m-1$, let $S_m^{(d)}(\Pi_N)$ be the spline space of piecewise polynomial functions on the grid Π_N

$$S_m^{(d)}(\Pi_N) = \{v \in C^d[0,b] : v|_{\sigma_j} \in \pi_m,\ j=1,\ldots,2N\}, \quad 0 \leqslant d \leqslant m-1,$$

$$S_m^{(-1)}(\Pi_N) = \{v : v|_{\sigma_j} \in \pi_m,\ j=1,\ldots,2N\}.$$

Here $v|_{\sigma_j}$ is the restriction of v onto the subinterval $\sigma_j = [t_{j-1}, t_j]$, $j=1,\ldots,2N$, and π_m denotes the set of polynomials of degree not exceeding m. Note that the elements of $S_m^{(-1)}(\Pi_N)$ may have jump discontinuities at the interior points $t_1,\ldots,t_{2N-1}$ of the grid Π_N.

In every subinterval $[t_{j-1}, t_j]$, $j=1,\ldots,2N$, we introduce $m \geqslant 1$ interpolation points

$$t_{jk} = t_{j-1} + \eta_k (t_j - t_{j-1}), \quad k=1,\ldots,m; \quad j=1,\ldots,2N, \tag{3.2}$$

where $\eta_1,\ldots,\eta_m$ are some fixed parameters which do not depend on j and N and satisfy the conditions

$$0 \leqslant \eta_1 < \cdots < \eta_m \leqslant 1. \tag{3.3}$$

To a given continuous function $v : [0,b] \to \mathbb{R}$ we assign a piecewise polynomial interpolation function $\mathscr{P}_N v \in S_{m-1}^{(-1)}(\Pi_N)$ which interpolates v at the points (3.2):

$$(\mathscr{P}_N v)(t_{jk}) = v(t_{jk}), \quad k=1,\ldots,m; \quad j=1,\ldots,2N.$$

Thus $(\mathscr{P}_N v)(t)$ is independently defined in every subinterval $[t_{j-1}, t_j]$, $j=1,\ldots,2N$, and may be discontinuous at the interior points $t_1,\ldots,t_{2N-1}$ of the grid Π_N; we may treat $\mathscr{P}_N v$ as two-valued function at these points. Note that in the case $\{\eta_1 = 0, \eta_m = 1\}$ $\mathscr{P}_N v$ is a continuous function on $[0,b]$.

We also introduce an interpolation operator $\mathscr{P}_N$ which assigns to every continuous function $v : [0, b] \to \mathbb{R}$ its piecewise polynomial interpolation function $\mathscr{P}_N v$.

From results proved in [16, pp. 115–119], we obtain the following Lemmas 3.1–3.3.

Lemma 3.1. *Let the interpolation nodes* (3.2) *with grid points* (3.1) *and parameters* (3.3) *be used. Then* $\mathscr{P}_N \in \mathscr{L}(C[0, b], L^\infty(0, b))$ *and*

$$\|\mathscr{P}_N\|_{\mathscr{L}(C[0,b],L^\infty(0,b))} \leqslant c, \quad N \in \mathbb{N},$$

with a positive constant c which is independent on N.

Lemma 3.2. *Let* $v \in C^{m,v}[0, b], m \in \mathbb{N}, v < 1$, *and let the interpolation nodes* (3.2) *with grid points* (3.1) *and parameters* (3.3) *be used. Then the following estimates hold*:

$$\|v - \mathscr{P}_N v\|_\infty \leqslant c \begin{cases} N^{-r(1-v)} & \text{for } 1 \leqslant r < \dfrac{m}{1-v}, \\ N^{-m}(1 + \log N) & \text{for } r = \dfrac{m}{1-v} = 1, \\ N^{-m} & \text{for } r = \dfrac{m}{1-v} > 1 \text{ or } r > \dfrac{m}{1-v}, r \geqslant 1; \end{cases}$$

$$\int_0^b |v(t) - (\mathscr{P}_N v)(t)| \, \mathrm{d}t \leqslant c \begin{cases} N^{-r(2-v)} & \text{for } 1 \leqslant r < \dfrac{m}{2-v}, \\ N^{-m}(1 + \log N) & \text{for } r = \dfrac{m}{2-v} \geqslant 1, \\ N^{-m} & \text{for } r > \dfrac{m}{2-v}, r \geqslant 1. \end{cases}$$

Here c is a positive constant which is independent of N and

$$\|v - \mathscr{P}_N v\|_\infty = \max_{1 \leqslant j \leqslant 2N} \sup_{t_{j-1} < t < t_j} |v(t) - (\mathscr{P}_N v)(t)|. \tag{3.4}$$

Lemma 3.3. *Let the conditions of Lemma* 3.2 *be fulfilled. Then*

$$\sup_{t_{j-1} < t < t_j} |v(s) - (\mathscr{P}_N v)(s)| \leqslant c(t_j - t_{j-1})^m \begin{cases} 1 & \text{if } m < 1 - v, \\ 1 + |\log t_j| & \text{if } m = 1 - v, \\ t_j^{1-v-m} & \text{if } m > 1 - v, \end{cases}$$

for $j = 1, \ldots, N$, *and*

$$\sup_{t_{j-1} < t < t_j} |v(s) - (\mathscr{P}_N v)(s)| \leqslant c(t_j - t_{j-1})^m \begin{cases} 1 & \text{if } m < 1 - v, \\ 1 + |\log(b - t_{j-1})| & \text{if } m = 1 - v, \\ (b - t_{j-1})^{1-v-m} & \text{if } m > 1 - v, \end{cases}$$

for $j = N + 1, \ldots, 2N$, *with a positive constant c which is independent of j and N.*

4. Collocation method

We know from Section 2 that the solution of problem {(1.1), (1.2)} has the form $u = J_0 v$ (see (2.15)), where v is the solution of Eq. (2.12) and J_0 is defined by the formula (2.6). This suggests to construct a collocation method for the numerical solution of problem {(1.1), (1.2)} as follows.

