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1. Introduction

In [2], I claim to prove the following theorem (Theorem 7 of [2]):

**Theorem 1.** The fine Isbell topology on \( C(X, \mathcal{P}_+ (Y)) \) is the topological modification of \( c_+ \).

Unfortunately, the proof for this alleged theorem is incorrect and the theorem is, in fact, false, unless one assumes that \( X \) is regular.

To see the error in the proof of [2], the reader can go to p. 343 where I define \( O = \bigcup_{x \in \pi x (M)} (\bigcap_{i \in S_x} U_{x, T_i}) \times (Y \setminus \bigcup_{i \in S_x} T_i) \). I claim the set \( O \) is disjoint from \( M \). However, simple examples show that these sets need not be disjoint.

We give a counter-example showing the theorem is indeed false for general topological spaces. We then state and prove a correct version of the theorem for regular spaces. Since the theorem is true for regular spaces and is only applied to regular spaces in [2], all other results of [2] are valid. In what follows we will use the notations and definitions from [2]. The only exception being that we will not use \( T(c_+) \) to denote the fine Isbell topology. The symbol \( T(c_+) \) will only denote the topological modification of \( c_+ \).

2. The counter-example

We will need the following propositions about compact families, both of which have simple proofs.
Proposition 2. ([2]) Let $C$ be a compact family on $X$ and $D \subseteq C$. The family $C_D = \{ C \in C : C \cap D \subseteq C \}$ is a compact family.

Proposition 3. ([1]) Let $C$ be a compact family on $X$ and $A$ be a closed set in $X$ such that $|A| \neq C$. The collection $A \cup C = \{ C \cap A : C \in C \}$ is a compact family on $A$.

Note that in [1] it is actually shown that the $X$-open supersets of $A \cup C$ form a compact family on $X$, however the same idea applies to show that $A \cup C$ is a compact family on $A$.

In [3] an example is given of two consonant zero-dimensional spaces $(X_0, \tau_0)$ and $(X_1, \tau_1)$ whose topological sum $X_0 \oplus X_1$ is not consonant. In [2], it is shown that there is a compact family $C$ on $X_0 \oplus X_1$ such that for every compact $K \subseteq X_0 \oplus X_1$ there is an open $U$ such that $K \subseteq U \subseteq C$. Let $(X, \tau)$ be the topological space whose underlying set is $X_0 \cup X_1$ and has base $\tau_0 \cup \{ X_0 \cup U : U \in \tau_1 \}$. Let $Y = \{ y_0, y_1, y_2 \}$ be the three point discrete space.

Let $S$ be the collection of all $f \in C(X, \mathcal{P}_+(Y))$ such that there exists a $C \in C$ such that $f(x) \subseteq \{ y_0 \}$ for every $x \in X_0 \cap C$ and $f(x) \subseteq \{ y_0, y_1 \}$ for every $x \in X_1 \cap C$.

Claim 1. $S \in T(c_+)$.

Proof. Let $f \in S$ and $\mathcal{F}$ be a filter on $C(X, \mathcal{P}_+(Y))$ such that $f \in \mathrm{lim}_{\mathcal{F}} \mathcal{F}$. Let $C \in C$ be such that $f(x) \subseteq \{ y_0 \}$ for every $x \in C \cap X_0$ and $f(x) \subseteq \{ y_1, y_2 \}$ for every $x \in C \cap X_1$. For every $x \in X_0 \cap C$ there is an $X$-open neighborhood $U_x$ of $x$ and an $F_x \in \mathcal{F}$ such that $g(u) \subseteq \{ y_0 \}$ for every $u \in U_x$ and $g \in F_x$. For every $x \in X_1 \cap C$ there is an $X$-open neighborhood $U_x$ of $x$ and an $F_x \in \mathcal{F}$ such that $g(u) \subseteq \{ y_0, y_1 \}$ for every $u \in U_x$ and $g \in F_x$. Since $\{ X_0 \cap U_x : x \in X_0 \cap C \} \cup \{ X_1 \cap U_x : x \in X_1 \cap C \}$ is an $(X_0 \oplus X_1)$-open cover of $C$, there exist finite sets $A_0 \subseteq X_0 \cap C$ and $A_1 \subseteq X_1 \cap C$ such that

$$C_1 := \left( \bigcup_{x \in A_0} (X_0 \cap U_x) \right) \cup \left( \bigcup_{x \in A_1} (X_1 \cap U_x) \right) \subseteq C.$$  

