Maximal functions and weighted norm inequalities on local fields

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**A B S T R A C T**

Weight functions are characterized so that Hardy–Littlewood maximal operator is bounded in certain spaces. The reverse weak type estimates with applications to some singular integrals and to the class \(L(1 + \log^+ L)\) of Zygmund are established. These results are also compared with the ones in Euclidean case which are obtained by K.F. Andersen and W.S. Young, thereby showing the differences between the two cases. We introduce a weak type estimate for a new class of maximal function and employ it to deduce a special result on singular operators over a local field which is obtained by K. Phillips and M. Taibleson.

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1. Introduction

It is well known that the wavelet integral transform is expressed as a pseudodifferential operator, the Radon transform as a wavelet integral transform, the pseudo-differential operator, in many cases, as an inverse Fourier transform, etc., and it is well known also that there are various characterizations of many interesting spaces, such as, Hardy, Besov, Besov–Hardy–Sobolev, etc., for instance, one can characterize Besov space by wavelet integral transform, Hardy space \(H^p\) by maximal functions (see [4] for the last statement and [22,8,9,31,48] for the other just mentioned ones). But maximal functions, singular integrals, oscillatory integral, Fourier integral operators, pseudoconvexity, wavelets, frames, harmonic analysis, Fourier transform, Radon transform, wavelet integral transform, and a lot of functional spaces, such as, besides the above mentioned ones, Hölder, Orlicz, Lizorkin, BMO (bounded mean oscillation), VMO (vanishing mean oscillation), etc., play an important role in science and technology, say, in mathematical physics, biophysics, even in psychology, cognitive sciences, generally, in social sciences [2,3,5,8,9,11–14,20–27,32,36,42,44,50,51]. The mutual influence between them, deterministic as well as stochastic, is more and more widespread, especially if they are studied not only on real and complex fields, but on local fields, etc. (see [8–10,15,41,46,47,49] and references therein). It is surprising, that via the eigenvectors of Vladimirov's pseudodifferential operator \(D^{\alpha}\), on \(p\)-adic field, S. Kozyrev recently discovered a class of wavelets [29], and from then on, a common theory of wavelet and pseudodifferential equations, linear, semilinear (non-linear, expected) on \(p\)-adic field, on ultrametric space, etc., has been developed [24,29–31,18]. It is interesting also to state that theory of pseudodifferential operators, is not only an extremely powerful tool to investigate linear, non-linear problems, but is just shown to be useful, too, even in mobile communications [43]. So the contents of this paper are some contributions to some above
mentioned areas, precisely to some problems of harmonic analysis over local fields. We only thus briefly introduce here some aspects of some areas among them, in which we are interested. Obviously, the harmonic analysis on local fields was constructed systematically by many authors (see, say, [45,46]). The classical singular operators of Calderón–Zygmund type and its maximal functions were considered by Keith Phillips and Mitchell Taibleson (see [37,38,45]). The theory of p-adic analysis and mathematical physics is investigated systematically in [49] by V.S. Vladimirov, I.V. Volovich, and E.I. Zelenov; and in [24] by A.Yu. Khrennikov. Pseudo-differential equations and stochastic over local fields were introduced in [28] by A.N. Kochubei.

More specifically, in this paper we first introduce an analogue of a theory of $A_r$ weights over local fields, and then specify the bounds of estimates. The proofs here are exploited from [16,17,34,35,39,41]. We study also in this section two problems characterizing weights so that the Hardy–Littlewood maximal operator is bounded in certain spaces. We characterize the largest integer $f$ which have compact support on $K$ so that $\|f\|_r < \infty$ for all $K$. We introduce systematically by many authors (see, say, [45,46]). The classical singular operators of Calderón–Zygmund type and its maximal functions were considered by Keith Phillips and Mitchell Taibleson (see [37,38,45]). The theory of $p$-adic analysis and mathematical physics is investigated systematically in [49] by V.S. Vladimirov, I.V. Volovich, and E.I. Zelenov; and in [24] by A.Yu. Khrennikov. Pseudo-differential equations and stochastic over local fields were introduced in [28] by A.N. Kochubei.

2. Preliminaries

$K$ is a local field if it is any totally disconnected, locally compact, non-discrete, complete field. The locally compact, non-discrete, complete fields have been completely characterized and are either connected real and complex number fields or totally disconnected. The discrete, complete fields have been completely characterized and are either connected real and complex number fields or totally disconnected. The $p$-adic fields are examples of local fields. Let $K$ be a fixed local field. Then there is an integer $q = p^r$, where $p$ is a prime and $r$ is a positive integer, and a norm $| \cdot |$ on $K$ such that for all $x \in K$ we have $|x| \geq 1$ and for each $x \in K^* \equiv K \setminus \{0\}$ we get $|x|^k = 1$ for some integer $k$. This norm is non-Archimedean, that is $|x + y| \leq \max(|x|, |y|)$ for all $x, y \in K$ and $|x| + |y| = \max(|x|, |y|)$ whenever $|x| \neq |y|$. Let $dx$ be the Haar measure on the locally compact, topological group $(K, +)$. This measure is normalized so that $\int_K dx = 1$, where $B_0 = \{x: |x| \leq 1\}$. We note also that there exists an additive character $\chi$ on $(K, +)$ such that $\chi$ is identically one on $B_0$, but is non-trivial on $B_1 = \{x: |x| \leq q\}$.

Let $\mathbb{K}^n$ be the $n$-dimensional vector space over $K$, hence $\mathbb{K}^n = \{x = (x_1, \ldots, x_n): x_i \in K\}$, $i = \overline{1, n}$. There is a norm defined on $\mathbb{K}^n$ by $|x| = \max(|x_1|, \ldots, |x_n|)$ for each $x = (x_1, \ldots, x_n) \in \mathbb{K}^n$. It is easy to see that this norm is non-Archimedean, i.e., $|x + y| \leq \max(|x|, |y|)$ for all $x, y \in \mathbb{K}^n$ and $|x + y| = \max(|x|, |y|)$ whenever $|x| \neq |y|$. $\mathbb{K}^n$ is a locally compact, Abelian, topological group under vector addition with the usual product topology (which coincides with the norm topology). A Haar measure is given by $dx = dx_1 \ldots dx_n$, where $dx_i$ is the normalized additive Haar measure of $K$, for $i = \overline{1, n}$. For each $a \in \mathbb{K}^*$, $d(a\alpha) = |a|^n dx$. Let $a \in K$ and $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{K}^n$. Let us define $\alpha \cdot x = (ax_1, \ldots, ax_n)$ and $(x, y) = x_1 y_1 + \cdots + x_n y_n$.

Fix $1 \leq \ell \leq \infty$. Let us denote by $L^\ell(\mathbb{K}^n)$ the space of all measurable functions $f$ from $\mathbb{K}^n$ to $\mathbb{C}$ so that the norm $\|f\|_\ell = (\int_{\mathbb{K}^n} |f(y)|^\ell \, dy)^{1/\ell} < \infty$ when $\ell < \infty$ and $\|f\|_\infty = \text{ess.sup.}|f| < \infty$. We denote also by $L^\ell_0(\mathbb{K}^n)$ the space of all non-negative functions $f$ which belong to $L^\ell(\mathbb{K}^n)$. Let be given any $(k, x) \in \mathbb{Z} \times \mathbb{K}^n$. Let $A$ be a measurable subset of $\mathbb{K}^n$, $u$ be a measurable function from $A$ to $\mathbb{C}$, and $f$ a function from $\mathbb{K}^n$ to $\mathbb{C}$. It is convenient to introduce the following

$$x + B_k = \{y \in \mathbb{K}^n: |y - x| \leq q^k\}, \quad B_k = 0 + B_k,$$

$$x + S_k = \{y \in \mathbb{K}^n: |y - x| = q^k\}, \quad S_k = 0 + S_k,$$

$$|A| = \int_A dx, \quad u(A) = \int_A u(x) \, dx, \quad f_k(x) = \frac{1}{|q^n|} \int_{x + B_k} f(y) \, dy.$$ 

