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THE CONGRUENCE THEORY OF CLOSURE PROPERTIES OF REGULAR TREE LANGUAGES

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Abstract. Boolean operations, tree homomorphisms and their converses, and forest product, in special cases σ-catenation, x-product, x-quotient and x-iteration, preserve the regularity of forests. These closure properties are proved algebraically by using congruences of term algebras which saturate the forests operated on and constructing, by means of them, a congruence which saturates the product forest. The index of the constructed congruence is finite, if the congruences saturating the forests to operate are of finite indexes. The cardinalities of ranked and frontier alphabets are arbitrary. The preservation of recognizability is a straightforward consequence of those congruence constructions and the Nerode type of congruence characterization for recognizable forests. Furthermore, the constructed congruences can also be applied directly to construct explictly tree automatas to recognize the product forests.

1. Introduction

The recognizable forests are defined to be forests recognized by finite tree automata. Deviating from the general custom to define the finite tree automata with finite ranked and frontier alphabets, here they are also allowed to be infinite. The sets of states are of course finite. The proofs of the closure properties can be done by constructing suitable finite recognizers and regular grammars. However, in consequence of the universal algebraic nature of recognizable forests, the proofs are here based on constructions of saturating congruences, and the formalism is a universal algebraic one. We now have a new method to construct the required automata to recognize product forests.

2. Preliminary definitions

Special notation

The set $\{1, 2, \dots\}$ is denoted by $\mathbf{N}$ and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. The notation $\bigcup(A_i \mid i \in I)$ is the union and $\bigcap(A_i \mid i \in I)$ the intersection of an indexed family $(A_i \mid i \in I)$. The set of all equivalence relations on A is denoted by $\mathrm{Eq}(A)$. For $\theta \in \mathrm{Eq}(A)$ and $a \in A$, $a\theta = \{b \mid a\theta b\}$ is the θ-class of an element a. The cardinality of the set of all equivalence classes modulo θ ($|A/\theta|$), the *index* of θ, is denoted by in θ. Furthermore, for any subset B of A, we introduce the notation

$$B\theta = \bigcup (b\theta \mid b \in B).$$

If $T = T\theta$ for $T \subseteq A$, it is said, that θ *saturates* T.

Definition 2.1. An *operator domain* is a pair consisting of a set Σ (of *operators*), and a mapping $r: \Sigma \to \mathbb{N}_0$ that assigns to every $\sigma \in \Sigma$ a finite *arity*, or *rank*, $r(\sigma)$. An operator domain is simply denoted by the set of its operators. For every number $m \in \mathbb{N}_0$, we use the notion

$$\Sigma_m = \{\sigma \in \Sigma \mid r(\sigma) = m\}$$

to symbolize the set of all *m-ary operators* in Σ.

Definition 2.2. A *Σ-algebra* $\mathscr{A}$ is a pair consisting of a nonempty set A and a mapping that assigns to every operator $\sigma \in \Sigma$ an *m-ary operation* on A, $\sigma^{\mathscr{A}}: A^m \to A$, where $m = r(\sigma)$ and A^m is the mth cartesian power of A. We write $\mathscr{A} = (A, \Sigma)$.

Definition 2.3. Let Σ be an operator domain and X a set (of *variables*), disjoint from Σ. The set of *ΣX-terms*, $F_\Sigma(X)$, [3] is the smallest set, such that

$$X \cup \Sigma_0 \subseteq F_\Sigma(X) \quad \text{and} \quad \sigma(t_1, \ldots, t_m) \in F_\Sigma(X)$$

whenever $m \in \mathbb{N}$, $\sigma \in \Sigma_m$ and $t_1, \ldots, t_m \in F_\Sigma(X)$.

Here ΣX-terms are regarded as formal representations of trees, and the set of variables is called a *frontier alphabet*, the operator domain a *ranked alphabet*, the ΣX-terms *ΣX-trees* and any subset of the set $F_\Sigma(X)$ a *ΣX-forest*. If we do not want to specify the alphabets, we shall speak simply about trees and forests. From now on, the cases where $X = \emptyset$ or $\Sigma_0 = \emptyset$ are not specifically handled, but the results hold in these cases, too.

Agreement. The symbols X and Y are reserved for frontier alphabets and the symbols Σ and Ω for ranked alphabets.

