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Semilocal and semiregular group rings

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Throughout this paper a ring will mean an associative ring with identity $1 \neq 0$. The Jacobson radical of a ring R is denoted by $J(R)$. A ring R is semilocal if $R/J(R)$ is Artinian. R is called semiperfect if $R/J(R)$ is Artinian and idempotents can be lifted modulo $J(R)$. An element a of a ring R is said to be regular (in the sense of von Neumann) if $a\,ra = a$ for some $r \in R$. If each element of R is regular, R is said to be a regular ring. A ring R is called semiregular if $R/J(R)$ is regular and idempotents can be lifted modulo $J(R)$. Regular rings and semiperfect rings are clearly semiregular. Further it is known that endomorphism rings of injective modules are semiregular. For a more detailed study of semiregular rings and related topics, we refer to [3] and [4]. Following [2] we call a ring R left weakly perfect if and only if R satisfies the minimum condition on principal right ideals which are not direct summands. Weakly perfect rings are semiregular [2, Theorem 3].

In Section 1, we prove that if ${}_RR$ is a direct summand of ${}_RS$ (where R is a subring of S) and if S is a semilocal ring then R is also a semilocal ring. Some important applications to group rings are given. Semiregular rings and group rings are studied in Section 2 and in Section 3 we have studied weakly perfect group rings.

Notation and basic facts involving group rings are taken from [7] and [6].

1. In general, subrings of semilocal rings need not be semilocal. Our Theorem 1 below is a useful observation in this direction. It is not hard to see that a ring R is semilocal if and only if $J(R)$ is a finite intersection of maximal right ideals of R.

THEOREM 1. Let S be a semilocal ring and let R be a subring of S with the same 1. Suppose that R is a direct summand of S as left R-modules. Then R is a semilocal ring.

PROOF. $\{WS+J(S)/J(S) \mid W$ is an intersection of finitely many maximal right ideals of $R\}$ is a non-empty family of right ideals of the Artinian ring $S/J(S)$ and therefore possesses a minimal member, say, $W_0S+J(S)/J(S)$. We claim that $J(R)=W_0$. Certainly $J(R)\subseteq W_0$, so if $J(R)\neq W_0$, then there exists a maximal right ideal M of R such that W_0 is not contained in M. So we have $R=W_0+M$. Further from the minimality of $W_0S+J(S)/J(S)$, we have

$$W_0S+J(S)=(W_0\cap M)S+J(S).$$

From above

$$\begin{aligned}1&=w_0+m,\ w_0\in W_0 \text{ and } m\in M.\\ &=s+t+m,\ s\in (W_0\cap M)S \text{ and } t\in J(S).\end{aligned}$$

Therefore $1+MS=t+s+m+MS=t+MS$, since $s+m\in MS$. This implies that $1-t\in MS$, but $t\in J(S)$, hence $1-t$ is invertible and $MS=S$. This is a contradiction, because MS will be proper as R is a direct summand of S as left R-modules. Therefore $J(R)=W_0$, which is a finite intersection of maximal right ideals of R. Thus R is semilocal.

Let AG denote the group ring of the group G over the ring A. Theorem 1 has some interesting applications to group rings. W. Burgess, J. Valette, J.M. Goursaud, S.M. Woods, J. Lawrence and others have contributed significantly to the problem of semilocal and semi-perfect group rings (for references, see [6]). Following corollary to Theorem 1 is very important and it also generalizes and simplifies some of the results obtained by Goursaud and Lawrence. Perhaps, first it has been observed by S.M. Woods.

COROLLARY 1. If the group ring AG is semilocal, then so is AH, for every subgroup H of G.

From this we obtain a completely different proof of S.M. Woods important Lemma [6], Chapter 10, Lemma 1.6].

COROLLARY 2. Let K be a field and let the group algebra KG be semilocal, then G is torsion.

PROOF. Let $x\in G$ and let $H=\langle x\rangle$, then by Corollary 1, KH is semilocal. Suppose if possible, H is infinite then $J(KH)=0$. So KH is Artinian and therefore H is finite, a contradiction. Thus H is finite and G is torsion.

COROLLARY 3. Let F be a subfield of the field K and let KG be semilocal, then FG is semilocal.

PROOF. Immediate from Theorem 1.

2. F-Semiperfect rings of [2] are precisely semiregular rings. Oberst and Schneider have shown that a ring R is F-semiperfect (semiregular) if and only if every finitely presented left (right) R-module has a projective cover [2]. The following characterization, although implicit in [3], has not previously been stated explicitly. Because of its importance, we give here a direct proof.

PROPOSITION 1. The following are equivalent for a ring R:

1) R is semiregular.
2) For every element a in R there exists an element b in R such that $(ab)^2 = ab$ and $a - aba \in J(R)$.
3) R/aR has a projective cover for every $a \in R$.

