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Fermion doubling and Berenstein–Maldacena–Nastase correspondence

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Abstract

We show that the string bit model suffers from doubling in the fermionic sector. The doubling leads to strong violation of supersymmetry in the limit $N \to \infty$. Since there is an exact correspondence between string bits and the algebra of BMN operators even at finite N, doubling is expected also on the side of super-Yang–Mills theory. We discuss the origin of the doubling in the BMN sector.

1. Introduction

Large N physics [1,2] plays an important role in the correspondence between Yang–Mills theory and strings (see [3] for a review). Recently, Berenstein–Maldacena–Nastase (BMN) advocated in [4–6] that IIB superstring theory on *pp*-wave background can be described in terms of a particular set of operators (BMN operators) having large R-charge J in super-Yang–Mills theory. The *pp*-wave background [7–9], appears as a Penrose limit of anti-de Sitter (AdS) space. Therefore, the BMN correspondence can be seen as a limit of the AdS/CFT correspondence [10,11]. The peculiarity of this background is that string theory can be solved there [12,13].

The BMN correspondence was conjectured to hold in the limit of large (infinite) J and N, the quantity $g_{\mathrm{BMN}} = J^2/N$ being the effective coupling of the string interaction. For finite values of J and N, however, the dynamics of BMN operators was shown to be equivalent to the string bit model [14–18]. The string bit model was introduced by Thorn [14], as a supersymmetric mechanical model describing fragmented superstring. For earlier works dealing with the discretization of one-dimensional superspace, see [19–21].

On the other hand, it is not difficult to see that the above string fragmentation corresponds to the lattice discretization of the string. In the Green–Schwarz approach, IIB superstring contains, besides the bosonic fields, also fermionic ones in target space representation.[2] It is well known, however, that the lattice

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[2] For a useful parametrization of the scalar superfields involved, see [22].

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formulation of systems that include fermions possesses a strong drawback, related to a fermion doubling problem (see, e.g., [23]). In particular, the lattice formulation of *supersymmetric theories* faces the problem of fermion doubling (see [24,25] and references therein for a review of the problem and subsequent developments). Thus, one may expect that the string bit model is spoiled by fermion doubling too. Below, we show that this is indeed the case.

Having in mind the abovementioned fact, together with the assumption that the string bit model describes *exactly* the BMN operator dynamics, we can conjecture that there are wrong fermionic modes in the BMN sector of super-Yang–Mills theory, which survive in the large N (or large J) limit. This conjecture is supported also by the fact that computations in the BMN approach are essentially the same as in matrix theory, while at the same time in the latter one can find traces of the fermionic doubling [26].

The plan of the Letter is as follows. Firstly, we briefly review the string bit model in both Hamiltonian and Lagrangian approaches. In Section 3 we analyze the fermionic spectrum and discover the low energy fermionic states corresponding to large lattice momenta (the edge of the Brillouin zone). Next, we solve the equations of motion for the bits, which allows one to quantize the model the same way, as it was done in the case of the continuous string. In Section 4 we discuss the fate of supersymmetry in the case of the fermionic doubling. Using a simplified version of sting bits as a toy-model, we show that the contribution of the fermionic mirror states leads to a strong violation of supersymmetry, in the continuum limit. After that, we show how doubling states can appear in the BMN correspondence and, finally, we discuss the results.

2. Bit string model

Let us shortly review the pp-wave IIB superstring bit model [15]. The superstring consisting of J bits is described in terms of phase space coordinates of its bits and their superpartners: $\{p_n^i, x_n^i, \theta_n^a, \tilde\theta_n^a\}$, where $n = 0, \dots, J-1$. The phase space variables satisfy the following (classical) commutation relations:

$$\left[p_n^i, x_n^i\right] = \delta^{ij}\delta_{mn}, \qquad \left\{\theta_n^\alpha, \theta_m^\beta\right\} = \frac{\mathrm{i}}{2}\delta^{ab}\delta_{mn},$$
$$\left\{\tilde\theta_n^a, \tilde\theta_m^b\right\} = \frac{\mathrm{i}}{2}\delta^{ab}\delta_{mn}. \tag{1}$$

