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Automorphic Forms, Fake Monster Algebras, and Hyperbolic Reflection Groups

Nils R. Scheithauer[1]

Department of Mathematics, University of California, Berkeley, California 94720
E-mail: nrs@math.berkeley.edu

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We construct two families of automorphic forms related to twisted fake monster algebras and calculate their Fourier expansions. This gives a new proof of their

tion groups of the root lattices of these algebras.

1. INTRODUCTION

Borcherds, Gritsenko, and Nikulin have shown that there are interesting relations between automorphic forms, generalized Kac–Moody algebras, and hyperbolic reflection groups. They can be summarized roughly as follows. First the denominator identities of nice generalized Kac–Moody algebras often define automorphic forms. Second many nice reflection groups of Lorentzian lattices are associated to automorphic forms whose singularities are at the reflection hyperplanes of the reflection group. Finally the root lattices of nice generalized Kac–Moody algebras often have nice hyperbolic reflection groups. In all 3 cases it is not known what the precise necessary conditions are. For example, it is not known when the denominator function of a generalized Kac–Moody algebra is an automorphic form.

In this paper we give some new examples for the above relations. We construct two families of automorphic forms of singular weight. The automorphic forms are the denominator functions of generalized Kac–Moody algebras similar to the fake monster algebras. In the bosonic case

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the reflection groups of their root lattices are associated to automorphic forms. We show that the reflection groups of the root lattices are similar to those of $II_{25,1}$ and $II_{9,1}$.

We describe the sections of this paper in more detail.

In the second section we recall some results about lattices and the singular theta correspondence.

We use this correspondence in Section 3 to construct a family of automorphic forms related to twisted fake monster algebras. This gives a new proof of their denominator identities and shows that they define automorphic forms.

In Section 4 we derive analogous results for twisted fake monster superalgebras. In addition we get new infinite product identities which are the denominator identities of generalized Kac–Moody superalgebras.

In the last section we describe the reflection groups of the root lattices of the generalized Kac–Moody algebras in this paper. In the bosonic case they are related to automorphic forms. In the super case we use Vinberg's algorithm to determine their fundamental domains.

2. LATTICES AND AUTOMORPHIC FORMS

In this section we fix some notations and recall some results on lattices and the singular theta correspondence [B3].

2.1. *Lattices*

Let M be a rational lattice with dual M'. We write $M(n)$ for the lattice obtained from M by multiplying all norms with n. A root α of M is a primitive vector of positive norm such that the reflection $\sigma_\alpha(x) = x - 2(x, \alpha)\,\alpha/\alpha^2$ is an automorphism of M. This implies that $2\alpha/\alpha^2$ is in M'. If M is even we define the level of M as the smallest positive integer n such that $n\lambda^2 \in 2\mathbb{Z}$ for all $\lambda \in M'$. It follows $nM' \subset M$. If α is a root of M then α^2 divides $2n$.

The lattice $II_{1,1}(n)$ consists of the elements $(m_1, m_2) \in \mathbb{Z}^2$ of norm $(m_1, m_2)^2 = -2nm_1m_2$. The lattice has determinant n^2 and the quotient of the dual $II_{1,1}(n)'$ by $II_{1,1}(n)$ is $\mathbb{Z}_n^2$. We write the elements of $II_{1,1}(n)'$ as $(m_1/n, m_2/n)$ with $m_i \in \mathbb{Z}$ so that $(m_1/n, m_2/n)^2 = -2m_1m_2/n$. $II_{1,1}(n)$ has level n and $II_{1,1}(n)'$ is isomorphic to $II_{1,1}(1/n)$.

Let E_8^p be the sublattice of E_8 fixed by an automorphism of cycle shape $1^m p^m$ where p is a prime such that $m = 8/(p+1)$ is an integer. Then E_8^p has level p and determinant p^m. The quotient $E_8^{p\prime}/E_8^p$ is $\mathbb{Z}_p^m$ and $E_8^{p\prime}(p) = E_8^p$. We give more details on these lattices in the last section.

Let Λ be the Leech lattice and Λ_p the sublattice fixed by an automorphism of cycle shape $1^m p^m$ where p is a prime such that $m = 24/(p+1)$ is an integer. For $p = 2$ resp. $p = 3$ the lattice is the Barnes–Wall resp. Coxeter–Todd lattice. Λ_p has similar properties as the Leech lattice. It has no roots and $\Lambda_p'(p) = \Lambda_p$. Furthermore Λ_p has level p and $\Lambda_p'/\Lambda_p = \mathbb{Z}_p^m$.

2.2. *The Singular Theta Correspondence*

Borcherds' singular theta correspondence [B3] gives a construction of automorphic forms from vector valued modular forms. We use slightly different notations here because we prefer to work with positive definite rather than with negative definite lattices.