We look for an approximate solution u_N of {(1.1), (1.2)} in the form

$$u_N(t) = (J_0 v_N)(t), \quad N \in \mathbb{N}, \tag{4.1}$$

where v_N satisfies the following conditions:

$$v_N \in S_{m-1}^{(-1)}(\Pi_N), \quad m \in \mathbb{N},$$
$$v_N(t_{jk}) = \sum_{i=0}^{n-1} a_i(t_{jk})(J_i v_N)(t_{jk}) + f(t_{jk}) + \int_0^b \sum_{i=0}^{n-1} K_i(t_{jk}, s)(J_i v_N)(s)\,\mathrm{d}s,$$
$$k = 1, \ldots, m; \quad j = 1, \ldots, 2N. \tag{4.2}$$

Here $\{J_i\}$ and $\{t_{jk}\}$ are given by the formulas (2.6) and (3.2), respectively.

Remark 4.1. If $v_N \in S_{m-1}^{(-1)}(\Pi_N)$ then

$$u_N^{(i)} = J_i v_N \in S_{m+n-i-1}^{(n-i-1)}(\Pi_N) \subset C^{n-i-1}[0, b], \quad i = 0, \ldots, n-1.$$

If $\eta_1 = 0$ and $\eta_m = 0$ (see (3.3)), then $v_N \in S_{m-1}^{(0)}(\Pi_N) \subset C[0, b]$ and

$$u_N^{(i)} = J_i v_N \in S_{m+n-i-1}^{(n-i)}(\Pi_N) \subset C^{n-i}[0, b], \quad i = 0, \ldots, n-1.$$

Last assertions follow from the equalities

$$u_N^{(i)}(t) = u_N^{(i)}(0) + \int_0^t u_N^{(i+1)}(s)\,\mathrm{d}s, \quad t \in [0, b], \ i = 0, \ldots, n-1.$$

Remark 4.2. The collocation conditions (4.2) form a system of equations whose exact form is determined by the choice of a basis in $S_{m-1}^{(-1)}(\Pi_N)$ (or in $S_{m-1}^{(0)}(\Pi_N)$ if $\eta_1 = 0$ and $\eta_m = 1$). For instance, in each subinterval $[t_{j-1}, t_j]$, $j = 1, \ldots, 2N$, we may use the Lagrange fundamental polynomial representation

$$v_N(t) = \sum_{k=1}^{m} c_{jk} \varphi_{jk}(t), \quad t \in [t_{j-1}, t_j], \ j = 1, \ldots, 2N,$$

where $c_{jk} = v_N(t_{jk})$ and

$$\varphi_{jk}(t) = \prod_{q=1, q \neq k}^{m} \frac{t - t_{jq}}{t_{jk} - t_{jq}}, \quad t \in [t_{j-1}, t_j], \ k = 1, \ldots, m; \quad j = 1, \ldots, 2N.$$

The conditions (4.2) then lead to a system of linear algebraic equations for the coefficients c_{jk}, $k = 1, \ldots, m$; $j = 1, \ldots, 2N$.

In the sequel by c we denote positive constants which are independent of N and may be different in different inequalities.

Theorem 4.1. *Let the conditions of Theorem* 2.1 *be fulfilled and let the interpolation nodes* (3.2) *with the grid points* (3.1) *and parameters* (3.3) *be used.*

Then there exists an $N_0 \in \mathbb{N}$ *such that, for* $N \geqslant N_0$, *and for every choice of collocation parameters* $0 \leqslant \eta_1 < \cdots < \eta_m \leqslant 1$, *the settings* (4.1) *and* (4.2) *determine a unique approximation* $u_N \in S_{m+n-1}^{(n-1)}(\Pi_N)$ *to* u, *the exact solution of the problem* $\{(1.1), (1.2)\}$. *The derivatives* $u_N^{(i)}$ *of* u_N *are approximations to* $u^{(i)}$, $i = 1, \ldots, n$. *Moreover, if* $N \geqslant N_0$, *then the following error estimates hold*:

$$\max_{0 \leqslant i \leqslant n-1} \|u^{(i)} - u_N^{(i)}\|_\infty \leqslant c \begin{cases} N^{-r(2-\nu)} & \text{for } 1 \leqslant r < \dfrac{m}{2-\nu}, \\ N^{-m}(1 + \log N) & \text{for } r = \dfrac{m}{2-\nu} \geqslant 1, \\ N^{-m} & \text{for } r > \dfrac{m}{2-\nu}, r \geqslant 1; \end{cases} \tag{4.3}$$

$$\|u^{(n)} - u_N^{(n)}\|_\infty \leqslant c \begin{cases} N^{-r(1-\nu)} & \textit{for } 1 \leqslant r < \dfrac{m}{1-\nu}, \\ N^{-m}(1+\log N) & \textit{for } r = \dfrac{m}{1-\nu} = 1, \\ N^{-m} & \textit{for } r = \dfrac{m}{1-\nu} > 1 \textit{ or } r > \dfrac{m}{1-\nu}, r \geqslant 1; \end{cases} \tag{4.4}$$

$$\max_{\substack{k=1,\dots,m \\ j=1,\dots,2N}} |u^{(n)}(t_{jk}) - u_N^{(n)}(t_{jk})| \leqslant c \begin{cases} N^{-r(2-\nu)} & \textit{for } 1 \leqslant r < \dfrac{m}{2-\nu}, \\ N^{-m}(1+\log N) & \textit{for } r = \dfrac{m}{2-\nu} \geqslant 1, \\ N^{-m} & \textit{for } r > \dfrac{m}{2-\nu}, r \geqslant 1. \end{cases} \tag{4.5}$$

Here c is a positive constant which does not depend on N, $u_N^{(n)} = v_N$ *is the solution of* (4.2) *and the norm* $\|\cdot\|_\infty$ *is defined by the formula* (3.4).