The reparametrization invariance of the string becomes, in the bit language, the invariance with respect to the symmetry group S_J of permutations of the labels n. The whole permutation group S_J is split into equivalence classes $[\gamma]$ of permutations having the same number of cycles with fixed lengths $J_1, J_2, \dots, J_s$, $\sum J_k = J$. Then, the Hilbert space of the quantized model can be split, according to this, into a direct sum of *twisted sectors* $\mathcal{H}_\gamma$, associated with each conjugacy class $[\gamma]$. The transformations inside a conjugacy class reduces to relabeling of bits in the cycles. Therefore, the twisted sector corresponding to the conjugacy class $[\gamma_s]$ with s cycles, can be identified with the s-string Hilbert space. In particular, one string sector corresponds to cyclic subgroups of S_J, and up to relabeling of n is given by

$$\gamma_1(n) = n + 1 \bmod J. \tag{2}$$

In the case of the s-string sector, one can introduce the following standard γ_s transformation by fixing the representant of the conjugacy class $[\gamma_1]$. For this, let us relabel $0 \leqslant n \leqslant J-1$ by sets of bits $\{n_k;\ 0 \leqslant n_k \leqslant J_k - 1,\ k = 1, \dots, s\}$ and define

$$\gamma_s = \gamma_1^{(1)}\gamma_1^{(2)}\cdots\gamma_1^{(s)},$$
$$\gamma_1^{(k)}(n_k) = n_k + 1 \bmod J_k. \tag{3}$$

Since the conjugation transformations preserve the cyclic structure of γ_s (including the lengths of the cycles), just changing the labels [27], in order to get the arbitrary representative γ_s' of the conjugacy class, it suffices to replace each bit label by a permutation: $n \mapsto \sigma(n)$. Among the permutations, however, there are some which do not change the cycles, thus leaving us with the same γ_s.

The Hamiltonian and supercharges describing the model read as follows [15]:

$$H = H_B + H_F, \tag{4a}$$
$$Q = \sum_{n=0}^{J-1}\left[a\left(p_n^i\gamma_i\theta_n - x_n^i\gamma_i\Pi\tilde\theta_n\right) + \left(x_{\gamma(n)}^i - x_n^i\right)\gamma_i\theta_n\right], \tag{4b}$$

$$\tilde{Q} = \sum_{n=0}^{J-1} \left[a\left(p_n^i \gamma_i \tilde{\theta}_n - x_n^i \gamma_i \Pi \theta_n\right) - \left(x_{\gamma(n)}^i - x_n^i\right) \gamma_i \tilde{\theta}_n \right], \tag{4c}$$

where

$$H_B = \sum_{n=0}^{J-1} \left[\frac{a}{2}\left(p_{in}^2 + x_{in}^2\right) + \frac{1}{2a}\left(x_{\gamma(n)}^i - x_n^i\right)^2 \right], \tag{5}$$

$$H_F = -\mathrm{i} \sum_{n=0}^{J-1} \left[(\theta_n \theta_{\gamma(n)} - \tilde{\theta}_n \tilde{\theta}_{\gamma(n)}) - 2a \tilde{\theta}_n \Pi \theta_n \right]. \tag{6}$$

In the Lagrangian form the action compatible with the Hamiltonian (4a) and the commutation relations (1) are given by

$$\begin{aligned} S = \sum_{n=0}^{J-1} \Bigg[& \frac{a}{2} \dot{x}_{in}^2 - \frac{1}{2a}\left(x_{\gamma(n)}^i - x_n^i\right)^2 - \frac{a}{2} x_{in}^2 \\ & + \mathrm{i}a(\theta_n \dot{\theta}_n + \tilde{\theta}_n \dot{\tilde{\theta}}_n) + \mathrm{i}(\theta_n \theta_{\gamma(n)} - \tilde{\theta}_n \tilde{\theta}_{\gamma(n)}) \\ & + 2\mathrm{i}a \tilde{\theta}_n \Pi \theta_n \Bigg]. \end{aligned} \tag{7}$$

The expressions (4a)–(4c) correspond to a discrete version of IIB superstring in pp-wave background [12,13], obtained by the most straightforward (naive) discretization. Since this naively discretized model contains fermions, one should expect problems typical of lattice fermions.