For every $x \in U_x \cap C_1$ and $g \in \bigcap_{x \in A_0} F_x$ we have $g(u) \subseteq \{ y_0 \}$ and for every $u \in X_1 \cap C_1$ and $g \in \bigcap_{x \in A_1} F_x$ we have $g(u) \subseteq \{ y_0, y_1 \}$. Thus, $\bigcap_{x \in A_0 \cup A_1} F_x \subseteq S$. Therefore, $S \in T(c_+)$. □

Define $f : X \to \mathcal{P}_+(Y)$ by

$$f(x) = \begin{cases} \{ y_0 \} & \text{if } x \in X_0, \\ \{ y_0, y_1 \} & \text{if } x \in X_1. \end{cases}$$

Notice that $f$ is upper semicontinuous and $f \in S$.

By way of contradiction, assume $S$ is open in the fine Isbell topology. In this case, there exist open subsets $O_1, \ldots, O_k$ of $X \times Y$ and compact families $D_1, \ldots, D_k$ on $X$ such that $f \in \bigcap_{i=1}^k [D_i, O_i] \subseteq S$. Let $1 \leq i \leq k$. We say $i$ is of type 1 provided that $D \cap X_1 \neq \emptyset$ for every $D \in D_i$ such that $f(D) \subseteq O_i$. We say $i$ is of type 2 provided that $i$ is not of type 1. If $i$ is type 1, then pick any $D_i \subseteq D_i$ such that $f(D) \subseteq O_i$. If $i$ is type 2, then we may pick a $D_i \subseteq D_i$ such that $f(D) \subseteq O_i$ and $D_i \subseteq X_0$.

Claim 2. For every $1 \leq i \leq k$ there exists a compact set $K_i \subseteq D_i$ such that $[E_i, O_i] \subseteq [D_i, O_i]$, where $E_i$ stands for the collection of all $X$-open supersets of $K_i$.

Proof. Let $1 \leq i \leq k$.

Case 1: $i$ is of type 2.

In this case, $f(D) \subseteq O_i$ and $D_i \subseteq X_0$. It is easily checked that $\{ U \subseteq D_i : U \subseteq D_i \}$ is a compact family on $D_i$. Since $X_0$ is consonant and open subsets of regular consonant spaces are again consonant, $D_i$ is consonant. Thus, there is a compact $K_i \subseteq D_i$ such that every $D_i$-open superset of $K_i$ is in $D_i$. Let $E_i$ denote the collection of $X$-open supersets of $K_i$. Let $W$ be an $X$-open superset of $K_i$. Since $W \cap D_i$ is a $D_i$-open superset of $K_i$, $W \cap D_i \subseteq D_i$. So, $W$ in $D_i$. Thus, every $X$-open superset of $K_i$ is in $D_i$. Since $E_i \subseteq D_i$, $[E_i, O_i] \subseteq [D_i, O_i]$.

Case 2: $i$ is of type 1.