The $x + B_k$ is the ball centered at $x$, with radius $q^k$ and $x + S_k$ is its boundary. It is interesting to note that $y \in x + B_k$ iff $x + B_k = y + B_k$. Let $f_k(x)$ be the mean of the function $f$ on the ball $x + B_k$. It is easy to see that

$$|x + B_k| = |B_k| = q^{nk}, \quad |x + S_k| = |S_k| = q^{nk} \left(1 - \frac{1}{q^n}\right).$$

For each $u \in \mathbb{K}^n$, $\chi_u$ defined by $\chi_u(x) = \chi((u, x))$ is a character on $\mathbb{K}^n$. Furthermore the mapping $u \mapsto \chi_u$ is a topological isomorphism of $\mathbb{K}^n$ onto its dual group, so we identify $\mathbb{K}^n$ with its dual group. The Fourier transform of a function $f \in L^1(\mathbb{K}^n)$ is defined by $\hat{f}(y) = \int_{\mathbb{K}^n} f(x) \overline{\chi_y(x)} \, dx$, where $\overline{\chi_y(x)} = \chi((-y, x))$. A function $f: \mathbb{K}^n \rightarrow \mathbb{C}$ is called locally constant if for each $x \in \mathbb{K}^n$ there exists such an integer $s = s(x)$ that $f(x + x') = f(x)$ for all $x' \in B_s$. Let $D(\mathbb{K}^n)$ be the family of all such $f$ which have compact support on $\mathbb{K}^n$. If $\varphi \in D(\mathbb{K}^n)$, then there exists an integer $s$ such that $\varphi(x + x') = \varphi(x)$ for all $x' \in B_s$. The largest integer $s = s(\varphi)$ as such is called the constancy of the function $\varphi$. The space $D'(\mathbb{K}^n)$ of Bruhat–Schwartz distributions on $\mathbb{K}^n$ is the strong conjugate space to the $D(\mathbb{K}^n)$. For each $f \in D'(\mathbb{K}^n)$ its Fourier transform $\hat{f}$ is defined in $D'(\mathbb{K}^n)$ by $\langle f, \varphi \rangle = \langle \hat{f}, \varphi \rangle$ for all $\varphi \in D(\mathbb{K}^n)$. The mappings $f \mapsto \hat{f}$ are linear homeomorphism of $D'(\mathbb{K}^n)$ onto $D'(\mathbb{K}^n)$ and
of $D'(\mathbb{K}^n)$ onto $D'(\mathbb{K}^n)$. The inverse map is denoted by $f \mapsto \tilde{f}$. It is given by $D(\mathbb{K}^n)$ by $\tilde{f}(x) = \int_{\mathbb{K}^n} f(y) \chi(-\langle x, y \rangle) \, dy$ and on of $D'(\mathbb{K}^n)$ by $\langle \tilde{f}, \varphi \rangle = \langle f, \varphi \rangle$ for all $\varphi \in D(\mathbb{K}^n)$. For other usual facts, we refer to [45,46,28]. Note also that, if $f \in L^1(\mathbb{K}^n)$ we can write

$$
\int_{\mathbb{K}^n} f \, dx = \sum_{y=-\infty}^{+\infty} \int_{\mathbb{K}^n} f \, dx,
$$

(2.1)

$$
\int_{\mathbb{K}^n} f(\alpha x) \, dx = \frac{1}{|\alpha|^n} \int_{\mathbb{K}^n} f \, dx, \quad \forall \alpha \in \mathbb{K}^n.
$$

(2.2)

The Hardy–Littlewood maximal function of $f \in L^1(\mathbb{K}^n)$ is defined by

$$
Mf(x) = \sup_{k \in \mathbb{Z}} \frac{1}{u(x + B_k)} \int_{x + B_k} |f(y)| \, dy.
$$

(2.3)

It is interesting to note that the centered maximal function and uncentered maximal function over a local field are equivalent.

We say that any non-negative, locally integrable function $u$ on $\mathbb{K}^n$ is a weight function (or a weight). A weight function $u$ has doubling property if there exists a constant $c_u > 0$ such that for any $(k, x) \in \mathbb{Z} \times \mathbb{K}^n$, $u(x + B_{k+1}) \leq c_u \cdot u(x + B_k)$. Let $u$ be a weight function, and define the maximal function

$$
M_u f(x) = \sup_{k \in \mathbb{Z}} \frac{1}{u(x + B_k)} \int_{x + B_k} |f(y)| u(y) \, dy.
$$

(2.4)

When $u = 1$ a.e. in $\mathbb{K}^n$, $M_u$ is the Hardy–Littlewood maximal operator, will be denoted by $M$.

Let $u$ be a weight function on $\mathbb{K}^n$ and $1 \leq \ell < \infty$. We say that $u$ is in $A_\ell$ class, if it satisfies the following condition

$$
\left( \int_{x + B_k} u(y) \, dy \right) \cdot \left( \int_{x + B_k} u(y)^{-\frac{1}{\ell - 1}} \, dy \right)^{\ell - 1} \leq c \cdot q^n k < +\infty, \quad \forall (k, x) \in \mathbb{Z} \times \mathbb{K}^n,
$$

(2.5)

or the equivalent condition

$$
\sup_{(k,x) \in \mathbb{Z} \times \mathbb{K}^n} u_k(x) \cdot \left( u_{-\frac{1}{\ell - 1}}(x) \right)^{\ell - 1} \leq c < +\infty.
$$

The function $u$ is said to be in the $A_\infty$ class, if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any $(k, x) \in \mathbb{Z} \times \mathbb{K}^n$ and any measurable subset $E$ of $x + B_k$ in which $|E| > \epsilon q^n k$, then $\sup E \leq \delta \cdot u(x + B_k)$. In [37] Keith Phillips proved a type of Calderón–Zygmund lemma over locally compact, Abelian, topological groups.

**Lemma 2.1.** Let $G$ be a locally compact, Abelian, topological group having a neighborhood basis of the identity of the form $\{H_m; m \in \mathbb{Z}\}$, where $H_m$'s are compact open subgroups of $G$ satisfying $H_{m+1} \subset H_m$ and $\bigcup_{m \in \mathbb{Z}} H_m = G$. Let $q_m = |H_{m+1}: H_m|$. For $f \in L^1_+(\mathbb{K}^n, \lambda)$, $1 \leq \ell \leq +\infty$ and $\lambda$ is the Haar measure and $t > 0$, there is a subset $P_t$ of $\mathbb{Z}^n \times \mathbb{Z}$ and a mapping $(m, k) \mapsto x_{m,k}$ of $P_t$ into $G$ such that $x_{m,k}H_k$: $(m, k) \in P_t$ is pairwise disjoint and the following inequalities hold

(i) $t \leq \frac{1}{x(H_t)} \int_{x_{m,k}H_k} f \, dl \leq q_k t$.

(ii) $\lambda(D_t) < +\infty$ and $\lim_{t \to \infty} \lambda(D_t) = 0$, where $D_t = \bigcup_{(m,k) \in P_t} x_{m,k}H_k$.

(iii) $f(x) \leq t$ a.e. in $(D_t)^c$.

(iv) $t \lambda(D_t) \leq \int_{D_t} f \, dl$. and if $\sup_{m \in \mathbb{Z}} q_m \leq c < +\infty$ then $\lambda(D_t) \leq \int_{D_t} f \, dl \leq ct \lambda(D_t)$.

For the details of the proof see [37]. In this paper we shall apply this lemma for two cases when $G = \mathbb{K}^n$ and $G = B_k$, the ball centered at 0. In both two cases $q_m = q^n$ for all $m \in \mathbb{Z}$.

The following lemma, which we shall use in the sequel, is the analogue of Vitali–Wiener covering lemma over local fields.