3. Fermion doubling

Let us analyze the fermionic spectrum in the free string bit model. In order to do this, let us consider the fermionic part (6) of the Hamiltonian and "diagonalize" it. For this, let us perform the bit (lattice) Fourier transform of the fields

$$\theta_n = \frac{1}{\sqrt{J}} \sum_{p=-J/2}^{J/2} \theta_p \mathrm{e}^{2\pi \mathrm{i} ln/J}, \tag{8a}$$

$$\tilde{\theta}_n = \frac{1}{\sqrt{J}} \sum_{p=-J/2}^{J/2} \tilde{\theta}_p \mathrm{e}^{2\pi \mathrm{i} pn/J}, \tag{8b}$$

where, by abuse of notations, we kept the same character for both the field and its Fourier transform, distinguishing them only by the labels

$$l, m, n, \ldots = 0, 1, \ldots, J-1 \quad (x\text{-representation}), \tag{9}$$

$$p, q, r, \ldots = -J/2, -J/2+1, \ldots, J/2 \quad (p\text{-representation}). \tag{10}$$

From (10) one can notice that for odd J the "momenta" $p, q, r, \ldots$ run through integer numbers, while for an even J value, they should be half-integer. This has no particular meaning and is a result of the choice for the origin of the momentum space, which in the present case was taken to be symmetric with respect to the inversion of momenta $p \to -p$.

Let us consider, for definiteness, the one string sector and fix the standard choice (2) for the "moduli" of γ permutation. As we discussed above, all other situations in the same class $[\gamma]$ are obtained from the standard one by all possible relabeling of bits $n' = n'(n)$. (The other multi-string sectors can be analyzed in a similar way, by fixing the "standard" γ-permutations to (3), and then "shuffling" the labels, in order to generalize the result to arbitrary γ_s.)

Plugging the transformations (8a), (8b) in the fermionic Hamiltonian (6), yields the expression

$$\begin{aligned} H_F &= \sum_p \sin\left(\frac{2\pi p}{J}\right)(\theta_{-p}\theta_p - \tilde{\theta}_{-p}\tilde{\theta}_p) + 2a\mathrm{i}\tilde{\theta}_{-p}\Pi\theta_p \\ &= (\theta_{-p} \quad \tilde{\theta}_{-p}) \begin{pmatrix} \sin\frac{2\pi p}{J} & \mathrm{i}a\Pi \\ -\mathrm{i}a\Pi & -\sin\frac{2\pi p}{J} \end{pmatrix} \begin{pmatrix} \theta_p \\ \tilde{\theta}_p \end{pmatrix}. \end{aligned} \tag{11}$$

It is not difficult to see that the spectrum of the Hamiltonian (11) reads

$$E_p^{(J)} = \pm\sqrt{J^2 \sin^2 \frac{2\pi p}{J} + 1}. \tag{12}$$

As expected, in the limit of large J, one can expand the sin function under the square root, in order to get the continuum energy levels of the fermions[3]

$$\omega_n = E_{p=n \ll J}^{(J)} \approx \pm\sqrt{(2\pi n)^2 + 1}, \tag{13}$$

obtained by Metsaev in [12].

Eq. (13) yields a correct, although incomplete, energy spectrum for the continuous superstring. Due

[3] Notice the difference in notations with the paper [12].

to the other zero of the sin function when its argument approaches $\pm\pi$, there are other low energy levels which survive in the continuum limit $J \to \infty$. They appear when the momentum p is in the vicinity of the edge of the *Brillouin zone*, $p \sim \pm J/2$. This will appear as a 2-fold degeneracy of each energy level in (13), $E_p = E_{J/2-p}$. This phenomenon has been is known for long time in lattice theories with fermions, where it is called fermion spectrum *doubling* (for more details see the textbook [23]).