Let M be an even lattice of signature (b^+, b^-) and

$$F(\tau) = \sum_{\gamma \in M'/M} f_\gamma(\tau)\, e^\gamma$$

be a function on the upper halfplane $\mathbb{H}$ with values in the group ring $\mathbb{C}[M'/M]$. F is a modular form of type ρ_M and weight m if the components satisfy

$$f_\gamma(T\tau) = e(-\gamma^2/2)\, f_\gamma(\tau)$$

$$f_\gamma(S\tau) = \frac{\sqrt{i}^{\,b^+ - b^-}}{\sqrt{|M'/M|}}\, \tau^m \sum_{\delta \in M'/M} e((\gamma, \delta))\, f_\delta(\tau)$$

under the standard generators $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ of $SL_2(\mathbb{Z})$. The first condition means that the exponents in the Fourier expansion of f_γ are all in $\mathbb{Z} - \gamma^2/2$. We say that F is holomorphic on $\mathbb{H}$ and meromorphic at the cusps if the components can be written

$$f_\gamma(\tau) = \sum_{n \in \mathbb{Q}} [f_\gamma](n)\, q^n$$

with coefficients $[f_\gamma](n) = 0$ for $n \ll 0$.

In the next section we will construct vector valued modular forms using products of Dedekind's eta function $\eta(\tau) = q^{1/24} \prod_{n>0} (1 - q^n)$. The following lemma will be useful to calculate their S-transformations.

LEMMA 2.1. *Let $f(\tau) = \eta((k\tau + j)/m)$ where j, k, and m are integers and let j', k', and m' be integers such that the matrix*

$$A = \begin{pmatrix} j/k' & -(jj' + kk')/km \\ m/k' & -j'/k \end{pmatrix}$$

is in $SL_2(\mathbb{Z})$, $km = k'm'$ *and* $m/k' > 0$. *Then the* S*-transformation of* f *is* $f(S\tau) = \varepsilon(A)\sqrt{m\tau/m'i}\,\eta((k'\tau+j')/m')$ *with* $\varepsilon(A)$ *as given in eq.* (74.93) *of* [R].

Proof. Let $F_m = \left(\begin{smallmatrix} 0 & -1 \\ m & 0 \end{smallmatrix}\right)$ be the Fricke involution. Then $\eta((k\tau+j)/m) = \eta(F_m ST^j SF_k\tau)$. The lemma now follows from $F_m ST^j SF_k S = AF_{m'}ST^{j'}SF_{k'}$ and the transformation formula of the eta function.

Borcherds' singular theta correspondence lifts a vector valued modular form F to an automorphic form (Theorem 13.3 in [B3]).

THEOREM 2.2. *Let* M *be an even lattice of signature* $(b^+, 2)$ *and* F *a modular form of weight* $1-b^+/2$ *and representation* ρ_M *which is holomorphic on* $\mathbb{H}$ *and meromorphic at the cusps and whose coefficients* $[f_\lambda](m)$ *are integers for* $m \leqslant 0$. *Then there is a meromorphic function* $\Psi_M(Z_M, F)$ *for* $Z \in P$ *with the following properties.*

(1) $\Psi_M(Z_M, F)$ *is an automorphic form of weight* $[f_0](0)/2$ *for the group* $\mathrm{Aut}(M, F)$ *with respect to some unitary character.*

(2) *The only zeros or poles of* Ψ_M *lie on the rational quadratic divisors* $\lambda^\perp$ *for* $\lambda \in M$ *with* $\lambda^2 > 0$ *and are zeros of order*

$$\sum_{\substack{0<x \\ x\lambda \in M'}} [f_{x\lambda}](-x^2\lambda^2/2)$$

or poles if this number is negative.

(3) Ψ_M *is a holomorphic function if the orders of all zeros are non-negative. If in addition* M *has dimension at least* 5, *or if* M *has dimension* 4 *and contains no* 2 *dimensional isotropic sublattice, then* Ψ_M *is a holomorphic automorphic form. If in addition* $[f_0](0) = b^+-2$ *then* Ψ_M *has singular weight and the only nonzero Fourier coefficients of* Ψ_M *correspond to norm* 0 *vectors in* $L = K/\mathbb{Z}z$ *with* $K = M \cap z^\perp$.

(4) *For each primitive norm* 0 *vector* z *in* M *and for each Weyl chamber* W *of* L *the restriction* $\Psi_z(Z, F)$ *has an infinite product expansion converging when* Z *is in the neighborhood of the cusp of* z *and* $Y \in W$ *which is up to a constant*

$$e((Z, \rho(L, W, F_L))) \prod_{\substack{\lambda \in L' \\ (\lambda, W) > 0}} \prod_{\substack{\delta \in M'/M \\ \delta | K = \lambda}} (1 - e((\lambda, Z) + (\delta, z')))^{[f_\delta](-\lambda^2/2)}.$$

A modular form can also give us some information on the reflection group of a lattice (Theorem 12.1 in [B3]).