Definition 2.4. If $F_\Sigma(X) \neq \emptyset$, the Σ-algebra $\mathscr{F}_\Sigma(X) = (F_\Sigma(X), \Sigma)$ defined so that

$$\sigma^{\mathscr{F}_\Sigma(X)}(t_1, \ldots, t_m) = \sigma(t_1, \ldots, t_m)$$

for all $m \in \mathbb{N}_0$, $\sigma \in \Sigma_m$ and $t_1, \ldots, t_m \in F_\Sigma(X)$, is called the *$\Sigma X$-term algebra*. If there is no danger of confusion, we write simply σ instead of $\sigma^{\mathscr{F}_\Sigma(X)}$.

Definition 2.5. For a ΣX-tree t, the *height* $\mathrm{hg}(t)$, the *root* $\mathrm{root}(t)$ and the *set of subtrees* $\mathrm{sub}(t)$ are defined as follows:

(i) if $t \in X \cup \Sigma_0$, then $\mathrm{hg}(t) = 0$, $\mathrm{root}(t) = t$, and $\mathrm{sub}(t) = \{t\}$;

(ii) if $t = \sigma(t_1, \ldots, t_m)$, where $m \in \mathbb{N}$, $\sigma \in \Sigma_m$ and $t_1, \ldots, t_m$ are ΣX-trees, then

$$\mathrm{hg}(t) = 1 + \max\{\mathrm{hg}(t_i) \mid 1 \leq i \leq m\},$$

$$\mathrm{root}(t) = \sigma \quad \text{and} \quad \mathrm{sub}(t) = \{t\} \cup \bigcup (\mathrm{sub}(t_i) \mid 1 \leq i \leq m).$$

We define for a ΣX-forest T, $\mathrm{root}(T) = \{\mathrm{root}(t) \mid t \in T\}$.

Definition 2.6. Let $(T_x \mid x \in X)$ be an X-indexed family of ΣX-forests. For each subset X' of the frontier alphabet X and each ΣX-tree t, we define the forest $t(x \leftarrow T_x \mid x \in X')$, mostly written simply as $t(x \leftarrow T_x)$, as follows:

(i) if $t \in X'$, then $t(x \leftarrow T_x) = T_t$;

(ii) if $t \in (X \setminus X') \cup \Sigma_0$, then $t(x \leftarrow T_x) = t$;

(iii) if $t = \sigma(t_1, \ldots, t_m)$, where $m \in \mathbb{N}$, $\sigma \in \Sigma_m$ and $t_1, \ldots, t_m$ are ΣX-trees, then

$$t(x \leftarrow T_x) = \{\sigma(s_1, \ldots, s_m) \mid s_i \in t_i(x \leftarrow T_x),\ i = 1, \ldots, m\}.$$

In the case $X' = \{x_1, \ldots, x_n\}$, we also use the notation $t(x_1 \leftarrow T_{x_1}, \ldots, x_n \leftarrow T_{x_n})$ for $t(x \leftarrow T_x \mid x \in X')$, and if $t = \sigma(x_1, \ldots, x_m)$, where $\sigma \in \Sigma_m$, we speak about the *σ-catenation* of the forests $T_{x_1}, \ldots, T_{x_m}$. For a ΣX-forest T, it is defined

$$T(x \leftarrow T_x) = \bigcup (t(x \leftarrow T_x) \mid t \in T).$$

Definition 2.7. Let S and T be ΣX-forests and $z \in X$. The *z-product* of S and T is the forest product

$$S \cdot_z T = T(z \leftarrow S).$$

The *z-quotient* of T by S is the forest

$$S^{-z}T = \{p \in F_\Sigma(X) \mid S \cdot_z p \cap T \neq \emptyset\},$$

where T is called the *dividend* forest. The *z-iteration* of T is the forest

$$T^{*z} = \bigcup (T^{n,z} \mid n \in \mathbb{N}_0),$$

where $T^{0,z} = \{z\}$, and

$$T^{n+1,z} = T^{n,z} \cup T^{n,z} \cdot_z T, \quad n \in \mathbb{N}_0.$$

Definition 2.8. For each number $m \in \mathbb{N}_0$, we introduce a new alphabet (distinct), $\Xi_m = \{\xi_1, \ldots, \xi_m\}$, which is assumed to be disjoint from all ranked alphabets used. Suppose we are given a mapping $h_X : X \to F_\Omega(Y)$, and for each $m \in \mathbb{N}_0$ a mapping

$h_m:\Sigma_m \to F_\Omega(Y \cup \Xi_m)$. The *tree homomorphism* determined by these mappings is the mapping [4] $h: F_\Sigma(X) \to F_\Omega(Y)$ defined such that

(i) $h(x) = h_X(x)$ for each $x \in X$, and

(ii) $h(\sigma(t_1, \ldots, t_m)) = h_m(\sigma)(\xi_1 \leftarrow h(t_1), \ldots, \xi_m \leftarrow h(t_m))$ whenever $m \in \mathrm{N}_0, \sigma \in \Sigma_m$ and $t_1, \ldots, t_m \in F_\Sigma(X)$.