The doubling can be related to a symmetry of the discrete system of string bits which relates fermionic modes of different chiralities [28]

$$\begin{pmatrix} \theta_n \\ \tilde{\theta}_n \end{pmatrix} \mapsto (-1)^n \begin{pmatrix} \Pi \tilde{\theta}_n \\ \Pi \theta_n \end{pmatrix}. \tag{14}$$

Thus, in the continuum limit, we obtained not just pp-wave IIB superstring but something more, i.e., the Green–Schwarz superstring with two fermionic sectors!

In this context one may ask, what happens to supersymmetry? The short answer is that the lattice theory in fact is not supersymmetric owing to the effects of discreetness. Also due to the doubling, the symmetry has few chances to be restored in the continuum limit!

4. A note on supersymmetry and doubling

In order to illustrate the behavior of supersymmetry on the lattice let us consider a simpler toy model example, which catches however the most important features generic for all supersymmetric models on the lattice.

Let us consider the model of "one-dimensional superstring" described by the continuum action

$$S = \int \mathrm{d}^2\sigma \left(\frac{1}{2} \partial_a X \partial_a X + \frac{\mathrm{i}}{2} \psi \hat{\partial} \psi \right), \tag{15}$$

where X and ψ are, respectively, a bosonic field and a Majorana–Weyl fermion on a two-dimensional cylinder. The action (15) is invariant with respect to the supersymmetry transformations

$$\delta X = -\mathrm{i}\epsilon\psi, \tag{16}$$

$$\delta\psi = \hat{\partial} X \epsilon, \tag{17}$$

where $\hat{\partial}$ is the two-dimensional Dirac operator, $\hat{\partial} = \gamma_a \partial_a$, and ϵ is the supersymmetry transformation parameter, which is a Majorana–Weyl spinor. In the canonical formalism the system is represented by the canonical variables $\Pi(\sigma) = (\partial L / \partial \dot{X})$, $X(\sigma)$ and ψ, satisfying

$$\begin{aligned} &\left[\Pi(\sigma), X(\sigma')\right]_{\mathrm{PB}} = \delta(\sigma - \sigma'), \\ &\left\{\psi_\alpha(\sigma), \psi_\beta(\sigma')\right\}_{\mathrm{PB}} = \frac{\mathrm{i}}{2} \gamma^0_{\alpha\beta} \delta(\sigma - \sigma'), \end{aligned} \tag{18}$$

and the Hamiltonian

$$H = \oint \mathrm{d}\sigma \left(\frac{1}{2} \Pi^2 + \frac{1}{2} (X')^2 + \mathrm{i} \psi \gamma_1 \psi' \right). \tag{19}$$

Supersymmetry is generated by the supercharge

$$Q = \oint \mathrm{d}\sigma \left[-\mathrm{i} \Pi \psi + \mathrm{i} X' \gamma_1 \gamma_0 \psi \right], \tag{20}$$

which satisfies the (classical) algebra,

$$\{Q, Q\} = -2H\gamma_0 + 2P\gamma_1, \tag{21}$$

where H is the Hamiltonian (19) and $P = \Pi X'$ denotes the shift generator. Just like the action, the Hamiltonian (19) is invariant with respect to the supersymmetry transformation

$$\delta H = \epsilon [Q, H] = 0. \tag{22}$$

Let us consider now a version of the above model in the case of a discrete spatial extension $\sigma \equiv \sigma_1$.[4] In order to do this, let us start with the supercharge[5]

$$Q = a \sum_{n=0}^{J} \left(-\mathrm{i} \Pi_n \psi_n + \frac{\mathrm{i}}{a} (X_{n+1} - X_n) \gamma_1 \gamma_0 \psi_n \right). \tag{23}$$

This expression is analogous to the supercharge (4b) and is a straightforward discretization of (20). The discrete Hamiltonian and the shift operator can be defined through the lattice version of Eq. (21). Indeed, for the Hamiltonian one has

$$\begin{aligned} H = a \sum_n \bigg(&\frac{1}{2} \Pi_n^2 + \frac{1}{2a^2} (X_{n+1} - X_n)^2 \\ &+ \frac{\mathrm{i}}{2a} \psi_n \gamma_1 \psi_{n+1} \bigg), \end{aligned} \tag{24}$$

[4] This model suffers from doubling, in the same way as the string bit model in the previous section.