Proof. The collocation conditions (4.2) have the operator equation representation

$$v_N = \mathscr{P}_N T v_N + \mathscr{P}_N f, \tag{4.6}$$

where T is given by the formula (2.13) and $\mathscr{P}_N$ is defined in Section 3. It follows from (2.11) and Lemma 2.3 that T is linear compact as an operator from $L^\infty(0,b)$ into $C[0,b]$. On the basis of Lemmas 3.1 and 3.2 we now obtain that (cf. [2])

$$\|T - \mathscr{P}_N T\|_{\mathscr{L}(L^\infty(0,b), L^\infty(0,b))} \to 0 \quad \text{as } N \to \infty. \tag{4.7}$$

Since equation $v = Tv$ has in $L^\infty(0,b)$ only the trivial solution $v=0$, then there exists an inverse operator $(I-T)^{-1} \in \mathscr{L}(L^\infty(0,b), L^\infty(0,b))$ where I is the identity operator. This together with (4.7) yields that there exists a number $N_0 \in \mathbb{N}$ such that for $N \geqslant N_0$ the operator $(I - \mathscr{P}_N T)$ is invertible in $L^\infty(0,b)$ and

$$\|(I - \mathscr{P}_N T)^{-1}\|_{\mathscr{P}(L^\infty(0,b), L^\infty(0,b))} \leqslant c, \quad N \geqslant N_0. \tag{4.8}$$

Thus, since $f \in C[0,b]$, then for $N \geqslant N_0$ Eq. (4.6) possesses a unique solution $v_N \in L^\infty(0,b)$. Actually (see (4.6)), $v_N \in S_{m-1}^{(-1)}(\Pi_N)$. Using v_N we find for $N \geqslant N_0$ a unique $u_N \in S_{m+n-1}^{(n-1)}(\Pi_N)$ in the form (4.1).

It follows from (2.12) and (4.6) that

$$(I - \mathscr{P}_N T)(v - v_N) = v - \mathscr{P}_N v. \tag{4.9}$$

On the basis of (4.8) we obtain from (4.9) that

$$\|v - v_N\|_\infty \leqslant c \|v - \mathscr{P}_N v\|_\infty, \quad N \geqslant N_0. \tag{4.10}$$

Due to Theorem 2.1 $v = u^{(n)} \in C^{m,\nu}[0,b]$. Now (4.10) together with $v = u^{(n)}$ and Lemma 3.2 yields the estimate (4.4).

Further, from (2.5) and (4.1) we obtain that

$$u^{(i)}(t) - u_N^{(i)}(t) = (J_i(v - v_N))(t), \quad t \in [0,b], \ i = 0, \dots, n-1. \tag{4.11}$$

From (2.6) it follows that

$$\max_{0 \leqslant t \leqslant b} |(J_i(v - v_N))(t)| \leqslant c \int_0^b |v(s) - v_N(s)|\, ds, \quad i = 0, \dots, n-1. \tag{4.12}$$

Using (2.11), (2.13), (4.12) and Lemma 3.1 we obtain that

$$\|\mathscr{P}_N T(v-\mathscr{P}_N v)\|_\infty \leqslant c\int_0^b |v(s)-(\mathscr{P}_N v)(s)|\,\mathrm{d}s, \quad N\geqslant N_0. \tag{4.13}$$

Since

$$(I-\mathscr{P}_N T)^{-1} = I + (I-\mathscr{P}_N T)^{-1}\mathscr{P}_N T, \quad N\geqslant N_0,$$

we get from (4.9) that

$$v - v_N = v - \mathscr{P}_N v + (I-\mathscr{P}_N T)^{-1}\mathscr{P}_N T(v-\mathscr{P}_N v), \quad N\geqslant N_0.$$

This together (4.8) and (4.13) yields

$$|v(t)-v_N(t)| \leqslant |v(t)-(\mathscr{P}_N v)(t)| + c\int_0^b |v(s)-(\mathscr{P}_N v)(s)|\,\mathrm{d}s,$$

where $t\in[0,b]$, $N\geqslant N_0$. Thus, we get from (4.11) and (4.12) that

$$\|u^{(i)}-u_N^{(i)}\|_\infty \leqslant c\int_0^b |v(s)-(\mathscr{P}_N v)(s)|\,\mathrm{d}s, \quad i=0,\ldots,n-1, \quad N\geqslant N_0.$$

This together with $v\in C^{m,\nu}[0,b]$ and Lemma 3.2 yields the estimate (4.3).

Let us prove the estimate (4.5). From (2.12) and (4.6) we obtain that

$$(I-\mathscr{P}_N T)(v_N-\mathscr{P}_N v) = \mathscr{P}_N T(\mathscr{P}_N v - v). \tag{4.14}$$

Using (4.8) and (4.13) we get from (4.14) the estimate

$$\|v_N-\mathscr{P}_N v\|_\infty \leqslant c\int_0^b |v(s)-(\mathscr{P}_N v)(s)|\,\mathrm{d}s, \quad N\geqslant N_0. \tag{4.15}$$

Further, we have $u^{(n)} = v\in C^{m,\nu}[0,b]$ and

$$|v_N(t_{jk})-v(t_{jk})| \leqslant \|v_N-\mathscr{P}_N v\|_\infty, \quad k=1,\ldots,m; \quad j=1,\ldots,2N. \tag{4.16}$$

This together with (4.15) and Lemma 3.2 yields the estimate (4.5). □

5. Superconvergence phenomenon

It follows from Theorem 4.1 that for method {(4.1), (4.2)} for every choice of collocation parameters $0\leqslant\eta_1<\cdots<\eta_m\leqslant 1$ a convergence of order $O(N^{-m})$ can be expected, using sufficiently large values of the grid parameter r. In the following we show that assuming a little more regularity of functions $f, a_i, K_i, i=0,1,\ldots,n-1$, by a careful choice of parameters $\eta_1,\ldots,\eta_m$ it is possible to prove a faster convergence of method {(4.1), (4.2)}.

Theorem 5.1. *Let in problem* {(1.1), (1.2)} $n\in\mathbb{N}$, $f, a_i\in C^{m+1,\nu}[0,b]$, $K_i\in W^{m+1,\nu}(\Delta)$, $i=0,\ldots,n-1$; $m\in\mathbb{N}$, $\nu\in\mathbb{R}$, $-\infty<\nu<1$; $\alpha_{ij},\beta_{ij}\in\mathbb{R}$, $i=0,\ldots,n-1$; $j=1,\ldots,n$. *Assume that both the homogeneous problem* {(2.1), (1.2)} *and homogeneous problem* {(2.2), (1.2)} *have only the trivial solution* $u=0$. *Moreover, let the interpolation nodes* (3.2) *with grid points* (3.1) *and parameters* (3.3) *be used and the parameters* $\eta_1,\ldots,\eta_m$ *in* (3.3) *be chosen so that the quadrature approximation*

$$\int_0^1 g(s)\,\mathrm{d}s \approx \sum_{k=1}^m w_k g(\eta_k), \quad 0\leqslant\eta_1<\cdots<\eta_m\leqslant 1, \tag{5.1}$$

with appropriate weights $w_k = w_k^{(m)}$, $k=1,\ldots,m$, *is exact for all polynomials* g *of degree* m.