THEOREM 2.3. *Let M be a Lorentzian lattice of dimension $1+b^+$ and F a modular form of weight $(1-b^+)/2$ and representation ρ_M which is holomorphic on $\mathbb{H}$ and meromorphic at the cusps and whose coefficients $[f_\lambda](m)$ are real for $m<0$. Suppose that if λ is a positive norm vector in M' and $[f_\lambda](-\lambda^2/2)\neq 0$ then reflection in $\lambda^\perp$ is in $\mathrm{Aut}(M,F,C)$. Then $\mathrm{Aut}(M,F,C)$ is the semidirect product of a reflection subgroup and a subgroup fixing the Weyl vector $\rho(M,W,F)$ of a Weyl chamber W. In particular if the Weyl vector has negative norm then the reflection group of M has finite index in the automorphism group and has only finitely many simple roots. If the Weyl vector has zero norm but is nonzero then the quotient of the automorphism group of M by the reflection group has a free abelian subgroup of finite index.*

3. AUTOMORPHIC FORMS AND THE FAKE MONSTER ALGEBRA

In this section we recall some results about the fake monster algebra and the twisted denominator identities. Then we give a new proof of the twisted denominator identities corresponding to certain automorphisms of prime order and show that they define automorphic forms of singular weight. The idea of the proof is to find appropriate lattices and modular forms and then apply the singular theta correspondence.

3.1. *The Fake Monster Algebra*

The fake monster algebra is a Lie algebra constructed by Borcherds describing the physical states of a bosonic string moving on a torus. It was the first explicit example of a generalized Kac–Moody algebra. We sketch two different constructions. The vertex algebra V of the even unimodular Lorentzian lattice $II_{25,1}$ carries an action of the Virasoro algebra. Let $P_n=\{v\in V \mid L_0 v=nv \text{ and } L_m v=0 \text{ for } m>0\}$. The vertex algebra induces a product on the space $P_1/L_{-1}P_0$ of physical states and turns it into a Lie algebra. The fake monster algebra G is the quotient of $P_1/L_{-1}P_0$ by the kernel of a natural bilinear form. The other construction of this algebra uses the BRST-operator. The vertex algebra of the integral lattice $II_{25,1}\oplus\mathbb{Z}$ is acted on by the BRST-operator Q satisfying $Q^2=0$. Here the space of physical states is given by the cohomology group $\mathrm{Ker}(Q)/\mathrm{Im}(Q)$. Again the vertex algebra induces the Lie bracket on this space. This Lie algebra is isomorphic to G.

The fake monster algebra has the following properties. The root lattice is the Lorentzian lattice $II_{25,1} = \Lambda \oplus II_{1,1}$ where Λ is the Leech lattice with elements $\alpha = (r, m, n)$ and norm $\alpha^2 = r^2 - 2mn$. A nonzero vector $\alpha \in II_{25,1}$ is a root if and only if $\alpha^2 \leqslant 2$. The multiplicity of a root α is given by $[1/\Delta](-\alpha^2/2)$ where $1/\Delta$ is the modular form $1/\Delta(\tau) = 1/\eta(\tau)^{24} = q^{-1} + 24 + 324q + 3200q^2 + \cdots$. The real simple roots of the fake monster algebra are the norm 2 vectors α in $II_{25,1}$ with $(\rho, \alpha) = -1$ where $\rho = (0, 0, 1)$ is the Weyl vector and the imaginary simple roots are the positive multiples $n\rho$ of the Weyl vector with multiplicity 24. The Weyl group W is the reflection group of $II_{25,1}$ and the positive roots are the roots α satisfying $(\alpha, \rho) < 0$ or $\alpha = n\rho$ for $n > 0$. The denominator identity is given by

$$e^{\rho} \prod_{\alpha \in II_{25,1}^{+}} (1 - e^{\alpha})^{[1/\Delta](-\alpha^2/2)} = \sum_{w \in W} \det(w)\, w\left(e^{\rho} \prod_{n>0} (1 - e^{n\rho})^{24} \right).$$

The sum in this identity defines the denominator function of the fake monster algebra. It is an automorphic form for $\mathrm{Aut}(II_{26,2})$.

The no-ghost theorem gives an action of the automorphism group of the Leech lattice on the fake monster algebra which can be used to calculate twisted denominator identities [B2].