PROOF. 1)$\Rightarrow$2). Let $a \in R$, then $a - axa \in J(R)$ for some $x \in R$. Now $(ax)^2 - ax \in J(R)$, so $e - ax \in J(R)$ for some $e \in R$ with $e^2 = e$. Put $1 - e + ax = u$ and $b = xeu^{-1}$, then $ab = ueu^{-1}$ is an idempotent. It is easily seen that $ax - ab \in J(R)$ and therefore $a - aba = a - axa + (ax - ab)a$ belongs to $J(R)$.

2)$\Rightarrow$3). Let $a \in R$, then $(ab)^2 = ab$ and $a - aba \in J(R)$ for some $b \in R$. So $R = abR \oplus (1 - ab)R = aR + (1 - ab)R$. It can be seen that $aR \cap (1 - ab)R \subseteq J(R)$ and hence $aR \cap (1 - ab)R$ is small (superfluous) in R. Thus $0 \to aR \cap (1 - ab)R \to \to (1 - ab)R \to R/aR \to 0$ is a projective cover of R/aR.

3)$\Rightarrow$1). Let $0 \to K \to P \to R/aR \to 0$ be a projective cover of R/aR. Also $0 \to aR \to R \to R/aR \to 0$ is exact, hence by [1, Lemma 2.3] $R = P \oplus P'$ with $P' \subseteq aR$ and $aR \cap P$ superfluous in P. In fact $aR \cap P$ will be superfluous in R and so $aR \cap P \subseteq J(R)$. Let $P' = eR$ and $P = (1 - e)R$ for suitable $e \in R$ with $e^2 = e$. Now $R = P' \oplus P = aR + (1 - e)R$, going modulo $J(R)$, we have $\bar{R} = \bar{a}\bar{R} + (\bar{1} - \bar{e})\bar{R}$. This sum is direct because $e \in aR$ and $aR \cap (1 - e)R \subseteq J(R)$. Thus $R/J(R)$ is regular.

Suppose $a^2 - a \in J(R)$, then put $f = e + ea(1 - e)$ where e is as above. It is easy to see that $f^2 = f$ and $f - a \in J(R)$.

REMARK 1. This proposition implies that each homomorphic image of a semiregular ring is semiregular.

We now turn to group rings. Let K be any field and let G be a group then $\omega(KG)$ will denote the augmentation ideal of the group algebra KG. If H is a subgroup of G then $\omega H = \omega(KH) \cdot KG$ as in [7].

The N^*-radical of a ring R is defined by

$$N^*(R) = \{\alpha \in R \mid \alpha S \text{ is nilpotent for all finitely generated subrings } S \subseteq R\}.$$

For the group algebra KG, we have

$$N^*(KG) = \{\alpha \in KG \mid \alpha\, KH \text{ is nilpotent for all finitely generated subgroups } H \text{ of } G\}.$$

For details see [6, chapter 8].

LEMMA 1. Let G be a group and let K be a field such that $J(KG) = N^*(KG)$. If $KG/J(KG)$ is regular then G is locally finite.

PROOF. If char $K=0$, then $N^*(KG)$ being nil, we have $J(KG)=0$ and KG is regular, so G is locally finite. Assume char $K=p$, then by [6, chapter 8, Theorem 2.6], we have

$$\begin{aligned} J(KG) &= N^*(KG) \\ &= J(K\Lambda^+(G))\cdot KG \\ &\subseteq \omega(K\Lambda^+(G))\cdot KG \\ &= \omega\Lambda^+(G). \end{aligned}$$

So we have,

$$\begin{aligned} K(G/\Lambda^+(G)) &\cong KG/\omega\Lambda^+(G) \\ &\cong KG/J(KG)/\omega\Lambda^+(G)/J(KG). \end{aligned}$$

But $KG/J(KG)$ is regular, hence the group algebra $K(G/\Lambda^+(G))$ is regular. This implies that $G/\Lambda^+(G)$ is locally finite, but $\Lambda^+(G)$ is always locally finite, hence G is locally finite.

REMARK 2. It is known that $J(KG)=N^*(KG)$ if G is locally finite or linear or solvable group. In general, it is an open problem [6, Chapter 8]. A group, whose every finitely generated subgroup is solvable, is called locally solvable. It is easily seen that if G is locally solvable then also $J(KG)=N^*(KG)$.

REMARK 3. If $J(KG)=N^*(KG)$, then $J(KG)$ is nil and so idempotents can be lifted modulo $J(KG)$. Thus if $J(KG)=N^*(KG)$ then KG is semiregular if and only if $KG/J(KG)$ is regular. If G is linear, solvable, or locally solvable then KG semiregular implies G is locally finite.

EXAMPLE 1. Let K be a field with char $K=p$ and let G be a universal locally finite group having p-elements then $J(KG)=0$ [6, Chapter 9, Theorem 4.8, Cor. 4.10] but KG is not regular. Thus G locally finite does not imply that KG is semiregular.

PROPOSITION 2. Let G be a nilpotent group and let K be any field then KG is semiregular if and only if G is locally finite.