Then the statements of Theorem 4.1 *are valid. Moreover, for all* $N \geqslant N_0$ *the following estimates hold*:

$$\max_{\substack{k=1,\dots,m \\ j=1,\dots,2N}} |u^{(n)}(t_{jk}) - u_N^{(n)}(t_{jk})| \leqslant c\theta_N(m, \nu, r), \tag{5.2}$$

$$\max_{i=0,\dots,n-1} \|u^{(i)} - u_N^{(i)}\|_\infty \leqslant c\theta_N(m, \nu, r). \tag{5.3}$$

Here c is a positive constant which is independent of N, the norm $\|\cdot\|_\infty$ *is given by the formula* (3.4) *and*

$$\theta_N(m, \nu, r) = \begin{cases} N^{-r(2-\nu)} & \text{for } 1 \leqslant r < \dfrac{m+1}{2-\nu}, \\ N^{-m-1}(1 + \log N) & \text{for } r = \dfrac{m+1}{2-\nu} \geqslant 1, \\ N^{-m-1} & \text{for } r > \dfrac{m+1}{2-\nu}, \quad r \geqslant 1. \end{cases} \tag{5.4}$$

Proof. We know from the proof of Theorem 4.1 that Eq. (4.6) has a unique solution $v_N \in S_{m-1}^{(-1)}(\Pi_N)$ for $N \geqslant N_0$. From (4.14), (4.8) and Lemma 3.1 it follows the estimate

$$\|v_N - \mathscr{P}_N v\|_\infty \leqslant c\|T(v - \mathscr{P}_N v)\|_\infty, \quad N \geqslant N_0, \tag{5.5}$$

where v is the solution of Eq. (2.12).

Using (2.6) and integration by parts we obtain that

$$\begin{aligned}(J_i(v - \mathscr{P}_N v))(t) &= \int_0^b \frac{\partial^i G(t,s)}{\partial t^i}[v(s) - (\mathscr{P}_N v)(s)]\,\mathrm{d}s \\ &= \frac{\partial^i G(t,b)}{\partial t^i}\int_0^b [v(\tau) - (\mathscr{P}_N v)(\tau)]\,\mathrm{d}\tau \\ &\quad - \left[\frac{\partial^i G(t,t+0)}{\partial t^i} - \frac{\partial^i G(t,t-0)}{\partial t^i}\right]\int_0^t [v(\tau) - (\mathscr{P}_N v)(\tau)]\,\mathrm{d}\tau \\ &\quad - \int_0^b \frac{\partial^{i+1} G(t,s)}{\partial t^i \partial s}\int_0^s [v(\tau) - (\mathscr{P}_N v)(\tau)]\,\mathrm{d}\tau\,\mathrm{d}s, \quad t \in [0,b], \quad i = 0,\dots,n-1.\end{aligned}$$

Thus, we get

$$\max_{0 \leqslant t \leqslant b} |(J_i(v - \mathscr{P}_N v))(t)| \leqslant c \max_{0 \leqslant t \leqslant b}\left|\int_0^t [v(\tau) - (\mathscr{P}_N v)(\tau)]\,\mathrm{d}\tau\right|, \quad i = 0,\dots,n-1. \tag{5.6}$$

From (5.6), (2.11) and (2.13) it follows

$$\|T(v - \mathscr{P}_N v)\|_\infty \leqslant c \max_{0 \leqslant t \leqslant b}\left|\int_0^t [v(\tau) - (\mathscr{P}_N v)(\tau)]\,\mathrm{d}\tau\right|. \tag{5.7}$$

From Theorem 2.1 we get that $v = u^{(n)} \in C^{m+1,\nu}[0,b]$. Using this we can show that

$$\max_{0 \leqslant t \leqslant b}\left|\int_0^t [v(\tau) - (\mathscr{P}_N v)(\tau)]\,\mathrm{d}\tau\right| \leqslant c\theta_N(m, \nu, r), \tag{5.8}$$

where $\theta_N(m, \nu, r)$ is given by the formula (5.4). This together with (4.16), (5.5) and (5.7) yields (5.2).

In order to prove (5.8) we choose $m+1$ parameters $0 \leqslant \tilde{\eta}_1 < \tilde{\eta}_2 < \cdots < \tilde{\eta}_{m+1} \leqslant 1$ such that $\{\eta_1, \ldots, \eta_m\} \subset \{\tilde{\eta}_1, \ldots, \tilde{\eta}_{m+1}\}$ and set

$$\tilde{t}_{jk} = t_{j-1} + \tilde{\eta}_k (t_j - t_{j-1}), \quad k = 1, \ldots, m+1; \quad j = 1, \ldots, 2N,$$

where $\{t_j\}$ are given by the formulas (3.1). Moreover, we introduce an operator $\tilde{\mathscr{P}}_N$ which assign to every continuous function $z : [0, b] \to \mathbb{R}$ its piecewise polynomial interpolation function $\tilde{\mathscr{P}}_N z \in S_m^{(-1)}(\Pi_N)$ such that

$$(\tilde{\mathscr{P}}_N z)(\tilde{t}_{jk}) = z(\tilde{t}_{jk}), \quad k = 1, \ldots, m+1; \quad j = 1, \ldots, 2N.$$

Due to Lemma 3.2

$$\int_0^b |v(\tau) - (\tilde{\mathscr{P}}_N v)(\tau)| \, d\tau \leqslant c\theta_N(m, v, r). \tag{5.9}$$