By definition of type 1, there is an $x \in D_i \cap X_1$. Since $f(D) \subseteq O_i$, there is an open neighborhood $L$ of $x$ and an open $V$ such that $f(x) \subseteq V \cap L \subseteq V \subseteq L$. Since $L \cap X_1 \neq \emptyset$, $X_0 \subseteq L$. Since $f(x) = \{ y_0, y_1 \}$, $y_0 \subseteq V$. Thus, $X_0 \times \{ y_0, y_1 \} \subseteq O_i$. Let $E \in (D_i)_0$. Since $E \cap D_i \subseteq D_i$ and $f(E) \subseteq O_i$, $f(E \cap D_i) \subseteq O_i$. Since $f(E \cap D_i) \subseteq O_i$, $E \cap D_i \cap X_1 \neq \emptyset$, by definition of type 1. Let $E \cap X_1 \neq \emptyset$. Thus, $(D_i)_0 \cap X_1$. Since $X_1$ is closed and $(D_i)_0 \cap X_1$ is a compact family on $X_1$ that does not have $\emptyset$ as an element. It is easy to check that the collection $A_i = \{ U \cap X_1 : U \cap X_1 \subseteq D_i \}$ is a compact family on $X_1$. Since $X_1$ is consonant and $D_i \cap X_1$ is open in $X_1$, $D_i \cap X_1$ is consonant. So, there is a nonempty compact $K_i \subseteq X_1 \cap D_i$ such that every $(D_i \cap X_1)$-open superset of $K_i$ is in $A_i \subseteq X_1 \cap (D_i)_0$. Let $E_i$ denote the collection of $X$-open supersets of $K_i$. Let $U$ be an $X$-open superset of $K_i$. Since $X_1 \cap U \neq \emptyset$ and $X_1 \cap U$ is an $X_1$-open superset of $K_i$, $X_0 \subseteq U$ and $X_1 \cap U \subseteq X_1$; $X_1 \cap U \subseteq (D_i)_0$, respectively. Since $X_1 \cap U \subseteq X_1 \cap (D_i)_0$ and $X_0 \subseteq U$, it follows that $U \subseteq (D_i)_0 \subseteq D_i$. Since $E_i \subseteq D_i$, $[E_i, O_i] \subseteq [D_i, O_i]$. □
By our choice of $C$, there is an $X_0 \oplus X_1$-open set $Z$ such that $\bigcup_{i=1}^k K_i \subseteq Z$ and $Z \not\in C$. Define $g : X \to \mathcal{P}_+(Y)$ by

$$g(x) = \begin{cases} 
\{y_0\} & \text{if } x \in Z \cap X_0, \\
\{y_0, y_1\} & \text{if } x \in X_0 \setminus Z, \\
\{y_0, y_1\} & \text{if } x \in Z \cap X_1, \\
Y & \text{if } x \in X_1 \setminus Z.
\end{cases}$$

It is easily checked that $g$ is upper semicontinuous.

**Claim 3.** $g \in S$.

**Proof.** Let $1 \leq i \leq k$.

Suppose $i$ is of type 2. In this case, $D_i \subseteq X_0$ and $K_i \subseteq D_i \cap Z$. Since $D_i \cap Z$ is open in $X_0 \oplus X_1$ and $D_i \cap Z \subseteq X_0$, $D_i \cap Z$ is open in $X$. Since $D_i \cap Z$ is open in $X$ and $K_i \subseteq D_i \cap Z$, $D_i \cap Z \subseteq K_i \cap D_i \subseteq D_i \subseteq D_i \cap Z$. Since $g((Z \cap D_i)) = f((Z \cap D_i)) \subseteq f(D_i \cap Z) \subseteq O_i$, $g((Z \cap D_i)) \subseteq O_i$. So, $g \in [D_i, O_i]$.

Suppose $i$ is of type 1. Since $Z \cap D_i$ is $(X_0 \oplus X_1)$-open, $X_0 \cap (Z \cap D_i)$ is $X$-open. Since $K_i \subseteq X_0 \cap (Z \cap D_i)$, $X_0 \cap (Z \cap D_i) \subseteq K_i \cap D_i \subseteq D_i \subseteq D_i \cap Z$. Suppose $x \in X_0 \cap (Z \cap D_i)$. In this case, $g(x) \subseteq \{y_0, y_1\}$. Thus, $g((X_0 \cap (Z \cap D_i))) \subseteq O_i$. So, $g \in [D_i, O_i]$.