If whenever $m \in \mathrm{N}_0$ and $\sigma \in \Sigma_m$, no letter of the alphabet Ξ_m appears more than once in $h_m(\sigma)$, we call h *linear*. Furthermore, h is said to be *alphabetic*, if $h(X) \subseteq Y$, and for each $m \in \mathrm{N}_0$ and $\sigma \in \Sigma_m$, $h_m(\sigma) = \omega(\xi_1, \ldots, \xi_m)$, where $\omega \in \Omega_m$.

Definition 2.9. A (*frontier-to-root*) *ΣX-recognizer* (a *tree recognizer*) [4] is a triple

$$\boldsymbol{A} = (\mathscr{A}, \alpha, A'),$$

where $\mathscr{A} = (A, \Sigma)$ is a Σ-algebra, $\alpha: X \to A$ is a mapping and A' ($\subseteq A$) is the *set of final states*. If A is finite, we call $\boldsymbol{A}$ a *finite recognizer*. In particular a ΣX-recognizer $(\mathscr{F}_\Sigma(X), 1_X, T)$, where T is a ΣX-forest and $1_X: X \to X$ is an identity mapping, is denoted by $\mathbf{F}_T$.

Definition 2.10. Let $\boldsymbol{A} = (\mathscr{A}, \alpha, A')$ be a ΣX-recognizer and $\hat{\alpha}$ the extension of α to a homomorphism from the term algebra $\mathscr{F}_\Sigma(X)$ to a Σ-algebra $\mathscr{A}$. It is said, that $\boldsymbol{A}$ *recognizes* a ΣX-forest T or that $\boldsymbol{A}$ is a ΣX-recognizer of T, if $T = \{t \in F_\Sigma(X) \mid \hat{\alpha}(t) \in A'\}$. A ΣX-forest, recognized by a finite tree recognizer, is said to be *recognizable*.

Definition 2.11. Let $\boldsymbol{A} = (\mathscr{A}, \alpha, A')$ be a ΣX-recognizer, and θ a congruence of $\mathscr{A}$ saturating the set of final states A'. The *quotient ΣX-recognizer* of $\boldsymbol{A}$ in respect of θ is the ΣX-recognizer

$$\boldsymbol{A}/\theta = (\mathscr{A}/\theta, \alpha_\theta, A'/\theta),$$

where $\alpha_\theta: X \to A/\theta$ is such a mapping that for each $x \in X$, $\alpha_\theta(x) = \alpha(x)\theta$.

Since a forest T is recognized by both $\mathbf{F}_T$ and each of its quotient recognizers [1, 2, 4], we obtain, after generalizing Nerode's theorem on regular languages and right congruences of the free monoid [5], the congruence characterization for recognizable forests [2, 4]. Finiteness of the index of a congruence saturating T is a sufficient and necessary condition for the recognizability of T.

3. The congruence theory of closure properties of regular tree languages

Theorem 3.1. *The union, intersection, difference and complement of two recognizable forests are recognizable* [5, 6].

Proof. Let θ, ρ, $\chi \in \mathrm{Eq}(A)$ and $\theta \subseteq \rho, \chi$. If S and T are subsets of A, such that S is saturated by ρ and T by χ, respectively, then obviously θ saturates the sets $S \cup T$, $S \cap T$, $S \setminus T$ and $A \setminus S$. Now the claim of the theorem follows from the congruence characterization for recognizable forests because a finite intersection of congruences of finite indexes is a congruence of finite index. □

Definition 3.2. For $\theta_1, \theta_2 \in \mathrm{Eq}(F_\Sigma(X))$ we define the equivalence relation on $F_\Sigma(X)$, $\theta_1 * \theta_2$, the *X-product* of θ_1 and θ_2, as follows. Let p and q be ΣX-trees. Then $(p, q) \in \theta_1 * \theta_2$, iff the conditions (i)–(iii) hold:

(i) $p\, \theta_2\, q$;

(ii) for each ΣX-tree s and all ΣX-forests S_x, $x \in X$, such that $q \in s(x \leftarrow S_x)$, there is a ΣX-tree t and ΣX-forests T_x, $x \in X$, for which $t\, \theta_1\, s$, $T_x\theta_2 = S_x\theta_2$ for all $x \in X$, and $p \in t(x \leftarrow T_x)$;

(iii) the condition (ii) holds, if we interchange p and q.