Further, the quadrature approximation (5.1) is exact for all polynomials of degree not exceeding m. This yields that the equality

$$\int_{t_{j-1}}^{t_j} g(\tau) \, d\tau = (t_j - t_{j-1}) \sum_{k=1}^{m} w_k g(t_{jk}), \quad j = 1, \ldots, 2N,$$

holds for all polynomials g of degree not exceeding m. Therefore

$$\int_{t_{j-1}}^{t_j} (\mathscr{P}_N v)(\tau) \, d\tau = \int_{x_{j-1}}^{t_j} (\tilde{\mathscr{P}}_N v)(\tau) \, d\tau, \quad j = 1, \ldots, 2N,$$

and

$$\left| \int_0^{t_j} [v(\tau) - (\mathscr{P}_N v)(\tau)] \, d\tau \right| = \left| \int_0^{t_j} [v(\tau) - (\tilde{\mathscr{P}}_N v)(\tau)] \, d\tau \right|$$
$$\leqslant \int_0^b |v(\tau) - (\tilde{\mathscr{P}}_N v)(\tau)| \, d\tau, \quad j = 1, \ldots, 2N.$$

This together with (5.9) yields

$$\max_{1 \leqslant j \leqslant 2N} \left| \int_0^{t_j} [v(\tau) - (\mathscr{P}_N v)(\tau)] \, d\tau \right| \leqslant c\theta_N(m, v, r). \tag{5.10}$$

Fix $t \in [0, b]$ and let $j \in \{1, \ldots, 2N\}$ such that $t \in [t_{j-1}, t_j]$. Actually, we consider only the case if $j \in \{1, \ldots, N\}$. For $j \in \{N+1, \ldots, 2N\}$ the argument is analogous. Due to Lemma 3.3

$$\left| \int_{t_{j-1}}^{t} [v(\tau) - (\mathscr{P}_N v)(\tau)] \, d\tau \right| \leqslant c(t_j - t_{j-1})^{m+1} \begin{cases} 1 & \text{if } m < 1 - v, \\ 1 + |\log t_j| & \text{if } m = 1 - v, \\ t_j^{1-v-m} & \text{if } m > 1 - v. \end{cases}$$

For $j = 1, \ldots, N$ we have

$$t_j = \frac{b}{2}\left(\frac{j}{N}\right)^r, \quad 0 < t_j - t_{j-1} \leqslant \frac{br}{2} j^{r-1} N^{-r} \leqslant \frac{br}{2N},$$

$$(t_j - t_{j-1})^{m+1} t_j^{1-v-m} \leqslant c j^{r(2-v)-m-1} N^{-r(2-v)}$$
$$\leqslant c \begin{cases} N^{-r(2-v)} & \text{if } r(2-v) < m+1, \\ N^{-m-1} & \text{if } r(2-v) \geqslant m+1. \end{cases}$$

Therefore,

$$\left|\int_{t_{j-1}}^{t} [v(\tau) - (\mathscr{P}_N v)(\tau)]\,\mathrm{d}\tau\right| \leqslant c\theta_N(m, v, r), \quad t \in [t_{j-1}, t_j]. \tag{5.11}$$

This together with (5.10) yields (5.8) implying the estimate (5.2).

Let us prove the statement (5.3). As well we have deduced the estimate (5.6) we get also

$$\begin{aligned}\|u^{(i)} - u_N^{(i)}\|_\infty &= \max_{0\leqslant t\leqslant b} |(J_i(v - v_N))(t)| \\ &\leqslant c \max_{0\leqslant t\leqslant b} \left|\int_0^t [v(\tau) - v_N(\tau)]\,\mathrm{d}\tau\right|, \quad i = 0, \ldots, n-1.\end{aligned}$$

In the following we prove that

$$\max_{0\leqslant t\leqslant b} \left|\int_0^t [v(\tau) - v_N(\tau)]\,\mathrm{d}\tau\right| \leqslant c\theta_N(m, v, r) \tag{5.12}$$

and so we get the estimate (5.3).

Fix $t \in [0, b]$, let $j \in \{1, \ldots, 2N\}$ be such that $t \in [t_{j-1}, t_j]$. We have

$$\left|\int_0^t [v(\tau) - v_N(\tau)]\,\mathrm{d}\tau\right| \leqslant \left|\int_0^{t_{j-1}} [v(\tau) - v_N(\tau)]\,\mathrm{d}\tau\right| + \left|\int_{t_{j-1}}^t [v(\tau) - v_N(\tau)]\,\mathrm{d}\tau\right|. \tag{5.13}$$

For the first term of the right-hand side of (5.13) we get

$$\begin{aligned}\left|\int_0^{t_{j-1}} [v(\tau) - v_N(\tau)]\,\mathrm{d}\tau\right| &\leqslant \left|\int_0^{t_{j-1}} [v(\tau) - (\tilde{\mathscr{P}}_N v)(\tau)]\,\mathrm{d}\tau\right| + \left|\int_0^{t_{j-1}} [(\tilde{\mathscr{P}}_N v)(\tau) - v_N(\tau)]\,\mathrm{d}\tau\right| \\ &\leqslant \int_0^b |v(\tau) - (\tilde{\mathscr{P}}_N v)(\tau)|\,\mathrm{d}\tau + \sum_{i=1}^{j-1} (t_i - t_{i-1}) \sum_{k=1}^m |w_k||v(t_{ik}) - v_N(t_{ik})|.\end{aligned}$$

This together with (5.2) and (5.9) yields that

$$\left|\int_0^{t_{j-1}} [v(\tau) - v_N(\tau)]\,\mathrm{d}\tau\right| \leqslant c\theta_N(m, v, r). \tag{5.14}$$

It remains to estimate the second term of the right-hand of (5.13). We have

$$\begin{aligned}\left|\int_{t_{j-1}}^t [v(\tau) - v_N(\tau)]\,\mathrm{d}\tau\right| &\leqslant \left|\int_{t_{j-1}}^t [v(\tau) - (\mathscr{P}_N v)(\tau)]\,\mathrm{d}\tau\right| \\ &\quad + \int_{t_{j-1}}^t |(\mathscr{P}_N v)(\tau) - v_N(\tau)|\,\mathrm{d}\tau, \quad t \in [t_{j-1}, t_j].\end{aligned} \tag{5.15}$$

By (5.5), (5.7) and (5.8) we get

$$\int_{t_{j-1}}^t |(\mathscr{P}_N v)(\tau) - v_N(\tau)|\,\mathrm{d}\tau \leqslant (t - t_{j-1})\|\mathscr{P}_N v - v_N\|_\infty \leqslant c\theta_N(m, v, r), \quad t \in [t_{j-1}, t_j].$$

