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A characterization of submanifolds by a homogeneity condition

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Abstract

A very short proof of the following smooth homogeneity theorem of D. Repovs, E.V. Shchepin and the author is presented. Let N be a locally compact subset of a smooth manifold M. Assume that for each two points $x, y \in N$ there exists a diffeomorphism $h: M \to M$ such that $h(x) = y$ and $h(N) = N$. Then N is a smooth submanifold of M.

MSC: primary 57R50; secondary 53A04, 54H11, 58A05

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Which shape could have a sheath so that it would be possible to draw a sabre out of it? Mathematical formulation of this question leads to the following notion. A subset N of the 3-dimensional (or m-dimensional Euclidean) space is called *Riemannian ambient homogeneous* if for each two points $x, y \in N$ there exists an isometry $h: \mathbb{R}^3 \to \mathbb{R}^3$ that maps x to y and N to N. It is well known that *each Riemannian ambient homogeneous curve in the* 3*-dimensional space is either a straight line or a circle or a spiral line.*

Which shape could have a cable so that it would be possible to draw a wire out of it (it is allowed to flex a wire but not to break it)? Mathematical formulation of this question leads to the following notion. A subset N of a smooth manifold M is called *smoothly ambient homogeneous* if *for each two points $x, y \in N$ there exists a diffeomorphism $h: M \to M$ that maps x to y and N to N*. (The theorem and applications below are interesting and non-trivial even for the case when $M = \mathbb{R}^m$ or even when M is the plane.) In this paper 'smooth' means 'differentiable'; see though remarks at the end.

Any smooth submanifold of a smooth manifold (in particular, a graph of a smooth function $\mathbb{R} \to \mathbb{R}$) is smoothly ambient homogeneous. In this note the converse is proved. The proof is simpler than in [10,11,9] although it uses the same ideas. (An elementary exposition for a particular case is given in [7].)

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Theorem. *Let N be a locally compact subset of a smooth manifold M. If N is smoothly ambient homogeneous then N is a smooth submanifold of M.*

Applications.

(1) If a graph of a continuous function $\mathbb{R}^{m-1} \to \mathbb{R}$ is smoothly ambient homogeneous, then the function is smooth. (A function having infinite derivative at some point is differentiable at this point.)
(2) The Cantor set cannot be smoothly ambient homogeneously embedded into $\mathbb{R}^m$.
(3) It is known that manifolds are homogeneous and that converse is false (the Cantor set is a counterexample). The theorem shows that the property of being a *submanifold* is equivalent to the *ambient homogeneity property.* Cf. [4].
(4) Using the theorem it is convenient to prove that some groups are Lie groups. E.g., it implies the Cartan theorem stating that *any closed subgroup of a Lie group is a Lie group.*
(5) For applications to the Hilbert–Smith conjecture see [11,8,12].
(6) The theorem allows to reduces of the following result [6, Theorem 3 on pp. 208–209] to its simpler case $m = 1$ (i.e. to the Cauchy equation $h(s + t) = h(s) + h(t)$): *if a one-parameter group $\{h^t\}_{t\in\mathbb{R}}$ of diffeomorphisms of an* m-dimensional manifold depends continuously on a parameter t, then it depends smoothly on the parameter. (V.I. Arnold suggested the result as a problem in 1980s.)

Proof of the theorem. The property of being a smooth submanifold in M is a local one. Hence we may assume that $M = \mathbb{R}^m$.

Let B_l^{m+1} be the interior of an m-ball of radius $1/l$ without the centre. Let $B_l^0 := \emptyset$. For $1 \leqslant k \leqslant m$ let B_l^k the open cone over the $(1/l^2)$-neighborhood of $(k-1)$-hemisphere in the $(m-1)$-sphere of radius $1/l$:

$$B_l^k := \left\{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid -l^2 x_k < |x| < 1/l \text{ and } l^2|x_i| < |x| \text{ for } k < i \leqslant m\right\},$$

where $|x| = \sqrt{x_1^2 + \cdots + x_m^2}$. (It is convenient to use the same notation B_l^k for different types of spaces.) Denote by O_m the group of orthogonal transformations of $\mathbb{R}^m$.

Take the greatest $k \geqslant 0$ such that

$$\text{for each } x \in N \text{ there exist } l \text{ and } A \in O_m \text{ such that } (x + AB_l^k) \cap N = \emptyset. \qquad (*)$$

(Informally this means that N is '$(m-k)$-dimensionally Lipschitz'.) Such k exists because $(*)$ is true for $k = 0$. See Fig. 1, in which $m = 2$ and $k = 1$.

If $k = m + 1$, then N consists of isolated points and the theorem is proved. Hence we may assume that $k \leqslant m$.