Theorem 3.3. *For a finite X, the X-product of equivalence relations of finite indexes is of finite index. If* $\theta \in \mathrm{Eq}(F_\Sigma(X))$ *saturates a* ΣX*-forest T, and for each* $x \in X$ $\theta_x \in \mathrm{Eq}(F_\Sigma(X))$ *saturates a* ΣX*-forest* T_x*, then* $T(x \leftarrow T_x)$ *is saturated by the X-product of* θ *and* $\bigcap(\theta_x \mid x \in X)$*. The X-product of congruences is a congruence.*

Proof. Let p be a ΣX-tree and $\theta_1, \theta_2 \in \mathrm{Eq}(F_\Sigma(X))$. Whenever $(S_X \mid x \in X)$ is a family of ΣX-forests, we obtain

$$|\{s\theta_1 \mid p \in s(x \leftarrow S_x),\ s \in F_\Sigma(X)\}| \leq \mathrm{in}\, \theta_1,$$

and whenever $y \in X$ and t is a ΣX-tree

$$|\{T_y\theta_2 \mid p \in t(x \leftarrow T_x),\ T_x \subseteq F_\Sigma(X),\ x \in X\}|$$
$$\leq |\{T/\theta_2 \mid T \subseteq F_\Sigma(X)\}| = 2^{\mathrm{in}\,\theta_2}.$$

If $q\, \theta_1 * \theta_2\, p$, then every possible "splitting" of the tree p, $(s\theta_1, (S_x\theta_2 \mid x \in X))$, for which $p \in s\theta_1(x \leftarrow S_x\theta_2)$, must also be a splitting of the tree q, and conversely. Hence by an upper limit of the cardinality of the subsets of possible splitting we obtain

$$\mathrm{in}\, \theta_1 * \theta_2 \leq \mathrm{in}\, \theta_2 (2^{\mathrm{in}\,\theta_1 (2^{\mathrm{in}\,\theta_2})^{|X|}}).$$

This yields the first claim of our theorem.

Let T, θ and T_x, θ_x, $x \in X$, be defined as in the theorem, and let us use $\tilde{\theta}$ to denote the equivalence relation $\bigcap(\theta_x \mid x \in X)$. Let a ΣX-tree p be in the forest $t(x \leftarrow T_x)$, where $t \in T$, and let a ΣX-tree q be in the same $\theta * \tilde{\theta}$-class as p. Therefore there exists a tree s and X-indexed forests $(S_x \mid x \in X)$, such that $s\, \theta\, t$, $S_x\tilde{\theta} = T_x\tilde{\theta}$, $x \in X$, and $q \in s(x \leftarrow S_x)$.

Because $t \in T$ and θ saturates T, s must be in T. Furthermore according to our statement that, for each $x \in X$, θ_x saturates T_x, we thus obtain $T_x = T_x\tilde{\theta} = S_x\tilde{\theta}$ whenever $x \in X$. Hence

$$q \in T(x \leftarrow S_x) \subseteq T(x \leftarrow S_x\tilde{\theta}) = T(x \leftarrow T_x).$$

Thus the forest $T(x \leftarrow T_x)$ is shown to be saturated by $\theta * \tilde{\theta}$.

Then, let θ_1 and θ_2 be congruences of $\mathscr{F}_\Sigma(X)$, $m \in \mathrm{N}$, $\sigma \in \Sigma_m$ and let $p_1, \dots, p_m$, $q_1, \dots, q_m$ be such ΣX-trees that $p_i\, \theta_1 * \theta_2\, q_i$, $i = 1, \dots, m$. We use the notation

$$p = \sigma(p_1, \dots, p_m) \quad \text{and} \quad q = \sigma(q_1, \dots, q_m).$$

Definition 3.2 yields $p_i\, \theta_2\, q_i$, $i = 1, \dots, m$, and since θ_2 is a congruence, condition (i) of Definition 3.2 holds for the pair (p, q).

Condition (ii) is proved by induction on the tree s. Let $q \in s(x \leftarrow S_x)$, where $\mathrm{hg}(s) = 0$. Since $\mathrm{hg}(q) > 0$, s must be in X. Obviously $q \in S_s$. On the other hand $p \in s(x \leftarrow T_x)$, where

$$T_x = \begin{cases} S_x & \text{for } x \neq s, \\ S_s \cup \{p\} & \text{for } x = s. \end{cases}$$

We still have $p\, \theta_2\, q$, hence $T_s\theta_2 = S_s\theta_2$. Thus, in consequence of the fact that $s\, \theta_1\, s$, condition (ii) is fulfilled for the pair (p, q).