This together with the estimates (5.11), (5.15), (5.14) and (5.13) yields (5.12) and therefore also (5.3). $\square$

Table 1
Results in case $\eta_1 = 0.1$ and $\eta_2 = 0.9$

N	$r=1$ ($\delta_1^{(0)}=2.83$)		$r=1.4$ ($\delta_{1.4}^{(0)}=4$)		$r=2$ ($\delta_2^{(0)}=4$)	
	$\varepsilon_N^{(0)}$	$\varrho_N^{(0)}$	$\varepsilon_N^{(0)}$	$\varrho_N^{(0)}$	$\varepsilon_N^{(0)}$	$\varrho_N^{(0)}$
4	7.1E − 4	4.03	7.6E − 4	4.01	1.2E − 3	3.74
8	1.8E − 4	4.00	1.9E − 4	4.02	2.9E − 4	3.94
16	4.4E − 5	3.99	4.7E − 5	4.01	7.4E − 5	3.99
32	1.1E − 5	3.99	1.2E − 5	4.00	1.8E − 5	4.00
64	2.8E − 6	3.99	3.0E − 6	4.00	4.6E − 6	4.00
N	$r=1$ ($\delta_1^{(1)}=2.83$)		$r=1.4$ ($\delta_{1.4}^{(1)}=4$)		$r=2$ ($\delta_2^{(1)}=4$)	
	$\varepsilon_N^{(1)}$	$\varrho_N^{(1)}$	$\varepsilon_N^{(1)}$	$\varrho_N^{(1)}$	$\varepsilon_N^{(1)}$	$\varrho_N^{(1)}$
4	8.5E − 3	2.66	6.1E − 3	3.23	6.2E − 3	3.42
8	3.2E − 3	2.70	1.8E − 3	3.39	1.7E − 3	3.71
16	1.2E − 3	2.74	5.1E − 4	3.51	4.3E − 4	3.86
32	4.2E − 4	2.77	1.4E − 4	3.59	1.1E − 4	3.93
64	1.5E − 4	2.79	3.9E − 5	3.65	2.8E − 5	3.96
N	$r=1$ ($\delta_1^{(2)}=1.41$)		$r=1.4$ ($\delta_{1.4}^{(2)}=1.62$)		$r=2$ ($\delta_2^{(2)}=2$)	
	$\varepsilon_N^{(2)}$	$\varrho_N^{(2)}$	$\varepsilon_N^{(2)}$	$\varrho_N^{(2)}$	$\varepsilon_N^{(2)}$	$\varrho_N^{(2)}$
4	3.2E − 1	1.44	2.4E − 1	1.65	1.6E − 1	2.03
8	2.2E − 1	1.42	1.5E − 1	1.63	7.9E − 2	2.01
16	1.6E − 1	1.42	9.0E − 2	1.63	3.9E − 2	2.00
32	1.1E − 1	1.42	5.6E − 2	1.63	2.0E − 2	2.00
64	7.9E − 2	1.41	3.4E − 2	1.62	9.8E − 3	2.00

6. Numerical experiments

Let us consider the following boundary value problem:

$$u''(t) = \sqrt{t}u'(t) + f(t) + \int_0^1 |t-s|^{-1/2}u(s)\,\mathrm{d}s, \qquad t \in [0,1],$$
$$u(0) = u(1) = 0. \tag{6.1}$$

Here the forcing function f is selected so that

$$u(t) = t^{5/2} + (1-t)^{5/2} - 1$$

is the exact solution of problem (6.1). Actually, this is a problem of the form {(1.1), (1.2)} where

$$n=2, \quad b=1, \quad a_0(t)=0, \quad a_1(t)=\sqrt{t}, \quad K_0(t,s)=|t-s|^{-1/2}, \quad K_1(t,s)=0,$$
$$f(t) = -\tfrac{5}{2}t^2 + \tfrac{5}{2}\sqrt{t}(1-t)^{3/2} - \psi(t) - \psi(1-t)$$

with

$$\psi(t) = \frac{5}{16}\pi t^3 + \frac{1}{24}\sqrt{1-t}(8+10t+15t^2) + \frac{5}{16}t^3 \ln\frac{1}{t}(2-t+2\sqrt{1-t}) - \frac{23}{4}\sqrt{t}.$$

It is easy to check that $a_1, f \in C^{m,\nu}[0,1]$ and $K_0 \in W^{m,\nu}(\Delta)$ with $\nu = 1/2$ and arbitrary $m \in \mathbb{N}$.

Problem (6.1) is solved numerically by collocation method {(4.1), (4.2)} in case $m=2$. In Tables 1–4 some of the results obtained for different values of parameters N, r, η_1 and η_2 are presented. The quantities $\varepsilon_N^{(i)}$ $(i=0,1,2)$ are the

Table 2
Results in case $\eta_1 = 0.1$ and $\eta_2 = 0.9$ at collocation points

N	$r=1$ $(\delta_1 = 2.83)$		$r=1.4$ $(\delta_{1.4} = 4)$		$r=2$ $(\delta_2 = 4)$	
	ε_N	ϱ_N	ε_N	ϱ_N	ε_N	ϱ_N
4	7.4E − 3	2.44	5.1E − 3	3.04	4.9E − 3	3.27
8	2.9E − 3	2.57	1.6E − 3	3.28	1.3E − 3	3.64
16	1.1E − 3	2.66	4.6E − 4	3.44	3.5E − 4	3.83
32	4.0E − 4	2.71	1.3E − 4	3.54	9.0E − 5	3.91
64	1.4E − 4	2.75	3.6E − 5	3.61	2.3E − 5	3.96