PROOF. From Remark 3, it follows that KG semiregular implies G is locally finite, because G is nilpotent. Conversely suppose G is locally finite and nilpotent. If char $K=0$, then KG is regular. So assume that char $K=p$. Let G^p be the unique Sylow p-subgroup of G, then G/G^p is a locally finite p'-group. Hence by [6, Chapter 7, Theorem 2.10], we have

$$J(KG)=J(KG^p)\cdot KG=\omega(KG^p)\cdot KG=\omega G^p.$$

Thus $K(G/G^p)\cong KG/J(KG)$. So $KG/J(KG)$ is regular, since G/G^p is a locally finite p'-group. Also G is locally finite so $J(KG)=N^*(KG)$. So KG is semiregular.

3. Let us recall [2] that a ring R is left weakly perfect if and only if R

satisfies the minimum condition on principal right ideals which are not direct summands, in other words, for every strictly descending chain $a_1R \supset a_2R \supset \supset a_3R \supset \ldots$, with $a_i \in R$, almost all a_n are regular elements. In this section we shall study weakly perfect group rings.

PROPOSITION 3. Let H be a subgroup of a group G and K be a field. If KG is left weakly perfect then so is KH.

PROOF. Let $a_1KH \supset a_2KH \supset a_3KH \supset \ldots$, $a_i \in KH$, be a strictly descending chain of principal right ideals in KH, then $a_1KG \supset a_2KG \supset a_3KG \supset \ldots$ will be a strictly descending chain in KG. Since KG is left weakly perfect, a_n is regular in KG for all $n \geq m$, for some fixed integer m. Let $a_n = a_n r a_n$, $r \in KG$ then $a_n = a_n(r_1 + r_2)a_n$, $r_1 \in KH$ and Supp $r_2 \cap H = \emptyset$. Thus $a_n - a_n r_1 a_n = a_n r_2 a_n = 0$, since $a_n - a_n r_1 a_n \in KH$ and Supp $a_n r_2 a_n \cap H = \emptyset$ as $a_n \in KH$. Hence a_n is regular in KH for all $n \geq m$ and KH is left weakly perfect.

LEMMA 2. Let R be a ring such that $J(R)$ is non-zero and left T-nilpotent. Then

1) There exists $\alpha \in J(R)$, $\alpha \neq 0$ with $\alpha J(R) = 0$.
2) $N(R) \neq 0$.
3) If $N(R)$ is nilpotent, then $J(R) = N(R)$.

PROOF. Same as the proof of Lemma 1.2 in [6, Chapter 10].

LEMMA 3. Let G be a group and let K be a field such that $J(KG)$ is left T-nilpotent then $J(KG) = N(KG) = N^*(KG)$ and $J(KG)$ is nilpotent.

PROOF. If $J(KG) = 0$, then the result is trivial. If char $K = 0$, then left T-nilpotency of $J(KG)$ will imply that $J(KG) = 0$. So we may assume that Char $K = p$ and $J(KG) \neq 0$. By Lemma 2 there exists $\alpha \in J(KG)$, $\alpha \neq 0$ with $\alpha J(KG) = 0$. Since $N(KG) \subseteq J(KG)$, so $\alpha N(KG) = 0$. $N(KG)$ is nilpotent [6, Chapter 8, Theorem 1.12]. Rest follows from Lemma 2.

THEOREM 2. Let G be a group and let K be a field. If char $K = 0$ then KG is left weakly perfect if and only if G is locally finite. If char $K = p$ and KG is left weakly perfect then G is a locally finite group having subgroups P and H such that

1) P is finite normal p-subgroup of H.
2) $|G:H| < \infty$.
3) $J(K(H/P)) = 0$.

PROOF. If KG is left weakly perfect, then $J(KG)$ is left T-nilpotent and $KG/J(KG)$ is regular [2, Theorem 3]. So if char $K = 0$, then $J(KG) = 0$ and KG is regular which implies that G is locally finite. Conversely if char $K = 0$ and G is locally finite then KG is regular. But regular rings are left weakly perfect, hence KG is left weakly perfect.

Now if char $K=p$ and KG is left weakly perfect, then $J(KG)$ is left T-nilpotent as seen above. So by Lemma 3, $J(KG)=N(KG)$ and $J(KG)$ is nilpotent. Also $KG/J(KG)$ is regular, hence by Lemma 1, G is locally finite. The result follows, now, from [6, Chapter 8, Corollary 1.14].

EXAMPLE 2. [5, Theorem 21.6]. Let $G=A\sim B$ where A and B are locally finite p-groups with $A\neq\langle 1\rangle$, and with B infinite. Further let char $K=p$ then

$$J(KG)=\omega G=\{\sum_x a_x\cdot x \mid \sum_x a_x=0\}$$

and $N(KG)=0$. For an example take $A=\mathbb{Z}_p$ and $B=\prod \mathbb{Z}_p$, an infinite direct product of copies of $\mathbb{Z}_p$.

This gives an example of a local ring KG which is not left weakly perfect.

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