Then let $q \in s(x \leftarrow S_x)$, where $\mathrm{hg}(s) > 0$. Now it follows from the definition of q, that for each index $i \in \{1, \dots, m\}$ there must exist such a ΣX-tree s_i that $q_i \in s_i(x \leftarrow S_x)$, and furthermore $s = \sigma(s_1, \dots, s_m)$. Therefore, on the basis of our statement $p_i\, \theta_1 * \theta_2\, q_i$, $i = 1, \dots, m$, we also must have for each $i \in \{1, \dots, m\}$ a ΣX-tree t_i and a family of forests $(T_{xi} \mid x \in X)$, such that $t_i\, \theta_1\, s_i$, $T_{xi}\theta_2 = S_x\theta_2$, $x \in X$, and $p_i \in t_i(x \leftarrow T_{xi})$.

By using the notion $t = \sigma(t_1, \dots, t_m)$, we obtain $t\, \theta_1\, s$, because θ_1 is a congruence. Furthermore

$$\bigcup (T_{xi} \mid 1 \leqslant i \leqslant m)\theta_2 = S_x\theta_2, \quad x \in X,$$

and

$$p \in t(x \leftarrow \bigcup (T_{xi} \mid 1 \leqslant i \leqslant m)).$$

Therefore t fulfills condition (ii). This finally shows, that condition (ii) holds for the pair (p, q).

Since $q_i\, \theta_1 * \theta_2\, p_i$, $i = 1, \dots, m$, it can be proved in a way analogous with the previous argument, that (ii) holds for the pair (q, p). Hence (iii) holds for the pair (p, q). This finally shows that $p\, \theta_1 * \theta_2\, q$, and hence the X-product preserves congruence. □

Corollary 3.4. *If X is finite, then $T(x \leftarrow T_x)$ is a recognizable forest whenever T and the forests T_x are recognizable* (*cf.* [4, 5, 6]).

Proof. A finite intersection of congruences of finite indexes is a congruence of finite index. So, according to the congruence characterization for recognizable forests, the claim of our corollary is a straightforward consequence of Theorem 3.3. □

Definition 3.5. Let $x \in X$ and $\theta \in \mathrm{Eq}(F_\Sigma(X))$. Let us define the equivalence relation on $F_\Sigma(X)$, θ^{*x}, the *x-power* of θ, as follows. For each ΣX-tree p and q, $(p, q) \in \theta^{*x}$, iff the conditions (i)–(iii) hold:

(i) $p\,\theta\,q$;

(ii) for each ΣX-tree s and ΣX-forest S, such that $q \in S^{*x} \cdot_x s$, there is a ΣX-tree t and a ΣX-forest T, for which $t\,\theta\,s$, $T\theta = S\theta$, and $p \in T^{*x} \cdot_x t$;

(iii) condition (ii) holds, if we exchange p and q.

Analogously with the proof of Theorem 3.3 we obtain Theorem 3.6, because Definitions 3.2 and 3.5 are much alike. Hence by the congruence characterization we obtain Corollary 3.7 directly.

Theorem 3.6. *Let $x \in X$. The x-powers preserve finiteness of index. The x-iteration of a forest is saturated by the x-power of such an equivalence relation that saturates the forest. The x-power of a congruence is a congruence.*

Corollary 3.7. *Let $x \in X$. The x-iterations preserve recognizability* (*cf.* [4, 5, 6]).

Definition 3.8. For all $x \in X$, $S \subseteq F_\Sigma(X)$ and $\theta \in \mathrm{Eq}(F_\Sigma(X))$, let us define an equivalence relation on $F_\Sigma(X)$, $\theta(x, S)$, the *x-S-quotient* of θ, such that for any p, $q \in F_\Sigma(X)$, $p\,\theta(x, S)\,q$, iff

$$(S \cdot_x p)/\theta = (S \cdot_x q)/\theta.$$

Theorem 3.9. *Let $x \in X$ and $S \subseteq F_\Sigma(X)$. The x-S-quotients preserve finiteness of index. For $T \subseteq F_\Sigma(X)$ the x-quotient $S^{-x}T$ is saturated by the x-S-quotient of such an equivalence relation that saturates T. The x-S-quotient of a congruence is a congruence.*

Proof. Let $x \in X$, $S \subseteq F_\Sigma(X)$ and $\theta \in \mathrm{Eq}(F_\Sigma(X))$. The first claim is evident, because in $\theta(x, S) \leq 2^{\mathrm{i}.\theta}$.

