The Odd Girth of the Generalised Kneser Graph

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Let \(X = \{1, 2, \ldots, n\}\) be a set of \(n\) elements and let \(X^{(r)}\) be the collection of all the subsets of \(X\) containing precisely \(r\) elements. Then the generalised Kneser graph \(K(n, r, s)\) (when \(2r - s \leq n\)) is the graph with vertex set \(X^{(r)}\) and edges \(AB\) for \(A, B \in X^{(r)}\) with \(|A \cap B| \leq s\).

Here we show that the odd girth of the generalised Kneser graph \(K(n, r, s)\) is

\[
2 \left\lceil \frac{r - s}{n - 2(r - s)} \right\rceil + 1
\]

provided that \(n\) is large enough compared with \(r\) and \(s\).

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1. INTRODUCTION

For \(X = \{1, 2, \ldots, n\}\) a set of \(n\) elements let \(X^{(r)}\) be the collection of all the subsets of \(X\) containing precisely \(r\) elements. Then the Kneser graph \(K(n, r)\) (when \(2r \leq n\)) is the graph with vertex set \(X^{(r)}\) and edges \(AB\) for \(A, B \in X^{(r)}\) with \(A \cap B = \emptyset\). The generalised Kneser graph \(K(n, r, s)\) (when \(2r - s \leq n\)) has the same vertex set \(X^{(r)}\), but the edge set consists of all \(AB\) where \(A, B \in X^{(r)}\) with \(|A \cap B| \leq s\). In particular, \(K(n, r, 0) = K(n, r)\).

Although simply defined, many of the properties of the Kneser graph are surprisingly difficult to characterise. For instance, the chromatic number \(\chi(K(n, r))\) remained an open question for many years, until it was finally solved by Lovász [2]. Later, Bárány gave a beautifully simple proof of this result [1], based on the classical Borsuk–Ulam Theorem.

A feature of the Kneser graph is that although it is always rich in even cycles, the parameters can be chosen so that the graph contains no short odd cycles. It is thus a natural question to ask the length of the shortest odd cycle in \(K(n, r)\). Poljak and Tuza, in [3], found an expression for the odd girth of \(K(n, r)\), where, as usual, the odd girth of a graph is the length of its shortest odd cycle.

Here we shall find an expression for the odd girth of the generalised Kneser graph \(K(n, r, s)\) when \(n\) satisfies a weak bound. This we shall do by, firstly, finding a general lower bound for the odd girth by generalising the work of Poljak and Tuza, and finally explicitly giving an embedding of an odd cycle to show that this lower bound is in fact the true odd girth, provided that \(n\) satisfies the required condition.

2. THE RESULTS

In [3], Poljak and Tuza proved the following theorem which gives the exact value of the odd girth of the Kneser graph \(K(n, r)\).

**Theorem A.** The odd girth of \(K(n, r)\) is

\[
2 \left\lceil \frac{r}{n - 2r} \right\rceil + 1.
\]
We shall generalise their theorem, giving the exact value of the odd girth of the generalised Kneser graph $K(n, r, s)$.

**Theorem 1.** The odd girth of $K(n, r, s)$ is at least

$$2 \left\lceil \frac{r - s}{n - 2(r - s)} \right\rceil + 1$$

and equality holds provided that $n \geq 2(r - s) + \binom{s(r - s)}{(r - 2s)}$.

To prove Theorem 1, let us define two new graph parameters. Let us define the bipartite-rank of a graph $G$ as

$$\text{rank}(G) = \max \{ e(B) : B \text{ a bipartite subgraph of } G \}$$

and the bipartite density as

$$\text{density}(G) = \frac{\text{rank}(G)}{e(G)}$$

where, as usual, $e(G)$ is the number of edges in $G$.

Our main result follows from an application of the following lemma, which generalises an observation of Poljak and Tuza. Let $K_i(n, r, s)$ be the subgraph of $K(n, r, s)$ with the same vertex set, but only those edges $AB$ for which $u_A > u_B = i$.