Take a sequence $\{A_l\}$ everywhere dense in O_m. Denote

$$B_l := A_l B_l^k \quad \text{and} \quad N_l := \left\{x \in N \mid (x + B_l) \cap N = \emptyset\right\}.$$

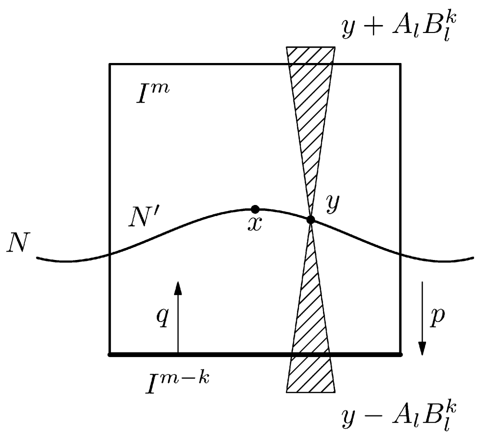

Fig. 1.

By $(*)$ we have $N = \bigcup_{l=1}^{\infty} N_l$. It is easy to check that N_l *is closed in* N (see the details in [11, Lemma 3.1]). Therefore by the Baire Category theorem some N_l contains a nonempty open in N set.

So there exists a point $x \in N$ and an m-dimensional cube I^m of diameter less than $1/l$ and with the centre x, for which $N' := N \cap I^m \subset N_l$. Then

$$\big[(y + B_l) \cup (y - B_l)\big] \cap N' = \emptyset \quad \text{for each } y \in N' \tag{$**$}$$

(Fig. 1). Indeed, if $z \in (y - B_l) \cup N'$, then $y \in (z + B_l) \cap N'$, which is impossible.

Since N is locally compact, we may assume that N' is compact. Also we may assume that $I^m = I^{m-k} \times I^k$ so that I^k is parallel to the k-dimensional plane $A_l(\mathbb{R}^k \times \vec{0})$. Let $p : I^m \to I^{m-k}$ be the projection.

Case 1. *$p(N')$ contains open in I^{m-k} set.* (This is automatically true for $k = m$, when everything is already clear by $(**)$.) We may assume that this set is I^{m-k} itself (by changing I^m to the product of a part of this set with I^k). By $(**)$ N' is a graph of a Lipschitz map $q : I^{m-k} \to I^k$ (Fig. 1). Therefore q has a differentiability point [3, Theorem 3.1.6]. Now the smooth ambient homogeneity implies that q is differentiable. Hence by smooth ambient homogeneity N is a smooth submanifold.

Case 2. *$p(N')$ does not contain any open in I^{m-k} set.* (Then $k < m$.) It follows that there exists a point $a \in I^{m-k} - p(N')$ close enough to the centre of I^{m-k}. Since $p(N')$ is compact, the distance from a to $p(N')$ is non-zero and there is a point $z \in N'$ such that $|a - p(z)|$ equals to this distance. Then the open ball $D \subset I^{m-k}$ with the centre a and radius $|a - p(z)|$ does not intersect $p(N')$. So $p^{-1}(D) \cap N' = \emptyset$. Clearly,

$$(z + B_l) \cup (z - B_l) \cup p^{-1}(D) \supset z + A_l B_s^{k+1} \quad \text{for some } s.$$

This and $(**)$ imply that $(z + A_l B_s^{k+1}) \cap N = \emptyset$. Since N is smoothly ambient homogeneous, it follows that for each $x \in N$ there exists a diffeomorphism $h : \mathbb{R}^m \to \mathbb{R}^m$ mapping z to x and N to N. Then

$$h\big(N \cap (z + A_l B_s^{k+1})\big) \supset x + A B_u^{k+1} \quad \text{for some } A \in O_m.$$

Hence $(*)$ remains true if we replace k by $k + 1$. This contradicts to the maximality of k. $\square$

Note that the Cantor set can be *continuously* or *Lipschitz* ambiently homogeneously embedded into the plane and $\mathbb{R}^m$ [5], see also [1,2]. Hence the analogues of the theorem in continuous or Lipschitz categories are false. V.I. Arnold conjectured that *the analogue of the theorem holds in analytic category.* It would be *very* interesting *to have a complete proof* that the analogue of the theorem holds in the C^r-category for each $r \geqslant 2$ (note that the argument for $r = 1$ in [11] needs some extra effort to make it work for the case $r \geqslant 2$).

The theorem (with an analogous proof) remains valid if by an ambient differentiable homogeneity we understand the following property: *for each two points $x, y \in N$ there exist their neighborhoods Ux and Uy in M and a diffeomorphism $h : Ux \to Uy$ that maps x to y and $Ux \cap N$ to $Uy \cap N$.*

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