**Lemma 2.2.**

(a) Let $u$ be a weight function with the doubling property, i.e., there exists a constant $c_u > 0$ such that $u(x + B_{k+1}) \leq c_u \cdot u(x + B_k)$ for all $(k, x) \in \mathbb{Z} \times \mathbb{K}^n$. For any measurable subset $E$ of $\mathbb{K}^n$, which is covered by a finite family of balls $\{x + B_k\}$, then we can select a disjoint subcollection $\{x_j + B_{k_j}; j = 1, 2, \ldots, m\}$ such that
Theorem 3.1. Let $u$ be a weight which has doubling property, then

(a) the maximal operator $M_u$ is of weak type $(1, 1),$

$$u\left(\{x \in \mathbb{R}^n : M_u f(x) > \alpha\}\right) \leq \frac{A}{\alpha} \cdot \|f\|_{L^1(u)}$$

for all $\alpha > 0$ and for any $f \in L^1(u).$ The constant $A$ can be chosen as $q^n$ when $u = 1$ a.e. in $\mathbb{R}^n.$

(b) For any $1 < \ell < +\infty,$ $|M_u f|_{L^\ell(u)} \leq A\|f\|_{L^\ell(u)}$ for all $f \in L^\ell(u),$ where $A_{\ell}$ can be chosen as $A_{\ell} = 2q^n \frac{\ell}{\ell - 1}$ when $u = 1$ a.e. in $\mathbb{R}^n.$

Remark. The part (a) of Theorem 3.1 can be proved easily by using Lemma 2.2. We can repeat the same arguments used in the $\mathbb{R}^n$ for the part (b) of Theorem 3.1 by using Lemma 2.1 (see [17,41]). So the proofs are intentionally omitted.

Now we shall prove the reverse Hölder inequality, for $A_{\ell}$ classes, over local fields.

Theorem 3.2. Let $u$ be in $A_{\ell}$ class, $1 < \ell < +\infty.$ Then the following reverse Hölder inequality holds

$$\left(\frac{1}{q^{nk}} \int_{x+B_k} (u(y))^{1+\delta} dy\right)^{\frac{1}{1+\delta}} \leq \frac{c}{q^{nk}} \int_{x+B_k} u(y) dy$$

(3.1)

for all $(k, x) \in \mathbb{Z} \times \mathbb{R}^n,$ with constants $c > 0,$ $\delta > 0$ independent of $(k, x).$

Proof. We fix $(k, x) \in \mathbb{Z} \times \mathbb{R}^n.$ For our convenience, we shall use $u_k(x)$ in place of $\frac{1}{q^{nk}} \int_{x+B_k} u(y) dy.$ Firstly, we want to claim the following

$$\left|\{y \in x + B_k : u(y) > \beta \cdot u_k(x)\}\right| \geq \alpha q^{nk}$$

(3.2)

holds for some $\alpha, \beta > 0.$ To see this, let $E = \{y \in x + B_k : u(y) \leq \beta u_k(x)\}$ and observe that

$$\frac{1}{\beta} \left(\frac{|E|}{q^{nk}}\right)^{\ell - 1} u_k(x) \cdot \left(\frac{1}{q^{nk}} \int_{E} (\beta u_k(x))^{-1+\gamma} dy\right)^{\ell - 1}$$

$$\leq u_k(x) \left(\frac{1}{q^{nk}} \int_{E} (u(y))^{-1+\gamma} dy\right)^{\ell - 1} \leq c < +\infty.$$

Taking $\beta$ small enough, we obtain (3.2). Now we shall prove that for any $\lambda > u_k(x)$ then

$$\int_{\{y \in x + B_k : u(y) > \lambda\}} u(y) dy \leq c\lambda \left|\{y \in x + B_k : u(y) > \beta \lambda\}\right|.$$  

(3.3)
To do that we shall use Lemma 2.1 for function \( u_\lambda(y) = u(x+y), \ G = B_k \). Then there is a family \( \{ x_{m,k_1} + B_k : (m, k_1) \in P_\lambda \} \) in which \( x_{m,k_1} + B_k \subset B_k \) and \( u(x+y) \leq \lambda \) for a.e. \( y \) belonging to \( B_k \setminus \bigcup_{(m,k_1) \in P_\lambda} x_{m,k_1} + B_k \) and \( \lambda \leq \frac{1}{q^{\alpha} - 1} \int_{x_{m,k_1} + B_k} \lambda^\delta \ d\lambda \leq q^{\lambda} \). So we have

\[
\int_{\{ y \in x + B_k : u(y) > \lambda \}} u(y) \, dy = \int_{\{ y \in B_k : u(x+y) > \lambda \}} u(y) \, dy \\
\leq \sum_{(m,k_1) \in P_\lambda} \int_{x_{m,k_1} + B_k} u(y) \, dy \leq q^\lambda \sum_{(m,k_1) \in P_\lambda} q^{nk_1}
\]

Now multiplying both sides of (3.3) by \( \lambda^{\delta - 1} \) we find that

\[
\int_{u_\lambda(x)}^{+\infty} \lambda^{\delta - 1} \left( \int_{\{ y \in x + B_k : u(y) > \lambda \}} u(y) \, dy \right) \leq c \int_0^{+\infty} \lambda^{\delta - 1} \left| \{ y \in x + B_k : u(y) > \lambda \} \right| \, d\lambda
\]

\[
\leq \frac{c}{1 + \delta} \int_{x + B_k} (u(y))^{1+\delta} \, dy.
\]

By Fubini theorem, the left-hand side equals

\[
\int_{\{ y \in x + B_k : u(y) > u_\lambda(x) \}} u(y) \left( \int_{u_\lambda(x)}^{u(y)} \lambda^{\delta - 1} \, d\lambda \right) \, dy
\]

\[
= \int_{\{ y \in x + B_k : u(y) > u_\lambda(x) \}} u(y) \left[ \frac{(u(y))^{\delta}}{\delta} - \frac{(u_\lambda(x))^{\delta}}{\delta} \right] \, dy
\]

\[
\geq \frac{1}{\delta} \int_{x + B_k} (u(y))^{1+\delta} \, dy - \frac{q^{nk_1}}{\delta} (u_\lambda(x))^{1+\delta}.
\]

Therefore,

\[
\left( \frac{1}{\delta} - \frac{c}{1 + \delta} \right) \frac{1}{q^{nk_1}} \int_{x + B_k} (u(y))^{1+\delta} \, dy \leq \frac{(u_\lambda(x))^{1+\delta}}{\delta}
\]

and taking \( \delta \) small enough we get the inequality (3.1). \( \square \)

**Corollary 3.3.** Suppose that \( u \) is in the class \( A_\ell, \ 1 < \ell < +\infty \). Then there exists \( 1 < s < \ell \) such that \( u \) is also in class \( A_s \).

**Proof.** Let \( v(x) = (u(x))^{\ell - 1} \). Obviously then \( v \in A_\ell \), where \( \frac{1}{s} + \frac{1}{\ell} = 1 \). Applying Theorem 3.2 to \( v \), we see that \( u \) is in the class \( A_s \) with \( s = \frac{\ell - 1}{\ell + 1} \). \( \square \)

Using Hölder’s inequality and Theorem 2 for

\[
\int_{G^n} \chi_E(x) u(x) \, dx
\]

we obtain

**Corollary 3.4.** If \( u \) is in the class \( A_\ell, \) for some \( 1 < \ell < +\infty \), then \( u \) is in the \( A_{s_\infty} \) class.

Now we could obtain the following analogue of Muckenhoupt’s theorem [34] over local fields.
**Theorem 3.5.** Let $1 < \ell < +\infty$ and $u$ be a weight function with doubling property. Then
\[
\|Mf\|_{L^{\ell}(u)} \leq A \cdot \|f\|_{L^{\ell}(u)} \tag{3.4}
\]
for all $f \in L^{\ell}(u)$, where $A$ is a constant, if and only if $u$ is in $A_{\ell}$ class.