Since $g \in S$, there is a $C \subseteq C$ such that $g(x) \subseteq \{y_0\}$ for every $x \in X_0 \cap C$ and $g(x) \subseteq \{y_0, y_1\}$ for every $x \in X_1 \cap C$. By the way we defined $g$, $C \subseteq C$. So, $C$ is a compact family. Let $\mathcal{L}$ be the collection of sets formed by taking finite unions of sets of the form $\bigcup_{i=1}^k K_i \subseteq Z$ and $Z \not\in C$. Define $j : X \to \mathcal{L}$ by

$$j_U(x) = \begin{cases} 
Y & \text{if } x \notin U, \\
((X \times Y) \setminus H)(x) & \text{if } x \in U.
\end{cases}$$

Since $U$ is open and $(X \times Y) \setminus H$ is upper semicontinuous, it follows that $j_U$ is upper semicontinuous.

Let $\mathcal{L}$ be the collection of all open sets $L$ such that $j_L \in S$. Notice that $X \in \mathcal{L}$, since $j_X = (X \times Y) \setminus H \in S$. So, $\mathcal{L}$ is not empty.

We now show that $\mathcal{L}$ is a compact family. Let $L \in \mathcal{L}$ and $U$ be an open set such that $L \subseteq U$. Since $j_U \leq j_L \in S$, $j_U \in S$. So, $U \in \mathcal{L}$. Thus, $\mathcal{L}$ is closed under open supersets. Since $\mathcal{L}$ is closed under open supersets, if $\emptyset \in \mathcal{L}$, then $\mathcal{L}$ is compact. So, we may assume $\emptyset \notin \mathcal{L}$. Suppose $L \in \mathcal{L}$ and $U$ is an open cover of $L$. Notice that $U \neq \emptyset$, since $L \neq \emptyset$. Without loss of generality, we may assume that $U$ is closed under finite unions. For each $U \in \mathcal{L}$ let $B_U = \{j_W : U \subseteq W$ and $W$ is open$\}$. Let $B = \{B_U : U \in \mathcal{L}\}$. Notice that $B$ is a filter base.
We claim that $j_L \in \lim_{c^+} B$. Let $x \in X$ and $V$ be an open set such that $j_L(x) \subseteq V$. Assume $x \notin L$. In this case, $V = Y$. Let $U_0 \in \mathcal{U}$ be arbitrary. Now, $g(z) \subseteq Y = V$ for every $g \in B_{U_0}$ and $z \in X$. Assume that $x \in L$. In this case, there is a $U_1 \in \mathcal{U}$ such that $x \in U_1$. Since $j_L$ is upper semicontinuous, there is an open set $U_2 \subseteq U_1 \cap L$ such that $x \in U_2$ and $j_L(z) \subseteq V$ for every $z \in U_2$. Let $g \in B_{U_1}$. There is an open $W$ such that $U_1 \subseteq W$ and $g = j_W$. For every $u \in U_2$ we have $g(u) = j_W(u) = j_L(u) \subseteq V$. So, $g(u) \subseteq V$ for all $u \in U_2$ and $g \in B_{U_1}$. Thus, $j_L \in \lim_{c^+} B$.

Since $j_L \in S$, there is a $U \in \mathcal{U}$ such that $B_U \subseteq S$. In particular, $j_U \in S$. So, $U \in \mathcal{L}$. Thus, $\mathcal{L}$ is compact.

Since $X \in \mathcal{L}$ and $f|X = f \ll O$, we have $f \in [\mathcal{L}, O]$. It remains to show that $[\mathcal{L}, O] \subseteq S$. Let $g \in [\mathcal{L}, O]$. There is an $L \in \mathcal{L}$ such that $g|L \ll O$. So, $g|L \subseteq O|L = ((X \times Y) \setminus \text{cl}(H))|L \subseteq ((X \times Y) \setminus H)|L$. Hence, $g \subseteq j_L$. Since $j_L \in S$, $g \in S$. Thus, $[\mathcal{L}, O] \subseteq S$. $
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References