Then let T be some ΣX-forest saturated by θ, a ΣX-tree p in the *x-quotient* $S^{-x}T$ and a ΣX-tree q in the same $\theta(x, S)$-class as p. It follows that $S \cdot_x p \cap T \neq \emptyset$ and $(S \cdot_x p)\theta = (S \cdot_x q)\theta$, hence $S \cdot_x q \cap T \neq \emptyset$, because $T = T\theta$. This yields that q is also in $S^{-x}T$. Thus we have shown the second claim of the theorem to be true.

Let θ then be a congruence of $\mathcal{F}_\Sigma(X)$, $m \in \mathbb{N}$, $\sigma \in \Sigma_m$ and let $p_1, \ldots, p_m, q_1, \ldots, q_m$ be such ΣX-trees that $p_i\,\theta(x, S)\,q_i$, $i = 1, \ldots, m$. If

$$p = \sigma(p_1, \ldots, p_m) \quad \text{and} \quad q = \sigma(q_1, \ldots, q_m),$$

then

$$\begin{aligned}(S \cdot_x p)/\theta &= \sigma(x_1, \ldots, x_m)(x_1 \leftarrow S \cdot_x p_1, \ldots, x_m \leftarrow S \cdot_x p_m)/\theta \\ &= \{\sigma(t_1, \ldots, t_m)/\theta \mid t_1 \in S \cdot_x p_1, \ldots, t_m \in S \cdot_x p_m\} \\ &= \{\sigma^{\mathcal{F}_\Sigma(X)/\theta}(u_1, \ldots, u_m) \mid u_1 \in S \cdot_x p_1/\theta, \ldots, u_m \in S \cdot_x p/\theta\} \\ &= \{\sigma^{\mathcal{F}_\Sigma(X)/\theta}(u_1, \ldots, u_m) \mid u_1 \in S \cdot_x q_1/\theta, \ldots, u_m \in S \cdot_x q_m/\theta\} \\ &= (S \cdot_x q)/\theta.\end{aligned}$$

It follows that $p\,\theta(x,S)\,q$, and therefore $\theta(x,S)$ is a congruence. $\square$

As a straightforward consequence of Theorem 3.9 and the congruence characterization, we obtain the next result.

Corollary 3.10. *Let $x \in X$. The x-quotients preserve recognizability, if the dividend forest is recognizable* (*cf.* [4, 6]).

Definition 3.11. For any ΩY-tree p and any tree homomorphism $h: F_\Sigma(X) \to F_\Omega(Y)$, we introduce the set of such ΣX-trees which h maps to p like a "generalized" alphabetic tree homomorphism:

$$\begin{aligned}\mathrm{al}(p,h) = \{t \in F_\Sigma(X) \mid\, & h(t) = p, (\forall x \in (X \cap \mathrm{sub}(t)))\, \mathrm{hg}(h(x)) = 0,\\ & (\forall \sigma \in (\Sigma_0 \cap \mathrm{sub}(t)))\, \mathrm{hg}(h_0(\sigma)) = 0, (\forall \sigma \in \mathrm{root}(\mathrm{sub}(t) \backslash (X \cup \Sigma_0)))\\ & h_{r(\sigma)}(\sigma) \in \{\omega(\xi_{i_1}, \ldots, \xi_{i_n}) \mid \omega \in \Omega_n, n \in \mathrm{N}, 1 \leqslant i_1, \ldots, i_n \leqslant r(\sigma)\}\}.\end{aligned}$$

The tree homomorphism $h: F_\Sigma(X) \to F_\Omega(Y)$ is called *regular for the equivalence relation* θ ($\in \mathrm{Eq}(F_\Sigma(X))$), if whenever p is an ΩY-tree, the converse of h satisfies the inclusion

$$h^{-1}(p) \subseteq \mathrm{al}(p,h)\theta.$$

For all equivalence relations $\theta \in \mathrm{Eq}(F_\Sigma(X))$ and tree homomorphism $h: F_\Sigma(X) \to F_\Omega(Y)$, let us define an equivalence relation on $F_\Omega(Y)$, $\theta(h)$, the *h-map* of θ, so that whenever $p, q \in F_\Omega(Y)$, then $p\,\theta(h)\,q$, iff

$$\mathrm{al}(p,h)/\theta = \mathrm{al}(q,h)/\theta.$$

Theorem 3.12. *Let $h: F_\Sigma(X) \to F_\Omega(Y)$ be a tree homomorphism. The h-maps preserve finiteness of index. If h is regular for an equivalence relation which saturates a forest T, then $h(T)$ is saturated by the h-map of the equivalence relation. If h is linear, the h-map of a congruence is also a congruence.*

Proof. Let $\theta \in \mathrm{Eq}(F_\Sigma(X))$ and $h: F_\Sigma(X) \to F_\Omega(Y)$ be a tree homomorphism. The first claim of our theorem is evident as in $\theta(h) \leqslant 2^{\mathrm{in}\,\theta}$.