**Proof.** Firstly, we prove that if \((3.4)\) holds for all $f \in L^{\ell}(u)$, then $u$ is in $A_{\ell}$ class. Fix $(k, x) \in \mathbb{Z} \times \mathbb{R}^n$. We observe that $Mf(y) \geq f_k(y) \cdot \Delta_k(y)$, where $\Delta_k(y)$ is the characteristic function of the ball $x + B_k$ and $f_k(x) = \frac{1}{q^{nk}} \int_{x+B_k} f(y) \, dy$. From \((3.4)\) for each $f \in L^{\ell}_+(u)$, we have
\[
\left( \int_{x+B_k} u(y) \, dy \right) \cdot \left( f_k(x) \right)^{\ell} \leq A \cdot \left( \int_{x+B_k} (f(y))^{\ell} u(y) \, dy \right).
\]
Thus
\[
f_k(x) \leq A \cdot \left( \frac{1}{\int_{x+B_k} u(y) \, dy} \right) \cdot \left( \int_{x+B_k} (f(y))^{\ell} u(y) \, dy \right)^{\frac{1}{\ell}}.
\]
Taking $f = u^{-1}$ we obtain $u_k(x), (u^{-1})_k(x) \leq c$. This shows $u$ is in $A_{\ell}$ class.

Now let us assume $u$ is in $A_{\ell}$ class and take $f \in L^{\ell}_+(u)$. By Hölder’s inequality, we must have
\[
f_k(x) = \frac{1}{q^{nk}} \int_{x+B_k} f(y) \, dy \leq \frac{1}{q^{nk}} \left( \int_{x+B_k} (f(y))^{\ell} u(y) \, dy \right)^{\frac{1}{\ell}} \cdot \left( \int_{x+B_k} \left( u(y) \right)^{-1} \, dy \right)^{\frac{1}{\ell}}
\]
so
\[
Mf(x) = \sup_{k \in \mathbb{Z}} f_k(x) \leq c \cdot \left( M_u \left( f^\ell \right)(x) \right)^{\frac{1}{\ell}}.
\]

By Theorem 3.1 we have
\[
\int_{\mathbb{R}^n} (Mf(x))^{\ell} u(x) \, dx \leq c_s \cdot \int_{\mathbb{R}^n} (M_u \left( f^\ell \right)(x))^{\ell} u(x) \, dx \leq c_s \int_{\mathbb{R}^n} |f(y)|^{\ell} u(y) \, dy \tag{3.5}
\]
for any $s > \ell$. From Corollary 3.4 there exists $\varepsilon > 0$ such that $u$ belongs to class $A_{\ell-\varepsilon}$, $\ell - \varepsilon > 1$. Applying \((3.5)\) for $s = \ell$ and $\ell - \varepsilon$ we obtain
\[
\int_{\mathbb{R}^n} (Mf(x))^{\ell} u(x) \, dx \leq c_{\ell} \cdot \int_{\mathbb{R}^n} |f(y)|^{\ell} u(y) \, dy,
\]
and this completes the proof. \(\square\)

A natural question posed by B. Muckenhoupt concerning the Hardy–Littlewood maximal function $M$ is that: what is the characterization of the weight $v$ for which $M$ is bounded from $L^{\ell}(u)$ to $L^{\ell}(v)$ for some non-trivial $u$? Another problem is characterizing weight $u$ for which there is non-trivial weight $v$ in the $\mathbb{R}^n$ setting the complete answer to the first problem due to Wo-Sang Young [50] and the second problem is solved independently by J.L. Rubio de Francia [21] and L. Carleson and P.W. Jones [6]. In what follows, we study the first problem over a local field and obtain a necessary condition on the weight $v \geq 0$ such that $M$ is bounded from $L^{\ell}(u)$ to $L^{\ell}(v)$ for some $u \leq \infty \, a.e.$ The result is as follows.

**Theorem 3.6.** Let be given $v$ a weight and $1 < \ell < \infty$. Assume that there is a weight $u < \infty \, a.e.$ such that
\[
\int_{\mathbb{R}^n} |Mf|^{\ell} u \, dx \leq C \int_{\mathbb{R}^n} |f|^{\ell} u \, dx
\]
for all $f \in L^{\ell}(u)$. Then
\[
\sum_{\gamma \in \mathbb{Z}} q^{\gamma(n+1)} \int_{S_0} v(\beta^{-\gamma} x) \, dx < \infty, \tag{3.6}
\]
where $\beta$ is the element such that $B_0 = \beta B_{-1}$ and the constant $C$ depends only on $n$ and $\ell$. 

We observe that \( \int_{\mathbb{R}^n} \frac{dx}{(1 + |x|^\gamma)^t} < \infty \). Indeed, we have
\[
\int_{\mathbb{R}^n} \frac{dx}{(1 + |x|^\gamma)^t} = \sum_{\gamma \in \mathbb{Z}} \int_{\gamma} \frac{dx}{(1 + |x|^\gamma)^t}.
\]
The last sum is split into two summands for \( \gamma > 0 \) and \( \gamma \leq 0 \), and replace \(-\gamma\) by \( \gamma \) we have
\[
\int_{\mathbb{R}^n} \frac{dx}{(1 + |x|^\gamma)^t} = \sum_{0 < \gamma \in \mathbb{Z}} \frac{q^{\gamma t}}{(1 + q^n)^t} \left( 1 - \frac{1}{q^n} \right) + \sum_{0 \leq \gamma \in \mathbb{Z}} \frac{q^{\gamma (t-1)}}{(1 + q^n)^t} \left( 1 - \frac{1}{q^n} \right) < \infty.
\]
From the above fact and the following proof we shall see that for any \( v \in L^\infty(\mathbb{R}^n) \) holds (3.6).

**Proof.** Since \( u < \infty \) and non-negative, there is a set \( E \) with positive measure on which \( u \) is bounded. Let us assume that \( E \subset B_r \) for some non-negative integer \( r \). Let \( f = \chi_E \), then we have \( f \in L^1(u) \). From the definition of the maximal operator \( M \), it is not hard to see that
\[
Mf(x) \geq \frac{1}{\max(q^n, |x|^\gamma)} \geq \frac{1}{1 + |x|^\gamma} \cdot |E| \quad (\forall x \in \mathbb{R}^n)
\]
(consider two cases \(|x| < q^n\) and \(|x| > q^n\)). So we have
\[
\int_{\mathbb{R}^n} |Mf|^t v^\gamma dx \geq \frac{|E|}{q^n} \int_{\mathbb{R}^n} \frac{v(x)}{(1 + |x|^\gamma)^t} dx.
\]
Now note that
\[
\int_{\mathbb{R}^n} \frac{v(x)}{(1 + |x|^\gamma)^t} dx = \sum_{\gamma \in \mathbb{Z}} \sum_{\gamma \in \mathbb{Z}} \frac{v(x)}{(1 + |x|^\gamma)^t} = \sum_{\gamma \in \mathbb{Z}} \frac{v(x)}{(1 + |x|^\gamma)^t} \int_{\mathbb{R}^n} v(\beta^\gamma x) dx.
\]
Thus,
\[
\sum_{\gamma \in \mathbb{Z}} \frac{v(x)}{(1 + |x|^\gamma)^t} \int_{\mathbb{R}^n} v(\beta^\gamma x) dx < \infty. \quad \square
\]

**4. A converse inequality for maximal functions**

For the maximal operator there is an inequality that presents some interest in itself and gives also the characterization of some important spaces in differentiation and in Analysis, say, the space \( L(1 + \log^+ L)^k \) of Zygmund. An inequality of the type described here appears in [40] by Elias M. Stein. Analogous results related to strong maximal function and the class \( L(\log^+ L)^k \) are studied by Favo, Gatto and Gutiérrez in [19] and by Andersen, Wo-Sang Young in [1]. We shall discuss here such analogous results over local fields, and applying them to deduce a more general result that was given in [7].