[5] We consider the following discretization of σ: $\sigma = an$, $n = 0, \ldots, J$, and $a = 2\pi L/J$.

while P appears to be the operator of the forward lattice shift, i.e., $P = \sum_n \Pi_n(X_{n+1} - X_n)$.

The above results agree perfectly with what can be expected from a naive discretization of the Hamiltonian (19). However, an unpleasant surprise comes next. The discrete Hamiltonian (24) fails to be exactly supersymmetric! Indeed, a straightforward computation yields

$$\frac{\delta H}{\delta\epsilon} = \sum_n \left[\frac{\mathrm{i}}{2}\left(-\Pi_n\gamma_0 + \frac{1}{a}(X_{n+1} - X_n)\gamma_1 \right) \times \gamma_1(\psi_{n+1} - 2\psi_n + \psi_{n-1}) \right]. \tag{25}$$

For slowly varying fields (which correspond to smooth functions in the continuum limit), this part of the supersymmetry variation is of order $\sim 1/J$ and thus it vanishes, as J approaches infinity. This occurs because the terms in (25) correspond to lattice analogs of second derivatives, multiplied by factors of order $a = 2\pi/J$. In the continuum limit they are supposed to give Lorentz non-invariant terms vanishing like

$$-\frac{\mathrm{i}a}{2}\oint \mathrm{d}\sigma\,(\Pi\gamma_0 - X'\gamma_1)\gamma_1\psi'' \sim O(1/J). \tag{26}$$

This is what would happen, if the doubler states would not come into the game. For the doubler states the fermionic factor in the r.h.s. of (25) is of the order of unity, while the summation adds a factor of order J making the non-invariant contribution divergent. This is in contrast with the situation of the "genuine" non-doubled part, where the fermionic factor is of order $1/J^2$, while the summation just reduces the decay by one power in J. In conclusion, the supersymmetry algebra on the lattice does not close, to ensure the supersymmetry of the Hamiltonian. Moreover, due to the contribution of doubler states, the non-invariant terms do not just fail to vanish in the continuum limit but, on the contrary, they even diverge!

We have considered a simplified toy model related to the bit string. However, this model catches, besides technical details, the crucial properties of the string bit model under study. The result, also, is not an unexpected one. Firstly, because the Poincaré algebra which is important part of the supersymmetry algebra is gravely affected by the discretization. (In our case it is, in fact, reduced to continuous shifts in time and discrete one in the spatial direction, while rotations are completely lost.) It would be at least strange if it were otherwise, because the string bit model can be (consistently if there was no doubling) formulated in any dimension and any background what comes in contradiction with the fact that consistent superstring theories can exist only in very special spaces and backgrounds.[6]

5. Bit string quantization (à la Metsaev)

Let us solve the equations of motion, following [12]. This will allow us to quantize the bit string and understand the phenomenon of doubling.

The equations of motion arising from the action (7) read

$$-\ddot{x}_n^i + \frac{1}{2a^2}(x_{\gamma(n)} - 2x_n + x_{\gamma^{-1}(n)}) - x_n^i = 0, \tag{27}$$

for the bosonic part, and

$$\dot{\theta}_n + \frac{1}{2a}(\theta_{\gamma(n)} - \theta_{\gamma^{-1}(n)}) + \Pi\tilde{\theta} = 0, \tag{28a}$$

$$\dot{\tilde{\theta}}_n - \frac{1}{2a}(\tilde{\theta}_{\gamma(n)} - \tilde{\theta}_{\gamma^{-1}(n)}) - \Pi\theta = 0, \tag{28b}$$

for fermions. Once again, let us limit ourselves to the one-string sector and fix the class $[\gamma_1]$ by the standard choice: $\gamma(n) = n + 1 \bmod J$. As we discussed earlier, the solution corresponding to an arbitrary element of the class is obtained by permutation of labels in the "standard" solution.