**Lemma 2.** Let $H$ be a subgraph of $K_i(n, r, s)$. Then

$$\text{density}(H) \geq \text{density}(K_i(n, r, s)).$$

**Proof.** Let $\text{Perm}(K_i(n, r, s)) \subset \text{Aut}(K_i(n, r, s))$ be the automorphisms that are defined on $K_i(n, r, s)$ by permutations of the ground set. Given a subset $A$ of the automorphism group $\text{Aut}(K_i(n, r, s))$, we shall consider $A$ as a probability space, in which every member has equal probability. Then let us define the expectation $E_A(.)$ in the natural way.

Consider the subgraph $H$ of $K_i(n, r, s)$, and let $B$ be a bipartite subgraph of $K_i(n, r, s)$ with $\text{rank}(K_i(n, r, s))$ edges. We shall consider the expected size of $e(H \cap f(B))$, the intersection of the edge sets of $H$ and the image of $B$ under a random automorphism, $f \in \text{Perm}(K_i(n, r, s))$. This expectation can easily be seen to be

$$E_{\text{Perm}}(e(H \cap f(B))) = \sum_{b \in E(B)} P_{\text{Perm}}(f(b) \in E(H)).$$

Now suppose that $|A \cap B| = i = |C \cap D|$. Then the number of permutations mapping the edge $AB$ to the edge $CD$ is $2! \cdot ((r - i)!)^2(n - 2r + i)!$.

Observe that

$$\frac{n!}{2! \cdot ((r - i)!)^2(n - 2r + i)!} = \frac{\binom{n}{r}}{(r - i)\binom{n - r}{r - i}} = e(K_i(n, r, s)).$$

Thus we have that, for any fixed $b, c \in E(K_i(n, r, s))$,

$$P_{\text{Perm}}(f(b) = c) = \frac{1}{e(K_i(n, r, s))}.$$
Using this in our expression for the expectation, we have

\[
E_{\text{Perm}}(e(H \cap f(B))) = \frac{e(H)e(B)}{e(K_i(n, r, s))},
\]

and hence

\[
density(H) = \frac{\text{rank}(H)}{e(H)} \geq \frac{E(e(H \cap f(B)))}{e(H)} = \frac{e(B)}{e(K_i(n, r, s))} = \text{density}(K_i(n, r, s)). \quad \square
\]

We are now in a position to prove Theorem 1.

**Proof of Theorem 1.** Firstly we prove the lower bound.

Take \( B \subset K_i(n, r, s) \) to be the bipartite subgraph of \( K_i(n, r, s) \) with bipartition \( (B_1, B_2) \), where

\[
B_1 = \{ F \in X^o : 1 \in F \}, \quad B_2 = \{ F \in X^o : 1 \not\in F \}.
\]

Then the number of edges in the bipartite graph \( B \) is

\[
e(B) = \binom{n-1}{r-1}\binom{r-1}{s}\binom{n-r}{r-s},
\]

and the number of edges in \( K_i(n, r, s) \) is

\[
e(K_i(n, r, s)) = \binom{n}{r}\binom{r}{s}\binom{n-r}{r-s} = \binom{n}{r}\left(\frac{s}{r-s}\right)\binom{r-1}{s}\binom{n-r}{r-s} = \binom{n}{r}\frac{r}{r-s}\binom{r-1}{s}\binom{n-r}{r-s}.
\]

Thus we have a lower bound for the density of \( K_i(n, r, s) \):

\[
density(K_i(n, r, s)) \geq \frac{e(B)}{e(K(n, r, s))} = 2\frac{\binom{n-1}{r-1}\binom{r-s}{r}}{\binom{n}{r}} = 2\frac{(r-s)}{n-r} = 2\frac{(r-s)}{n}.
\]

Now we apply Lemma 2. For \( C_{2k+1} \) to be a subgraph of \( K_i(n, r, s) \), we must have

\[
\frac{2(r-s)}{n} = \text{density}(C_{2k+1}) = \frac{2k}{2k+1}.
\]

We need now only notice that \( K(n, r, s) \) has \( C_{2k+1} \) as a minimal odd cycle iff \( K_i(n, r, s) \) has \( C_{2k+1} \) as a minimal odd cycle. To see this, let \( A \) have the two neighbours \( B \) and \( C \) in an odd cycle of \( K(n, r, s) \), and suppose that \( |A \cap B| = l < s \). We may simply change
We observe that, with these definitions, we may define the even process for appropriate vertices, we clearly can find a copy of the cycle as a subgraph of $K_{r}(n, r, s)$.