Let us recall that
\[
Mf(x) = \sup_{k \in \mathbb{Z}} \frac{1}{q^n} \int_{x + B_k} |f(y)| dy,
\]
and we denote \( u(A) = \int_A u(y) dy \). Let us recall that a weight function \( u \) is a non-negative, locally integrable function on \( \mathbb{R}^n \). A weight function \( u \) has doubling property if there exists a constant \( c_u > 0 \) such that for any \((k, x) \in \mathbb{Z} \times \mathbb{R}^n\) then
\[
u(x + B_{k+1}) \leq c_u \cdot u(x + B_k).
\]

Firstly, we prove a reverse weak type estimate for the Hardy–Littlewood maximal function for balls, and secondly the similar results for \( \mathbb{R}^n \). The last result of this section is an application to the class \( L(1 + \log^+ L)^k \) of Zygmund.

**Theorem 4.1.** Let be given \((s, x) \in \mathbb{Z} \times \mathbb{R}^n\) and two functions \( u \) and \( v \) which are non-negative and integrable on \( x + B_s \).

(a) Assume that there exists a constant \( c_1 > 0 \) so that
\[
u(x + y + B_k) \geq c_1 \cdot \esssup_{z \in y + B_k} v(x + z),
\]
for all \( y + B_k \subset B_s \). Then
\[
u\left( \left\{ y \in x + B_s : Mf(y) > \lambda \right\} \right) \supset \frac{c_1}{q^n} \int_{\left\{ y \in x + B_s : Mf(y) > \lambda \right\}} |f(y)| |v(y)| dy.
\]
for all \( f \in L^1(x + B_s) \) and for all \( \lambda \geq \frac{1}{q^n} \int_{x + B_s} |f(y)| dy \).
(b) Conversely, if (4.1) holds for all \( f = \chi_E \), the characteristic function of measurable \( E \subset \mathbb{R}^n \) with \( 0 < |E| < +\infty \), and for all \( 0 < \lambda < 1 \), then

\[
\frac{u(x + y + B_k)}{q^{nk}} \geq \frac{c_1}{q^n} \text{ess.sup } v(x + z),
\]

for all \( y + B_k \subset B \).

Note that in the part (b) of Theorem 4.1, the hypothesis conditions on \( f \) and \( \lambda \) are weaker than in the Euclidean case, and especially they do not need the doubling condition on the weight \( \mu \) which is different from the Euclidean case obtained by K.F. Andersen and W.S. Young in [1]. In [1], Theorem 1, page 257, \( u \) requires doubling property, namely \( |u(2Q)| \leq c|u(Q)| \) for all cubes \( Q \).

**Proof.** (a) Let \( \lambda \geq \frac{1}{q^n} \int_{x+B_k} |f(x)| \, dy \) and \( f \in L^1_+ (x + B_+) \). From Lemma 2.1 there are pairwise disjoint \( \{x_{m,k} + B_k \subset B_k \} \) for which \( \bigcup_{(m,k) \in P} x_{m,k} + B_k \subset B_k \). Let \( \lambda \leq \frac{1}{q^n} \int_{x+B_k} f(x) \, dy \leq q^n \lambda \) and \( f(x) \lesssim \lambda \) a.e. in \( x \in B_k \setminus \bigcup_{(m,k) \in P} (x_{m,k} + B_k) \).

For any \( y \in x + x_{m,k} + B_k \), obviously \( y + B_k = x + x_{m,k} + B_k \). We have

\[
Mf(y) \geq \frac{1}{q^n} \int_{x+B_k} f(z) \, dz = \frac{1}{q^n} \int_{x+x_{m,k}+B_k} f(z) \, dz
\]

that is

\[
\bigcup_{x + x_{m,k} + B_k} \{ y \in x + B_k : Mf(y) > \lambda \}.
\]

We get

\[
u = \{ y \in x + B_k : Mf(y) > \lambda \} \geq \sum_{(m,k) \in P} u(x + x_{m,k} + B_k)
\]

\[
\geq \frac{1}{q^n} \sum_{(m,k) \in P} \frac{1}{q^k} \int_{x_{m,k} + B_k} f(x) \cdot u(x + x_{m,k} + B_k) \, dy
\]

\[
\geq \frac{c_1}{q^n} \sum_{(m,k) \in P} \text{ess.sup } v(x + z) \int_{x_{m,k} + B_k} f(x) \, dz
\]

\[
\geq \frac{c_1}{q^n} \sum_{(m,k) \in P} \int_{x_{m,k} + B_k} f(x) v(x) \, dz.
\]

For almost everywhere \( y \in B_k \) and \( f(x) \lambda, y \) belongs to \( \bigcup_{(m,k) \in P} x_{m,k} + B_k \). This implies

\[
\sum_{(m,k) \in P} \int_{x_{m,k} + B_k} f(x) v(x) \, dz \geq \int_{y \in B_k} f(x+y) v(x+y) \, dy \geq \int_{y \in B_k} f(y) v(y) \, dy.
\]

So

\[
u = \{ y \in x + B_k : Mf(y) > \lambda \} \geq \frac{c_1}{q^n} \int_{y \in B_k} f(y) v(y) \, dy.
\]

(b) Let be given \( \epsilon > 0 \) and \( y + B_k \subset B_k \). There exists a measurable \( E \subset y + B_k \), with \( 0 < |E| < q^m \) and \( v(x + t) \geq \text{ess.sup}_{z \in y + B_k} v(x + z) - \epsilon \) for \( t \in E \). Take \( f = \chi_{x+ E} \) the characteristic function of \( x + E \). For each \( z \notin x + y + B_k \), we will prove that \( Mf \lesssim \lambda = \frac{|E|}{q^m} \). Indeed, for \( j \in \mathbb{Z} \)

\[
\frac{1}{q^{mj}} \int_{z + B_j} |f(t)| \, dt = \frac{1}{q^{mj}} \int_{z + B_j} \chi_{x + E} \, dt.
\]

If \( j \geq k \) then \( q^{nj} \geq q^{mk} \) so \( \frac{1}{q^{nj}} \int_{z + B_j} |f(t)| \, dt \leq \frac{|E|}{q^{nj}} = \lambda \).
If \( j < k \) let \( z = x + z' \), since \( z \notin x + y + B_k \) then \( z' \notin y + B_k \). It means that \( |z' - y| \geq q^{k+1} \). If there is \( t \in z' + B_j \cap y + B_k \) then 
\[ |z' - y| = |z' - t - t - y| \leq \max(|z' - t|, |t - y|) \leq q^{k} < q^{k+1}, \]
which is a contradiction. So \( z' + B_j \cap y + B_k = \emptyset \), and so \( z + B_j \cap x + y + B_k = \emptyset \). From \( E_{\epsilon} \subset y + B_k \) we get \( z + B_j \cap x + E_{\epsilon} = \emptyset \). In that case \( \frac{1}{q^n} \int_{z+B_j} |f(t)| \, dt = 0 \).