The solution to the equations of motion is obtained in a way analogous to that of [12], except the discrete Fourier transform (8a), (8b) is used. In particular, the bosonic part of the solution looks as follows:

$$x_n^i(\tau) = X^i\cos\tau + P^i\sin\tau + \sum_{l=\pm 1,\dots,\pm[J/2]} \frac{1}{\omega_l}\big(\alpha_l^{1i}\hat{\varphi}_{l;n}^1(\tau) + \alpha_l^{2i}\hat{\varphi}_{l;n}^2(\tau)\big), \tag{29}$$

where α_l^{ai} are string mode operators while $\varphi_{l;n}^a(\tau)$ are respective modes of the string

$$\hat{\varphi}_{l;n}^1(\tau) = \exp\big(-\mathrm{i}(\hat{\omega}_l\tau - 2\pi ln/J)\big), \tag{30a}$$

[6] Strictly speaking, this is related not only to the fermion doubling problem but also to violations of conformal symmetry on the lattice.

$$\hat{\varphi}^2_{l;n}(\tau) = \exp\bigl(-\mathrm{i}(\hat{\omega}_l \tau + 2\pi l n/J)\bigr), \tag{30b}$$

and

$$\hat{\omega}_l = \operatorname{sgn} l \sqrt{\hat{k}_l^2 + 1}, \quad \hat{k}_l^2 = \frac{2}{a^2}\left(1 - \cos\frac{2\pi l}{J}\right). \tag{31}$$

Once again, it is not difficult to see that, as $J \to \infty$, $a = 1/J \to 0$, one recovers the solution of [12].

Let us turn now to the fermionic sector. The solution in this sector reads

$$\theta_n(\tau) = \cos\tau\,\Theta + \sin\tau\,\Pi\tilde{\Theta} + \sum_l c_l\bigl(\check{\varphi}^1_{n;l}(\tau)\theta^1_l + \mathrm{i}(\check{\omega}_l - \check{k}_l)\check{\varphi}^2_{n;l}(\tau)\Pi\theta^2_l\bigr), \tag{32a}$$

$$\tilde{\theta}_n(\tau) = \cos\tau\,\tilde{\Theta} + \sin\tau\,\Pi\Theta + \sum_l c_l\bigl(\check{\varphi}^2_{n;l}(\tau)\theta^2_l - \mathrm{i}(\check{\omega}_l - \check{k}_l)\check{\varphi}^1_{n;l}(\tau)\Pi\theta^1_l\bigr), \tag{32b}$$

where, as in the bosonic case, the sum is performed over $l = \pm 1, \dots, \pm[J/2]$, and the fermionic modes $\check{\varphi}^a_{n;l}(\tau)$ are given by the same expressions (30a), (30b), except that the hatted $\hat{\omega}_l$ and $\hat{k}_l$ are replaced by the "checked" ones $\check{\omega}_l$, $\check{k}_l$, given by

$$\check{\omega}_l = \operatorname{sgn} l \sqrt{\check{k}_l^2 + 1}, \quad \check{k}_l = \frac{2}{a}\left(\sin\frac{2\pi l}{J}\right). \tag{33}$$

The peculiarity of the fermionic solution (32a), (32b) is that, owing to the presence of a $\sin^2$ factor (instead of cos, as in the bosonic case), very high fermionic modes $l \sim J/2$ possess the same energy as the modes in the region $l \ll J$. In fact, the modes of the same energy come in pairs $(l, J/2 - l)$, in total accord with the discussion of the previous section. The canonically quantized model is obtained by replacing the Poisson brackets of the oscillator modes generators α^a_l and θ^a_l (where $a = 1, 2$) with the commutation relations [12,13]

$$\bigl[P^i, X^j\bigr] = -\mathrm{i}\delta^{ij},$$

$$\bigl[\alpha^{ai}_l, \alpha^{bj}_m\bigr] = \frac{1}{2}\hat{\omega}\delta^{ab}\delta^{ij}\delta_{m+n,0}, \tag{34}$$

for bosonic modes, and

$$\bigl\{\theta^{a\alpha}_l, \theta^{b\beta}_m\bigr\} = -\frac{1}{4}\delta^{ab}\delta^{\alpha\beta}\delta_{m+n,0}, \tag{35}$$

for fermionic ones.