Hence solving inequality (1) for $k$ gives a lower bound for the odd girth of $K(n, r, s)$,

$$2k + 1 \geq 2\left\lfloor \frac{r-s}{n-2(r-s)} \right\rfloor + 1,$$

which holds for any valid parameters $n, r$ and $s$.

To show that in fact we have equality, provided that $n$ is large enough, we must find a copy of $C_{2k+1}$ embedded in an appropriate Kneser graph.

Before describing the construction, we must first observe some relationships between $n, r, s$ and $k$. Firstly, we may assume that

$$n \geq 2(r-s) + \left\lceil \frac{(r-s)}{k} \right\rceil.$$  \hfill (2)

For suppose the contrary. Then $n < 2(r-s) + (r-s)/k$, and so

$$\left\lfloor \frac{r-s}{n-2(r-s)} \right\rfloor = k > \left\lceil \frac{r-s}{(r-s)/k} \right\rceil = k,$$

giving a contradiction.

In similar vein, it is easy to see that $n \geq 2(r-s) + \left(\frac{s(r-s)}{(r-2s)}\right)$ is sufficient to ensure that we have

$$\frac{(r-s)}{k} \geq s.$$  \hfill (3)

We can now demonstrate the construction.

Let $Y = \{s+1, \ldots, n\}$. We define sets $X_{1}, X_{2}, \ldots, X_{2k+1}$, which will form a $(2k+1)$-cycle in $K(n, r, s)$, as follows.

Let us choose sets $A \in Y^{(r-s)}$ and $B \in (Y \setminus A)^{(r-s)}$ and set

$$X_{1} = \{1, 2, \ldots, s\} \cup A \quad \text{and} \quad X_{2k+1} = \{1, 2, \ldots, s\} \cup B.$$  

We partition each of $A$ and $B$ into $k$ disjoint parts, each part being of cardinality either $\lceil (r-s)/k \rceil$ or $\lceil (r-s)/k \rceil$ in such a way that $A = A_{1} \cup A_{2} \cup \cdots \cup A_{k}$, $B = B_{1} \cup B_{2} \cup \cdots \cup B_{k}$ and $|A_{i}| = |B_{i}|$, for $i = 1, \ldots, k$. Now let

$$X_{2j+1} = \{1, 2, \ldots, s\} \cup \bigcup_{i=1}^{j} B_{i} \cup \bigcup_{i=j+1}^{k} A_{i}, \quad 1 \leq j \leq k - 1.$$  

We observe that, with these definitions,

$$r + \lceil (r-s)/k \rceil \geq |X_{2j+1} \cup X_{2j+3}| \geq r + \lceil (r-s)/k \rceil, \quad 0 \leq j \leq k - 1,$$

and so, by (2),

$$|\{1, \ldots, n\} \setminus (X_{2j+1} \cup X_{2j+3})| \geq n - r - \left\lfloor \frac{(r-s)}{k} \right\rfloor \geq r - 2s.$$  

Also, by (3), we have that

$$|X_{2j+1} \setminus X_{2j+3}| = |X_{2j+3} \setminus X_{2j+1}| = |A_{j+1}| \geq \lfloor (r-s)/k \rfloor \geq s.$$  

Thus, we may define the even $X$’s by choosing

$$C_{i} \in (X_{2j+1} \setminus X_{2j+3})^{(s)}, \quad D_{i} \in (X_{2j+3} \setminus X_{2j+1})^{(s)}.$$
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Figure 1. An example of $C_5$ embedded into $K(10, 6, 2)$.

\[ E_j \in \{(1, \ldots, n) \setminus (X_{2j+1} \cup X_{2j+3})\}^{(r-2s)}, \]

for each $0 \leq j \leq k-1$ and defining

\[ X_{2j+2} = C_j \cup D_j \cup E_j, \quad 0 \leq j \leq k-1. \]

Then $X_1, \ldots, X_{2k+1}$ form an embedding of $C_{2k+1}$ in $K(n, r, s)$ and the result follows.

In Figure 1 we show an example of this construction to embed a 5-cycle in the graph $K(10, 6, 2)$.

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References


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