From these arguments above we have 
\[
M_f(z) = \sup_{j \in \mathbb{Z}} \int_{z+B_j} |f(t)| \, dt \leq \frac{|E_\epsilon|}{q^{nk}} = \lambda.
\]
So we obtain \( \{ z \in x + B_j : M_f(z) > \lambda \} \subset x + y + B_k \), implying 
\[
\begin{align*}
\forall \epsilon &> 0 \exists z \in B_k \quad |z| \leq \epsilon \\
\Rightarrow &\quad \sup_{z \in B_k} |z| < \epsilon
\end{align*}
\]
and then 
\[
\sup_{z \in y + B_k} |z| < \epsilon.
\]

With \( \epsilon \to 0^+ \) we get 
\[
\frac{u(x + y + B_k)}{q^{nk}} \geq \frac{c_1}{q^n} \sup_{z \in x + y + B_k} v(z).
\]
for all \( y + B_k \subset B_{\epsilon} \). \( \square \)

**Theorem 4.2.** Let \( u, v \) be weight functions on \( \mathbb{R}^n \).

(a) If there exists a constant \( c = c(u, v) > 0 \) such that 
\[
\frac{u(x + B_k)}{q^{nk}} \geq \frac{c}{q^n} \sup_{z \in x + B_k} v(z),
\]
for all \( (k, x) \in \mathbb{Z} \times \mathbb{R}^n \), then 
\[
u \left( \{ x \in \mathbb{R}^n : M_f(x) > \lambda \} \right) \geq \frac{c}{q^n} \sup_{z \in x + B_k} v(z), \tag{4.2}
\]
for all \( f \in L^1(\mathbb{R}^n) \) and all \( \lambda > 0 \).

(b) Conversely, if \( (4.2) \) holds for \( f = \chi_E \), the characteristic function of measurable \( E \subset \mathbb{R}^n \) with \( 0 < |E| < +\infty \), and for all \( 0 < \lambda \leq 1 \), then 
\[
\frac{u(x + B_k)}{q^{nk}} \geq \frac{c}{q^n} \sup_{z \in x + B_k} v(z), \tag{4.3}
\]
for all \( (k, x) \in \mathbb{Z} \times \mathbb{R}^n \).

Similarly as above, we note that the conditions in part (b) are more weaker than those in the Euclidean case, and moreover, \( u \) does not require the doubling condition.

**Proof.** (a) Take \( f \in L^1(\mathbb{R}^n) \), and \( \lambda > 0 \), and \( m, k \in \mathbb{Z}_+ \). Let \( f^k(x) = f(x) \) if \( |f(x)| \leq q^k \) and \( x \in B_k \), \( f^k(x) = 0 \) otherwise. For any \( \lambda > 0 \) by Theorem 4.1:

\[
u \left( \{ x \in B_m \mid Mf^k(x) > \lambda \} \right) \geq \frac{c}{q^n} \int_{x \in B_m} f^k(x) v(x) \, dx,
\]
with \( m \) large enough \( \frac{1}{q^m} \int_{\mathbb{R}^n} f^k \leq \lambda \) and \( m \geq k \). Then \( \{ f^k \}_{k \geq 1} \) is non-negative, increasing to \( f \) pointwise and \( Mf^k \uparrow Mf(x) \) when \( k \to \infty \), so by the monotone convergence theorem, with \( m \to \infty \) and then \( k \to \infty \), we obtain (4.2).
(b) Given any \((k, x) \in \mathbb{Z} \times \mathbb{R}^n\) and \(\varepsilon > 0\). There exists a measurable \(E_\varepsilon \subset x + B_k\), with \(0 < |E_\varepsilon| < q^nk\) and \(v(z) > \text{ess.sup}_{y \in x + B_k} v(y) - \varepsilon\) for a.e. \(z \in E_\varepsilon\). Let \(f = \chi_{E_\varepsilon}\) and \(\lambda = \frac{|E_\varepsilon|}{q^nk}\). For each \(y \notin x + B_k\), we get \(|x - y| \geq q^{k+1}\). For \(j \in \mathbb{Z}\),

\[
\frac{1}{q^n j} \int_{y + B_j} |f(z)| \, dz = \frac{1}{q^n j} |y + B_j \cap E_\varepsilon|.
\]

If \(j \geq k\) then \(\frac{1}{q^n j} \int_{y + B_j} |f(z)| \, dz < \frac{|E_\varepsilon|}{q^n j} = \lambda\). If \(j < k\) we once again show that \(E_\varepsilon \cap y + B_j = \emptyset\). Indeed if there is \(z \in y + B_j \cap x + B_k\), then \(|x - y| \leq \max(|x - z|, |z - y|) \leq q^k < q^{k+1}\), a contradiction. So \(E_\varepsilon \cap y + B_j \subset x + B_k \cap y + B_j = \emptyset\). That means

\[
Mf(y) = \sup_{j \in \mathbb{Z}} \frac{1}{q^n j} \int_{y + B_j} |f(z)| \, dz \leq \lambda,
\]

thus \(\{y \in \mathbb{R}^n : Mf(y) > \lambda\} \subset x + B_k\), implying

\[
u(x + B_k) \geq \nu(\{y \in \mathbb{R}^n : Mf(y) > \lambda\}) \geq \frac{c}{q^n} \int_{E_\varepsilon} v(x) \, dx \geq \frac{c}{q^n} q^n (\text{ess.sup}_{y \in x + B_k} v(y) - \varepsilon).
\]

With \(\varepsilon \to 0^+\), we obtain (4.3). \(\square\)

**Theorem 4.3.** Assuming that \(u, v\) are weight functions and \((x, a) \in \mathbb{Z} \times \mathbb{R}^n\) so that

\[
\frac{\nu(x + y + B_k)}{q^nk} \geq c \cdot \text{ess.sup}_{z \in y + B_k} v(x + z),
\]

for any ball \(y + B_k \subset B_j\). Then \(\int_{x + B_k} Mf(y)u(y) \, dy < +\infty\), implying that

\[
\int_{x + B_k} |f(x)| \cdot \left(1 + \log^+ |f(y)|\right) v(y) \, dy < +\infty,
\]

for all \(f\) which supp \(f \subset x + B_k\).

Note that in [7], page 302, J.A. Chao showed a similar result on local fields in case without weights and on some sphere.

**Proof.** If \(v = 0\) a.e. in \(x + B_k\) then this theorem is trivial. So let us assume that \(v(x) > 0\) on some set of positive measure in \(x + B_k\). Then (4.4) shows that \(u(x + B_k) > 0\). Let \(f \geq 0\), which is supported in \(x + B_k\). If \(f \neq 0\) a.e. in \(y + B_k\) then \(Mf(y) \geq \frac{1}{q^n j} \int_{y + B_j} f > 0\) for any \(y \in x + B_k\) and the hypothesis

\[
\int_{x + B_k} Mf(y)u(y) \, dy < +\infty
\]

implies \(f\) and \(u\) are integrable on \(x + B_k\). From (4.4) it is clear that so is \(v\). Now we have

\[
\int_{x + B_k} f(y) \log f(y) v(y) \, dy = \int_{\{y \in x + B_k : f(y) > 1\}} f(y) v(y) \, dy \frac{f(y)}{\lambda} \int_1 \frac{d\lambda}{\lambda}
\]

\[
= \int_1 \frac{d\lambda}{\lambda} \int_{\{y \in x + B_k : f(y) > 1\}} f(y) v(y) \, dy \leq \frac{q^n}{c} \int_{x + B_k} Mf(y) \, dy < +\infty.
\]

This shows \(f\) is in \(L(1 + \log^+ L)\) class on \(x + B_k\). \(\square\)

5. Singular kernels and weak type estimates for a class of maximal functions

For each functions \(\zeta\) from \(\mathbb{R}^n\) to \(\mathbb{C}\) we consider the following conditions

(i) \(\|\zeta\|_{\infty} \leq c < +\infty\).