A note is in order. The solution we found in this section corresponds to a particular choice of the cyclic permutation $\gamma(n)$. As proposed in [15], the physical states of the string bit model are those symmetrized with respect to conjugations of γ, $h^{-1}\gamma h$, or averaged over the conjugacy class of γ. As we noted earlier, going to a different γ, in the same conjugacy class, is equivalent to a permutation of the labels $n \to h(n)$ [27]. Therefore, a solution with a different $\gamma' = h^{-1}\gamma h$ is still given by Eqs. (29) and (32), where now the functions $\varphi_{n;l}$ are replaced by $\varphi_{h(n);l}$. Then, a physical state with B bosonic and F fermionic modes symmetrized over the permutations generically looks as follows:

$$\frac{1}{J!}\sum_{h\in S_J} \alpha^{h^{-1}\gamma h}_{l_1} \cdots \alpha^{h^{-1}\gamma h}_{l_B} \theta^{h^{-1}\gamma h}_{l_1} \cdots \theta^{h^{-1}\gamma h}_{l_F} |0\rangle, \tag{36}$$

where the labels correspond to raising operators. The ground state is unique and invariant with respect to the permutation group S_J, so we do not have to twist the vacuum.

6. BMN correspondence

So far, we observed that the bit string model contains a number of problems like fermion doubling and supersymmetry violation. On the other hand, the bit string model is equivalent to BMN sector of the super-Yang–Mills model at any finite J. This equivalence would imply that the fermionic subsector of the BMN operators is badly defined, at finite J. (Since there is no definition for the BMN correspondence at $J = \infty$, this would signal a self-consistency problem in the BMN correspondence.) Hence, in this section we proceed to the analysis of the implications of fermion doubling at the level of the BMN operators.

The BMN correspondence [4] relates a class of operators in $N = 4$ super-Yang–Mills model, which have a large R-charge ($J \to \infty$), to states in the closed superstring on the pp-wave background. The string "semantics" of the BMN language is as follows. The light-cone superstring vacuum in the BMN language is given by the operator

$$\frac{1}{\sqrt{J}\,N^{J/2}}\operatorname{tr}\bigl[Z^J\bigr] \leftrightarrow |0, p_+\rangle, \tag{37}$$

where Z is the complex scalar component, $Z = (\phi^5 + \mathrm{i}\phi^6)/\sqrt{2}$.

The excited string states correspond to the insertion of "impurities" under the trace (37), according to the following rule:

$$D_\mu Z \leftrightarrow \alpha^{\dagger\mu}, \quad \mu = 1, \dots, 4, \tag{38}$$

$$\phi^{j-4} \leftrightarrow \alpha^{\dagger i}, \quad i = 5, \dots, 8, \tag{39}$$

$$\chi^a_{J=1/2} \leftrightarrow \theta^{\dagger\alpha}, \quad \alpha = 1, \dots, 8, \tag{40}$$

where $\alpha^\dagger$ and $\theta^\dagger$ are, respectively, bosonic and fermionic standard oscillator raising operators. Also, in order to get non-zero string modes, the insertions should be accompanied by a factor $e^{2\pi i kn/J}$, where k is the position of the insertion in the row of Z's. Hence, e.g., a double fermionic insertion corresponds to

$$\frac{1}{\sqrt{J}\,N^{J/2+1}} \sum_{k=0}^{J-1} \mathrm{tr}\big[\chi^\alpha_{J=1/2} Z^k \chi^\beta_{J=1/2} Z^{J-l}\big] e^{2\pi i kl/J} \leftrightarrow \theta_l^{\dagger\alpha} \theta_{-l}^{\dagger\beta} |0, p_+\rangle. \tag{41}$$

Thus, the BMN correspondence (*language*) is formulated in terms of *words* or strings of products of Z's, with insertions of impurities and operators on the space of allowed words. The main operators acting on words are the position X and the shift P. X gives the position j (up to a cyclic permutation) of an insertion in the chain of Z's, while P performs a permutation of the impurity in the jth position to the $(j+1)$th one.