Table 3
Results in case $\eta_1 = (3-\sqrt{3})/6$ and $\eta_2 = (3+\sqrt{3})/6$

N	$r=1$ $(\delta_1^{(0)} = 2.83)$		$r=1.4$ $(\delta_{1.4}^{(0)} = 4.29)$		$r=2$ $(\delta_2^{(0)} \approx 8)$	
	$\varepsilon_N^{(0)}$	$\varrho_N^{(0)}$	$\varepsilon_N^{(0)}$	$\varrho_N^{(0)}$	$\varepsilon_N^{(0)}$	$\varrho_N^{(0)}$
4	5.2E − 5	5.62	1.6E − 5	12.0	3.6E − 5	10.3
8	9.2E − 6	5.62	1.3E − 6	12.4	2.8E − 6	12.9
16	1.6E − 6	5.63	1.2E − 7	11.3	2.0E − 7	14.3
32	2.9E − 7	5.64	1.0E − 8	11.3	1.3E − 8	15.1
64	5.1E − 8	5.65	9.1E − 10	11.3	8.3E − 10	15.5
N	$r=1$ $(\delta_1^{(1)} = 2.83)$		$r=1.4$ $(\delta_{1.4}^{(1)} = 4.29)$		$r=2$ $(\delta_2^{(1)} \approx 8)$	
	$\varepsilon_N^{(1)}$	$\varrho_N^{(1)}$	$\varepsilon_N^{(1)}$	$\varrho_N^{(1)}$	$\varepsilon_N^{(1)}$	$\varrho_N^{(1)}$
4	1.4E − 3	2.84	5.8E − 4	4.36	5.1E − 4	6.63
8	5.0E − 4	2.82	1.4E − 4	4.30	7.2E − 5	7.16
16	1.8E − 4	2.82	3.2E − 5	4.29	9.6E − 6	7.48
32	6.2E − 5	2.83	7.4E − 6	4.29	1.2E − 6	7.70
64	2.2E − 5	2.83	1.7E − 6	4.29	1.6E − 7	7.84
N	$r=1$ $(\delta_1^{(2)} = 1.41)$		$r=1.4$ $(\delta_{1.4}^{(2)} = 1.62)$		$r=2$ $(\delta_2^{(2)} = 2)$	
	$\varepsilon_N^{(2)}$	$\varrho_N^{(2)}$	$\varepsilon_N^{(2)}$	$\varrho_N^{(2)}$	$\varepsilon_N^{(2)}$	$\varrho_N^{(2)}$
4	4.0E − 1	1.43	3.0E − 1	1.64	2.0E − 1	2.01
8	2.8E − 1	1.42	1.9E − 1	1.63	1.0E − 1	2.00
16	2.0E − 1	1.42	1.2E − 1	1.62	5.0E − 2	2.00
32	1.4E − 1	1.42	7.1E − 2	1.62	2.5E − 2	2.00
64	1.0E − 1	1.41	4.4E − 2	1.62	1.2E − 2	2.00

Table 4
Results in case $\eta_1 = (3-\sqrt{3})/6$ and $\eta_2 = (3+\sqrt{3})/6$ at collocation points

N	$r=1$ $(\delta_1 = 2.83)$		$r=1.4$ $(\delta_{1.4} = 4.29)$		$r=2$ $(\delta_2 \approx 8)$	
	ε_N	ϱ_N	ε_N	ϱ_N	ε_N	ϱ_N
4	1.3E − 3	2.67	5.6E − 4	4.07	4.0E − 4	6.56
8	4.8E − 4	2.74	1.3E − 4	4.18	5.4E − 5	7.36
16	1.7E − 4	2.78	3.2E − 5	4.24	7.0E − 6	7.68
32	6.2E − 5	2.80	7.4E − 6	4.27	9.0E − 7	7.83
64	2.2E − 5	2.82	1.7E − 6	4.28	1.1E − 7	7.91

approximate values of the norms $\|u_N^{(i)} - u^{(i)}\|_\infty$ $(i=0,1,2)$ calculated as follows:

$$\varepsilon_N^{(i)} = \{\max |u_N^{(i)}(\tau_{jk}) - u^{(i)}(\tau_{jk})| : k=0,\ldots,10;\ j=1,\ldots,2N\}.$$

Here $i=0,1,2$ and

$$\tau_{jk} = t_{j-1} + \frac{k}{10}(t_j - t_{j-1}), \quad k=0,\ldots,10; \quad j=1,\ldots,2N,$$

with grid points $\{t_j\}$, defined by the formula (3.1).

In Tables 2 and 4 the errors

$$\varepsilon_N = \{\max |u_N''(t_{jk}) - u''(t_{jk})| : k=1,2;\ j=1,\ldots,2N\}$$

of $u_N'' = v_N$ at the collocation points (3.2) for $m=2$ are presented. In Tables also the ratios

$$\varrho_N^{(i)} = \varepsilon_{N/2}^{(i)}/\varepsilon_N^{(i)}, \quad i=0,1,2, \quad \varrho_N = \varepsilon_{N/2}/\varepsilon_N,$$

characterizing the observed convergence rate, are presented. Moreover, in order to facilitate the comparison of numerical experiments with theoretical results we have in tables used the notations $\delta_r^{(i)}$ $(i=0,1,2)$ and δ_r for the ratios regarding the theoretical rate of convergence of proposed algorithms established by Theorems 4.1 and 5.1. These ratios are defined in the following way. It follows from (4.3) and (4.5) for $n=2$, $m=2$, $v=1/2$ and $N \geqslant N_0$ that

$$\max_{i=0,1} \|u^{(i)} - u_N^{(i)}\|_\infty \leqslant \varphi_r(N),$$

$$\max_{k=1,2;\, j=1,\ldots,2N} \|u''(t_{jk}) - u_N''(t_{jk})\| \leqslant \varphi_r(N),$$

where

$$\varphi_r(N) = c \begin{cases} N^{-3r/2} & \text{if } 1 \leqslant r < 4/3, \\ N^{-2}(1+\log N) & \text{if } r = 4/3, \\ N^{-2} & \text{if } r > 4/3. \end{cases}$$

Define the ratios $\delta_r^{(0)}$, $\delta_r^{(1)}$ and δ_r associated with $\varphi_r(N)$ by

$$\delta_r^{(0)} = \delta_r^{(1)} = \delta_r = \varphi_r(N/2)/\varphi_r(N).$$

Thus, for $N \geqslant N_0$,

$$\delta_r^{(0)} = \delta_r^{(1)} = \delta_r = \begin{cases} 2^{3r/2} & \text{if } 1 \leqslant r < 4/3, \\ 2^2 \left(\dfrac{1+\log(N/2)}{1+\log N} \right) & \text{if } r = 4/3, \\ 2^2 & \text{if } r > 4/3. \end{cases}$$

In a similar way we introduce the ratio $\delta_r^{(2)}$ associated with the error estimate (4.4) for $1 \leqslant r < 4$: $\delta_r^{(2)} = 2^{r/2}$.