(ii) \(\|\zeta(x)\| \leq \frac{c}{|x|^n}\) a.e. in \(x \neq 0\).
Proof. Assume (iv) holds. Let $\beta \neq 0$, $|y| = q^{k-2}$ where $k \in \mathbb{Z}$. We have

\[
\int_{|x| \geq q^j |y|} |\zeta(x-y) - \zeta(x)| \, dx \leq \int_{|x| \geq q^j} c|y| |x|^{n+1} = c q^{k-2} \sum_{j=k}^{+\infty} \int_{q^j}^{+\infty} \frac{dx}{q^{j(n+1)}} \\
\leq c \sum_{j=k}^{+\infty} q^{j-k+2} q^n (1 - \frac{1}{q^n}) \frac{1}{q^{j(n+1)}} \leq c \left( 1 - \frac{1}{q^n} \right) \sum_{j=1}^{+\infty} \frac{1}{q^{j-k+2}} \\
\leq c \left( 1 - \frac{1}{q^n} \right) \sum_{j=1}^{+\infty} \frac{1}{q^j} \leq +\infty.
\]

So (iv) implies (v). For any $j \in \mathbb{Z}$, $j \geq 1$ and $y \in S_0$, obviously $|\beta^j y| = q^{-j} < 1$, so we obtain

\[
\sup_{|y|=1} \int_{S_0} |\zeta(x+\beta^j y) - \zeta(x)| \, dx \\
\leq \sup_{|y|=1} \int_{S_0} \frac{c}{q^j} = c \left( 1 - \frac{1}{q} \right) \sum_{j=1}^{+\infty} \frac{1}{q^j} \leq +\infty.
\]

Thus (iv) implies (iii). \(\square\)

We shall need the following lemma and its proof for our last application. In fact this lemma can be deduced from the proof of Theorem 3.1 in [38], but let us here state and prove it more clearly.

Lemma 5.2. Assume that $\zeta$ satisfies (ii), (iii), $\eta(x) = |\zeta(x)|^m$ in $L^\infty(\mathbb{R}^n)$, $\int_{S_0} \zeta(x) \, dx = 0$ and homogeneous condition

\[\zeta(\beta^j x) = q^{mn} \zeta(x) \quad \forall (s, x) \in \mathbb{Z} \times \mathbb{R}^n.\]

For each $k \in \mathbb{Z}$ let $\zeta_k(x) = \zeta(x) \cdot \xi_k(x)$ where $\xi_k$ is the characteristic function of $(B_k)^c$, the complement of $B_k$. Then $\{\|\xi_k\|: k \in \mathbb{Z}_+\}$ are uniformly bounded.

Proof. For each $m \in \mathbb{Z}$ we put $\zeta_{m,k} = \zeta_k \cdot \xi_{B_{-m}}$, where $\xi_{B_{-m}}$ is the characteristic function of $B_{-m}$. We firstly show that $\zeta_k \cdot \zeta_{m,k} \in L^1$ for all $m, k \in \mathbb{Z}$. Indeed,

\[\zeta_{m,k} = \zeta \cdot \xi_{B_{-k}} \cdot \zeta_{B_{-m}} = \zeta \cdot \xi_{B_{-k} \cap B_{-m}}.\]

So we have

\[\|\zeta_{m,k}\|_1 \leq \|\zeta_k\|_1 = \int_{(B_{-k})^c} |\zeta(x)| \, dx \leq \|\eta\|_\infty \cdot \int_{(B_{-k})^c} \frac{dx}{|x|^n} = \|\eta\|_\infty \cdot \sum_{j=k+1}^{+\infty} \int_{\mathbb{R}^n} \frac{dx}{q^{j(n+1)}} < +\infty,
\]

for all $k, m \in \mathbb{Z}$. Now for each $z \in \mathbb{R}^n$, let us consider $x \to \zeta_0(x-z) - \zeta_0(x)$ in $L^1$. Clearly

\[F(\zeta_0(x-z) - \zeta_0(x))(y) = \int_{\mathbb{R}^n} (\zeta_0(x-z) - \zeta_0(x)) \cdot \chi \left( \frac{y}{x} \right) \, dx = \left( \chi \left( \frac{y}{z} \right) - 1 \right) \cdot \tilde{\zeta}_0(y),\]

a.e. in $y \in L^1$. For each $z \in B_1$ we have
\[
\int_{\mathbb{R}^n} |\zeta_0(x - z) - \zeta_0(x)| \, dx = \int_{\mathbb{R}^n} |\zeta(x - z) - \zeta(x)| \, dx
\]
\[
= \sum_{j=1}^{+\infty} \int_{S_j} |\zeta(x - z) - \zeta(x)| \, dx = \sum_{j=1}^{+\infty} \int_{S_0} |\zeta(x - z\beta^j) - \zeta(x)| \, dx
\]
\[
= \sum_{j=1}^{+\infty} \int |\zeta(x + \beta^j y) - \zeta(x)| \, dx < +\infty.
\]

So \(\sup_{z \in \mathcal{B}_1} \|\zeta_0(\cdot - z) - \zeta_0(\cdot)\|_1 < +\infty\). This implies the family functions \(F(\zeta_0(\cdot - z) - \zeta_0(\cdot)) \in C_0\) are uniformly bounded when \(|z| \leq q\). For each \(y\), \(|y| \geq 1\), the function \(z \mapsto |\chi((y, z) - 1)|\) is continuous on \(\mathcal{B}_1\) which is a compact set, so it attains maximum \(M (M > 0)\) at some \(z_y \in \mathcal{B}_1\). It is not hard to see that the value \(M\) does not depend on \(y\), \(|y| \geq 1\). So for almost every where \(y\), \(|y| \geq 1\), we have
\[
|\tilde{\zeta}_0(y)| = M^{-1} |F(\zeta_0(\cdot - z_y) - \zeta_0(\cdot))(y)|,
\]
then \(|\tilde{\zeta}_0(y)| \leq c < +\infty\) a.e. in \(|y| \geq 1\). Let \(s \in \mathbb{Z}_+\), then
\[
\tilde{\zeta}_0(y) - \tilde{\zeta}_0(\beta^{-s} y) = \int_{q^s \cdot 1 \leq |x| \leq q^{s+1}} \tilde{\zeta}(x) \chi((x, y)) \, dx
\]
\[
= \sum_{j=1}^{s} \int_{S_0} \tilde{\zeta}(x) \chi(\beta^{-j}(x, y)) \, dx.
\]
So for all \(y \in S\), we get \(|\beta^{-j}(x, y)| \leq q^j \cdot q^{-s} \leq 1\) for all \(1 \leq j \leq s\). This implies that \(\tilde{\zeta}(y) = \tilde{\zeta}(\beta^{-s} y)\) for a.e. in \(y \in S_{-s}\) (\(s \in \mathbb{Z}_+\)). Then, \(|\tilde{\zeta}(y)| \leq c\) a.e. in \(y \in \mathbb{R}^n\) or \(\zeta_0 \in L^\infty\). From the equality \(\tilde{\zeta}_0(y) = \tilde{\zeta}_0(\beta^k y)\) a.e. in \(y \in \mathbb{R}^n\), it follows that \(\{\tilde{\zeta}_0\}_{k \geq 0}\) are uniformly bounded in \(L^\infty\). \(\square\)

Now let us consider \(\{m\}_{m \geq 1}\), that are kernels from \(L^1_{\text{loc}}\) which satisfy the following condition
\[
\sup_{y \neq 0} \sum_{m=1}^{+\infty} \int_{|x| \geq q^2 |y|} |\zeta_m(x - y) - \zeta_m(x)| \, dx \leq c_2 < +\infty, \tag{5.1}
\]
and let \(Tf(x) = \sup_{m \geq 1} |\zeta_m \ast f(x)|\). The following is our main theorem.