Let then $f: F_\Sigma(X) \to F_\Omega(Y)$ be a tree homomorphism, regular for θ. Let T be a ΣX-forest saturated by θ, p as a ΩY-tree in the forest $f(T)$ and q a ΩY-tree in the same $\theta(f)$-class as p. Since $\mathrm{al}(p,f) \subseteq f^{-1}(p)$ we obtain on the basis of Definition 3.11 that

$$\mathrm{al}(q,f)\theta = \mathrm{al}(p,f)\theta = f^{-1}(p)\theta.$$

Since $f^{-1}(p) \cap T \neq \emptyset$ and $T = T\theta$, $\mathrm{al}(q,f) \cap T \neq \emptyset$, and therefore besides p, the tree q is in the forest $f(T)$. This concludes the proof of the second claim of our theorem.

Then suppose θ is a congruence of $\mathscr{F}_\Sigma(X)$ and $h: F_\Sigma(X) \to F_\Omega(Y)$ is a linear tree homomorphism. Furthermore, let $m \in \mathbb{N}$, $\omega \in \Omega_m$ and the ΩY-trees $p_1, \ldots, p_m$, $q_1, \ldots, q_m$ be such that $p_i\, \theta(h)\, q_i$, $i = 1, \ldots, m$. This agrees with the following notation:

$$p = \omega(p_1, \ldots, p_m), \qquad q = \omega(q_1, \ldots, q_m),$$

and

$$R = \operatorname{root}(\{\omega(\xi_{i_1}, \ldots, \xi_{i_m}) \in h_n(\Sigma_n) \mid m, n \in \mathbb{N}, \omega \in \Omega_m, 1 \leq i_1, \ldots, i_m \leq n\}).$$

Next we show that if $t_1, \ldots, t_m$ are ΩY-trees and $t = \omega(t_1, \ldots, t_m)$, then $\operatorname{al}(t, h) \neq \emptyset$, iff

$$\omega \in \Omega_m \cap R \quad \text{and} \quad \operatorname{al}(t_i, h) \neq \emptyset, \quad i = 1, \ldots, m. \tag{1}$$

First suppose $\operatorname{al}(t, h) \neq \emptyset$ and $u \in \operatorname{al}(t, h)$. Thus there must exist such $n \in \mathbb{N}$, $\rho \in \Sigma_n$ and such ΣX-trees $u_1, \ldots, u_n$ and indexes $i_1, \ldots, i_m \in \{1, \ldots, n\}$ that

$$h_n(\rho) = \omega(\xi_{i_1}, \ldots, \xi_{i_m}),$$

$$u_{i_j} \in \operatorname{al}(t_j, h), \quad j = 1, \ldots, m,$$

$$u = \rho(u_1, \ldots, u_n).$$

So, condition (1) holds.

Next presume (1). Now there must exist $k \in \mathbb{N}$, $\tau \in \Sigma_k$, indexes $j_1, \ldots, j_m \in \{1, \ldots, k\}$ and ΣX-trees $v_{j_1}, \ldots, v_{j_m}$, such that

$$h_k(\tau) = \omega(\xi_{j_1}, \ldots, \xi_{j_m}),$$

$$v_{j_i} \in \operatorname{al}(t_i, h), \quad i = 1, \ldots, m.$$

The m-tuple $(v_{j_1}, \ldots, v_{j_m})$ is then extended to a k-tuple $(v_1, \ldots, v_k)$ by choosing each v_j, $j \notin \{j_1, \ldots, j_m\}$, arbitrarily from a set $\operatorname{al}(t_i, h)$, $i = 1, \ldots, m$. Because h is linear, we obtain

$$\begin{aligned} h(\tau(v_1, \ldots, v_k)) &= \omega(\xi_{j_1}, \ldots, \xi_{j_m})(\xi_1 \leftarrow h(v_1), \ldots, \xi_k \leftarrow h(v_k)) \\ &= \omega(h(v_{j_1}), \ldots, h(v_{j_m})) \\ &= \omega(t_1, \ldots, t_m) = t. \end{aligned}$$

Now $\tau(v_1, \ldots, v_k) \in \operatorname{al}(t, h)$, and therefore $\operatorname{al}(t, h) \neq \emptyset$.