From this it follows that in Tables 1 and 2

$$\delta_1^{(0)} = \delta_1^{(1)} = \delta_1 = 2^{3/2} \approx 2.83,$$

$$\delta_r^{(0)} = \delta_r^{(1)} = \delta_r = 2^2 = 4 \quad \text{for } r > 4/3,$$

$$\delta_r^{(2)} = 2^{r/2} \quad \text{for } 1 \leqslant r \leqslant 2.$$

In Tables 3 and 4 results for Gaussian parameters $\eta_1 = (3-\sqrt{3})/6$ and $\eta_2 = (3+\sqrt{3})/6$ are presented. Since the corresponding Gaussian quadrature formula (5.1) for $m=2$ is exact for all polynomials of degree 3, we can apply

Theorem 5.1. In a similar way as above we introduce the ratios $\delta_r^{(0)}$, $\delta_r^{(1)}$ and δ_r associated with the error estimates (5.2) and (5.3) for $n=2$, $m=2$, $v=1/2$ and $N \geqslant N_0$:

$$\delta_r^{(0)} = \delta_r^{(1)} = \delta_r = \begin{cases} 2^{3r/2} & \text{if } 1 \leqslant r < 2, \\ 2^3 \left(\dfrac{1+\log(N/2)}{1+\log N} \right) & \text{if } r = 2, \\ 2^3 & \text{if } r > 2. \end{cases}$$

From this it follows that in Tables 3 and 4

$$\delta_r^{(0)} = \delta_r^{(1)} = \delta_r = 2^{3r/2} \quad \text{for } 1 \leqslant r < 2,$$

$$\delta_2^{(0)} = \delta_2^{(1)} = \delta_2 = 2^3(1+\log(N/2))(1+\log N)^{-1} \approx 8, \qquad N \geqslant N_0.$$

Note that Theorem 5.1 does not refine the estimate for the error $\|u'' - u_N''\|_\infty$. Therefore we apply the estimate (4.4) that holds for all values of parameters $0 \leqslant \eta_1 < \eta_2 \leqslant 1$. Thus, in Table 3 we have used the same values for $\delta_r^{(2)}$ as in Table 1.

From Tables 1–4 we can see that in most cases the numerical results are in good accordance with the theoretical estimates of Theorems 4.1 and 5.1. In Tables 1 and 3 only the decrease of $\varepsilon_N^{(0)}$ in some cases is considerable faster than it is indicated by the error estimates (4.3) and (5.3). This phenomenon is worth examining independently.

References

[1] H. Brunner, Collocation Methods for Volterra Integral and Related Functional Equations, Cambridge Monographs on Applied and Computational Mathematics, vol. 15, Cambridge University Press, Cambridge, UK, 2004.

[2] H. Brunner, A. Pedas, G. Vainikko, Piecewise polynomial collocation methods for linear Volterra integrodifferential equations with weakly singular kernels, SIAM J. Numer. Anal. 39 (2001) 957–982.

[3] H. Brunner, A. Pedas, G. Vainikko, A spline collocation method for linear integro-differential equations with weakly singular kernels, BIT 41 (2001) 891–900.

[4] M. Ganesh, I.H. Sloan, Optimal order spline methods for nonlinear differential and integro-differential equations, Appl. Numer. Math. 29 (1999) 445–478.

[5] M. Ganesh, A. Spence, Orthogonal collocation for a nonlinear integro-differential equation, IMA J. Numer. Anal. 18 (1998) 191–206.

[6] R.J. Hangelbrock, H.G. Kaper, G.K. Leaf, Collocation methods for integro-differential equations, SIAM J. Numer. Anal. 14 (1977) 377–390.

[7] S.M. Hosseini, S. Shahmorad, Tau numerical solution of Fredholm integro-differential equations with arbitrary polynomial bases, Appl. Math. Model. 27 (2003) 145–154.

[8] Q. Hu, Interpolation correction for collocation solution for Fredholm integro-differential equation, Math. Comput. 67 (1998) 987–999.

[9] R. Kangro, I. Parts, Superconvergence in the maximum norm of a class of piecewise polynomial collocation methods for solving linear weakly singular Volterra integro-differential equations, J. Integral Equations Appl. 15 (2003) 403–427.

[10] É.B. Karpilovskaja, The convergence of collocation type methods for ordinary integro-differential equations (Russian), Izv. Vysš, Učebn. Zaved. Math. 11 (1968) 105–113.

[11] K. Maleknejad, Y. Mahmoudi, Taylor polynomial solution of high-order nonlinear Volterra–Fredholm integro-differential equations, Appl. Math. Comput. 145 (2003) 641–653.

[12] Nguyen Minh Chuong, Nguyen Van Tuan, Spline collocation methods for Fredholm–Volterra integro-differential equations of high order, Vietnam J. Math. 25 (1997) 15–24.

[13] I. Parts, A. Pedas, Spline collocation methods for weakly singular Volterra integro-differential equations, in Numerical Mathematics and Advanced Applications, Enumath 2001, Springer, Milano, 2003, pp. 919–928.

[14] I. Parts, A. Pedas, Collocation approximations for weakly singular Volterra integro-differential equations, Math. Model. Anal. 8 (2003) 315–328.

[15] T. Tang, Superconvergence of numerical solutions to weakly singular Volterra integro-differential equations, Numer. Math. 61 (1992) 373–382.

[16] G. Vainikko, Multidimensional Weakly Singular Integral Equations, Lecture Notes in Mathematics, vol. 1549, Springer, Berlin, 1993.

[17] W. Volk, The numerical solution of linear integrodifferential equations by projection methods, J. Integral Equations 9 (1985) 171–190.

[18] W. Volk, The iterated Galerkin methods for linear integro-differential equations, J. Comput. Appl. Math. 21 (1988) 63–74.

[19] A.-M. Wazwaz, A reliable algorithm for solving boundary value problems for higher-order integro-differential equations, Appl. Math. Comput. 118 (2001) 327–342.