**Theorem 5.3.** If \(T\) could be extended to bounded operator from \(L^\ell(\mathbb{R}^n)\) to \(L^\ell(\mathbb{R}^n)\), \(1 \leq \ell < +\infty\), then \(T\) is of weak type \((1, 1)\)
\[
\left| \{ \{x: T \{f\} > \lambda\} \right| = \frac{C_T}{\lambda} \cdot \|f\|_1 \quad (\forall f \in L^1(\mathbb{R}^n), \lambda > 0),
\]
where \(C_T > 0\) is some constant and can be chosen as
\[
C_T = \frac{2}{(\ell - 1)^{-1}} \cdot q^{2(\ell - 1)} \cdot \|T\|_\ell + 4c_2.
\]

**Proof.** From Lemma 2.1, for any \(f \in L^1(\mathbb{R}^n), \alpha > 0\), there exist a subset \(P\) of \(\mathbb{Z}_+ \times \mathbb{Z}\) and a mapping \((m, k) \mapsto x_{m, k}\) such that \(\{x_{m, k} + B_k\}: (m, k) \in P\) is pairwise disjoint, and \(\alpha \leq \frac{1}{|B_k|} \int_{x_{m, k} + B_k} f(y) \, dy \leq \alpha q^n\). More than that, if \(D_\alpha = \bigcup_{(m, k) \in P} x_{m, k} + B_k\), then \(|D_\alpha| < +\infty\), and \(\lim_{\alpha \to +\infty} |D_\alpha| = 0, f(x) \leq \alpha \) a.e. in \((D_\alpha)^c\), and \(\alpha |D_\alpha| \leq \int_{D_\alpha} f \leq q^n \cdot \alpha \cdot |D_\alpha|\). Now take \(\alpha > 0\) (we shall choose \(\alpha > 0\) small enough later). Let us put \(f(x) = g(x) + b(x)\), and \(b_{m, k}(x) = b(x)X_{x_{m, k} + B_k}\), where \(X_{x_{m, k} + B_k}\) is the characteristic function of \(x_{m, k} + B_k\), for all \((m, k) \in P\). It is easy to see that \(b(x) = 0\) for each \(x \in (D_\alpha)^c\).
\[
\int_{\mathbb{R}^n} b_{m, k}(y) \, dy = 0 \quad \text{for all} \quad (m, k) \in P_\alpha,
\]
and \(Tf(x) \leq Tg(x) + Tb(x)\) for a.e. \(x \in \mathbb{R}^n\)
\[
\left| \{ \{x: T \{f\} > \lambda\} \right| \leq \left| \{ \{x: T \{g\} > \frac{\lambda}{2}\} \right| + \left| \{ \{x: T \{b\} > \frac{\lambda}{2}\} \right|.
\]
For the function \(g\): Firstly, we shall prove \(g \in L^\ell(\mathbb{R}^n)\). Indeed,
we have  

For all  

(b) If  

then  

\[
\|\|g\|\|^\ell = \int_{(D_a)^c} |g(x)|^\ell \, dx + \int_{D_a} |g(x)|^\ell \, dx 
\]

\[
\leq \alpha^{\ell-1} \int_{(D_a)^c} |f(x)| \, dx + \sum_{(m,k) \in P_a} \int_{x_{m,k} + B_k} \frac{1}{|B_k|^{\ell-1}} \left( \int_{x_{m,k} + B_k} |f(y)| \, dy \right)^\ell \, dx 
\]

\[
\leq \lambda^{\ell-1} \int_{(D_a)^c} |f(x)| \, dx + \sum_{(m,k) \in P_a} \left( \alpha \cdot q^n \right)^{\ell-1} \int_{x_{m,k} + B_k} |f(y)| \, dy 
\]

\[
\leq \lambda^{\ell-1} \int_{D_a^c} |f(x)| \, dx + \left( \alpha \cdot q^n \right)^{\ell-1} \int_{D_a} |f(x)| \, dx < +\infty. 
\]

So  

\[
\left\{ x : Tg(x) > \frac{\lambda}{2} \right\} \leq \frac{2\ell}{\lambda^\ell} \cdot \|g\|\|^\ell \cdot \int Tg(x) \, dx 
\]

\[
\leq \frac{2\ell}{\lambda^\ell} \cdot C_\ell \cdot \|g\|\|^\ell \leq \frac{2\ell \cdot C_\ell \cdot \alpha^{\ell-1} \cdot q^{n(\ell-1)}}{\lambda^\ell} \cdot \|f\|_1. 
\]

For the function  

we have  

(a)  

(b)  

For all  

we obtain  

\[
\int (D_a^*)^c \cdot Tb(x) \, dx \leq \sum_{(m,k) \in P_a} \left( x_{m,k} + B_{k+1} \right)^c \int Tb_{m,k}(y) \, dy 
\]

\[
\leq \sum_{(m,k) \in P_a} \left( x_{m,k} + B_{k+1} \right)^c \int dy \sup \left( x_{m,k} + B_k \right) \int \left| \chi_j(y-z) - \chi_j(y-x_{m,k}) \right| \cdot |b_{m,k}(z)| \, dz 
\]

\[
= \sum_{(m,k) \in P_a} \left( x_{m,k} + B_{k+1} \right)^c \int dy \cdot \sum_{j=1}^{+\infty} \int \left| \chi_j(y-z) - \chi_j(y-x_{m,k}) \right| \cdot |b_{m,k}(z)| \, dz 
\]

\[
\leq \sum_{(m,k) \in P_a} \left( x_{m,k} + B_{k+1} \right)^c \int dz \left( \sum_{y-x_{m,k} \geq q^j} + \sum_{y-x_{m,k} < q^j} \left| \chi_j(y-z) - \chi_j(y-x_{m,k}) \right| \cdot |b_{m,k}(z)| \, dy \right) 
\]

\[
\leq \sum_{(m,k) \in P_a} \left( x_{m,k} + B_{k+1} \right)^c \int |b_{m,k}(z-x_{m,k})| \, dz \left( \sum_{j=1}^{+\infty} \left| \chi_j(y-z) - \chi_j(y-x_{m,k}) \right| \cdot |b_{m,k}(z)| \, dy \right) 
\]

\[
\leq \sum_{(m,k) \in P_a} \left( x_{m,k} + B_{k+1} \right)^c \int |b_{m,k}(z)\, dz = c_2 \sum_{(m,k) \in P_a} \left( \int_{x_{m,k} + B_k} |f(z)| \, dz + \int_{x_{m,k} + B_k} |g(z)| \, dz \right) 
\]

\[
\leq c_2 \sum_{(m,k) \in P_a} \left( \int_{x_{m,k} + B_k} |f(z)| \, dz + \int_{x_{m,k} + B_k} |g(z)| \, dz \right) 
\]
This implies a special result that is similar to a result contained in Theorem 3.1 of K. Phillip and M. Taibleson (see [38]).

Remark. Now we shall provide an application of the above theorem to a known result obtained by K. Phillips and M. Taibleson for singular integrals. Let us choose \( \zeta \in L_1^1(\mathbb{R}^n) \). Then (5.1) is equivalent to

\[
\sup_{y \neq 0} \int_{|x - y| \geq q^2 |y|} |\zeta(x - y) - \zeta(x)| \, dx \leq c_2 < +\infty, \tag{5.2}
\]

and

\[
T_k f(x) = \int_{|x - y| \geq q^{-k}} \zeta(x - y) f(y) \, dy.
\]

for each \( f \in L^\ell, \, 1 < \ell < +\infty \).

If in addition we assume that \( \zeta \) satisfies both (5.2) and conditions in Lemma 5.2, then now from Lemma 5.2 we can conclude that

\[
\|T_k f\|_2 \leq B_2 \cdot \|f\|_2, \quad \forall f \in L^2(\mathbb{R}^n), \quad \forall k \in \mathbb{Z}_+.
\]

where the constant \( B_2 \) does not belongs to \( f \) and \( k \in \mathbb{Z}_+ \). This means that \( \lim_{k \to +\infty} T_k f = Tf \) exists in \( L^2 \)-norm, and \( T \) also satisfies \( \|T f(x)\|_2 \leq B_2 \|f\|_2 \) for all \( f \in L^2(\mathbb{R}^n) \). So the operator \( T^* f = |Tf| \) exists also in \( L^2(\mathbb{R}^n) \) and \( \|T^* f\|_2 \leq B_2 \|f\|_2 \). This implies \( T^* \) must be of weak type of \((1,1)\) by Theorem 5.3. So we have proved that \( T^* \) is of weak type \((1,1)\). This is a special result that is similar to a result contained in Theorem 3.1 of K. Phillip and M. Taibleson (see [38]).

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