One can define the scalar product of words $\Psi \sim \cdots Z\phi Z \cdots \psi \cdots$, which is given by

$$(\Psi, \Psi') = \langle \overline{\Psi} \Psi' \rangle_{N, J \to \infty}. \tag{42}$$

(The necessary properties required for this to be a scalar product follow from the planar properties of the correlator in the large N limit, [4].) As it can be seen, the shift operator P is not self-adjoint, with respect to the BMN scalar product, and there is an adjoint operator P^+, which corresponds to the backward shift

$$P^+ : j \to j - 1. \tag{43}$$

Since the bosonic interaction comes through the term $\sim g^2_{\mathrm{YM}}\, \mathrm{tr}[Z, \phi][\overline{Z}, \phi]$, this produces in the bosonic part of the effective Hamiltonian a term proportional to P^+P (due to the cyclic property of the trace this is the same as PP^+). On the other hand, the fermionic interactions in super-Yang–Mills theory are linear in the shifts

$$\sim \big(\chi \Gamma_Z [Z, \chi] + \chi \Gamma_{\overline{Z}} [\overline{Z}, \chi]\big). \tag{44}$$

This leads to a contribution proportional to the symmetric part of the shift operator $\sim 1/2(P + P^+)$ in the fermionic part of the effective Hamiltonian. As we observed above, when analyzing the bit string model, this leads to the fermionic spectrum doubling. In terms of the shift operators, this is explained by the existence of such zero modes of $1/2(P + P^+)$, which correspond to highly oscillating modes on the lattice string.

7. Discussion

In this Letter we addressed the problem of finite N effects in the BMN correspondence. For finite N and J, the set of BMN operators maps into the Hilbert space of J string bits. As we have shown above, the fermionic spectrum of the string bit model is doubled. An immediate effect of doubling is the failure to get a supersymmetric limit, as $J \to \infty$.

We considered a free theory and, on this level, one can explicitly separate the contribution of the doubler states, in order to get the correct spectrum of IIB string, as J and N go to infinity. We *believe* that this can also be done on the tree level of interacting closed superstrings. However, as the experience of the lattice shows, in the case of bit loops the doubling states mix with the correct modes.

In spite of above problems, the study of supersymmetric models on the lattice have achieved, during the last several years, a considerable progress (see [24,25,29] for a review).[7] One can hope to apply the technique developed in this approach to string bits too. This is accompanied, however, by the fact that, beyond the typical lattice problem with fermion doubling, there are specifical string problems, related to conformal invariance violation by the string discretization. One can also expect the duality symmetries to be violated too.

Returning to the BMN correspondence, one can see that there is a class of unwanted fermionic states, given by the fermionic doublers, which survive in the (formal) BMN limit. In fact, the BMN sector is known to contain some "extra" states which are conjectured to decouple because of large masses acquired due

[7] While this work was in progress a paper [30] studying topological models on the lattice appeared.

to the interactions [4]. On the other hand, the above arguments about decoupling, used in [4] would hardly apply to fermion doublers, since they propagate and interact exactly in the same way, as the genuine fermionic modes.

The above fact can also signal the presence of the same problem in the fermionic spectrum of the AdS/CFT correspondence, when one tries to obtain such correspondence starting from large but finite values of N. In order to be able to say something more precise, one has to study this topic too. We hope to do this in the future. Perhaps one can avoid the problems, working directly in the model with $N = \infty$, which is the super-Yang–Mills model in non-commutative space. However, in this case, a procedure allowing to get rid of non-planar contributions must be devised and implemented.

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