Suppose then $\omega \in \Omega_m \setminus R$ or $j \in \{1, \ldots, m\}$ is such an index that $\operatorname{al}(p_j, h) = \emptyset$. As $p_i\, \theta(h)\, q_i$, $i = 1, \ldots, m$, we have

$$\operatorname{al}(p, h) = \operatorname{al}(q, h) = \emptyset,$$

hence $p\, \theta(h)\, q$.

Next let $\omega \in \Omega_m \cap R$ and $\text{al}(p_i, h) \neq \emptyset$, $i = 1, \ldots, m$, whence the sets $\text{al}(p, h)$ and $\text{al}(q, h)$ are not empty. Choose $s \in \text{al}(p, h)$. Now we must have $n \in \mathbb{N}$, $\sigma \in \Sigma_n$, ΣX-trees $s_1, \ldots, s_n$ and indexes $i_1, \ldots, i_m \in \{1, \ldots, n\}$ such that

$$h_n(\sigma) = \omega(\xi_{i_1}, \ldots, \xi_{i_m}),$$

$$s_{i_j} \in \text{al}(p_j, h), \quad j = 1, \ldots, m,$$

$$s = \sigma(s_1, \ldots, s_n).$$

On the basis of our statement

$$\text{al}(p_j, h)/\theta = \text{al}(q_j, h)/\theta, \quad j \in \{1, \ldots, m\}.$$

Hence for each index i_j, there must be such a tree $\bar{s}_{i_j} \in \text{al}(q_j, h)$ that $s_{i_j}\, \theta\, \bar{s}_{i_j}$. Let us write $a_{i_j} = \bar{s}_{i_j}$ for $j = 1, \ldots, m$, and $a_i = s_i$ for $i \notin \{i_1, \ldots, i_m\}$. Since θ is a congruence, we obtain $s\, \theta\, \sigma(a_1, \ldots, a_n)$, and evidently $\sigma(a_1, \ldots, a_n) \in \text{al}(q, h)$, because h is linear. Therefore $\text{al}(p, h)/\theta \subseteq \text{al}(q, h)/\theta$.

On the basis of the fact that $p_i\, \theta(h)\, q_i$, $i = 1, \ldots, m$, an analogous argument with the previous one yields $\text{al}(q, h)/\theta \subseteq \text{al}(p, h)/\theta$, as θ is symmetric. So, finally we have $\text{al}(p, h)/\theta = \text{al}(q, h)/\theta$. It follows that $p\, \theta(h)\, q$, and therefore $\theta(h)$ is a congruence. □

Corollary 3.13. *Suppose T is a recognizable ΣX-forest and $h: F_\Sigma(X) \to F_\Omega(Y)$ a linear tree homomorphism. If h is also regular for a congruence of finite index, which saturates T, then $h(T)$ is also recognizable. Particularly alphabetic tree homomorphisms preserve recognizability* (*cf.* [4, 5, 6]).

Proof. Let $h: F_\Sigma(X) \to F_\Omega(Y)$ be an alphabetic tree homomorphism. Hence $\text{al}(p, h) = h^{-1}(p)$ for every ΩY-tree p, and thus h is regular for every equivalence relation on $F_\Sigma(X)$. Obviously h is also linear, and therefore the claims of the corollary are consequences of the congruence characterization and Theorem 3.12. □

By using tree automata, we obtain a known and more general result according to which linear tree homomorphisms always preserve recognizability [4].

Theorem 3.14. *Let $h: F_\Sigma(X) \to F_\Omega(Y)$ be a tree homomorphism and T a ΩY-forest. If T is recognizable, then $h^{-1}(T)$ is also recognizable* (*cf.* [4, 6]).

Proof. Suppose $h: F_\Sigma(X) \to F_\Omega(Y)$ is a tree homomorphism and T a recognizable ΩY-forest. According to the congruence characterization there exists a congruence θ of $\mathcal{F}_\Omega(Y)$ which saturates T and is of finite index. The product $h \cdot \theta \cdot h^{-1}$ of relations h, θ and h^{-1} is obviously a congruence of $\mathscr{F}_\Sigma(X)$ and satisfies $\text{in}(h \cdot \theta \cdot h^{-1}) \leq \text{in}\,\theta$. Let p be some ΣX-tree in the forest $h^{-1}(T)$ and let q be a ΣX-tree in the same $h \cdot \theta \cdot h^{-1}$-class as p. Then $h(p)\,\theta\, h(q)$ and $h(p) \in T$, whence $h(q) \in T$, as $T = T\theta$. It follows that q is also in $h^{-1}(T)$. Hence $h \cdot \theta \cdot h^{-1}$ saturates $h^{-1}(T)$. The congruence characterization implies now that $h^{-1}(T)$ is recognizable. □

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