We consider the case that the automorphism has cycle shape $1^m p^m$ where p is a prime such that $m = 24/(p+1)$ an integer (cf. [B2, N]). Then the corresponding twisted denominator identity is

$$\begin{aligned} e^{\rho} \prod_{\alpha \in L^{+}} (1 - e^{\alpha})^{[f](-\alpha^2/2)} \prod_{\alpha \in pL'^{+}} (1 - e^{\alpha})^{[f](-\alpha^2/2p)} \\ = \sum_{w \in W} \det(w)\, w\left(e^{\rho} \prod_{n>0} (1 - e^{n\rho})^{m} (1 - e^{pn\rho})^{m} \right). \end{aligned}$$

with $L = \Lambda_p \oplus II_{1,1}$ and $f(\tau) = 1/\eta(\tau)^m \eta(p\tau)^m$. The Weyl vector is $\rho = (0, 0, 1)$ and the Weyl group W is the reflection group of L. It is generated by the norm 2 vectors in L and the norm $2p$ vectors in $pL' \subset L$. This identity is also the untwisted denominator identity of a generalized Kac–Moody algebra. The real simple roots of this algebra are the roots α satisfying $(\rho, \alpha) = -\alpha^2/2$ and the imaginary simple roots are the positive multiples $n\rho$ of the the Weyl vector with multiplicity $2m$ if p divides n and m else. The root lattice of the algebra is L and the multiplicity of a root α is given by $[f](-\alpha^2/2)$ if α is in L but not in pL' and by $[f](-\alpha^2/2) + [f](-\alpha^2/2p)$ if α is in $pL' \subset L$.

3.2. *Automorphic Forms*

Now we give a proof of the twisted denominator identities of the fake monster algebra corresponding to automorphisms of cycle shape $1^m p^m$ using the singular theta correspondence. First we work out the case $p=2$ explicitly. The general case will be a simple generalization of this example.

Let

$$f(\tau)=1/\eta(\tau)^8\,\eta(2\tau)^8=q^{-1}+8+52q+256q^2+1122q^3+4352q^4+\cdots.$$

Then f is a modular form for $\Gamma_0(2)$ of weight -8 with singularities at the cusps 0 and $i\infty$. The Fourier expansion of

$$f(\tau/2)=1/\eta(\tau/2)^8\,\eta(\tau)^8=q^{-1/2}+8+52q^{1/2}+256q+\cdots$$

can be decomposed into two series with integral and half-integral exponents in q. Define

$$g_0(\tau)=(f(\tau/2)+f((\tau+1)/2))/2=8+256q+4352q^2+\cdots$$

and

$$g_1(\tau)=(f(\tau/2)-f((\tau+1)/2))/2=q^{-1/2}+52q^{1/2}+1122q^{3/2}+\cdots.$$

We will use the functions f, g_0, and g_1 to construct the vector valued modular form F. Therefore we need their transformation properties under the generators of $SL_2(\mathbb{Z})$. The T-transformations are clear. Lemma 2.1 implies the following S-transformations

$$\begin{aligned}f(S\tau)&=2^4 f(\tau/2)/\tau^8\\ f((S\tau)/2)&=f(\tau)/2^4\tau^8\\ f((S\tau+1)/2)&=f((\tau+1)/2)/\tau^8\end{aligned}$$

so that

$$\begin{aligned}f(S\tau)&=2^4(g_0(\tau)+g_1(\tau))/\tau^8\\ g_0(S\tau)&=(f(\tau)/2^4+g_0(\tau)-g_1(\tau))/2\tau^8\\ g_1(S\tau)&=(f(\tau)/2^4-g_0(\tau)+g_1(\tau))/2\tau^8.\end{aligned}$$

Another way to express this is to say that f generates a 3 dimensional representation of $SL_2(\mathbb{Z})$ of weight -8. With respect to the basis $\{f, g_0, g_1\}$ the T-matrix and S-matrix are given by

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad S = \tfrac{1}{2} \begin{bmatrix} 0 & 2^5 & 2^5 \\ 2^{-4} & 1 & -1 \\ 2^{-4} & -1 & 1 \end{bmatrix}.$$

We describe the discriminant form of the Barnes–Wall lattice Λ_2. (Note that the description in [N] is false.) Λ_2'/Λ_2 has 135 nonzero elements of norm 0 mod 1 and 120 elements with norm $1/2$ mod 1. We call the first of type 0 and the latter of type 1. The following tables show how many elements in Λ_2'/Λ_2 have given inner product with γ. For γ of type 0 we have

	0_0	0	1
0	1	71	56
$\frac{1}{2}$	0	64	64

and for γ of type 1

	0_0	0	1
0	1	63	64
$\frac{1}{2}$	0	72	56

The lattice

$$M = \Lambda_2 \oplus II_{1,1}(2) \oplus II_{1,1}$$

is an even lattice of level 2, determinant 2^{10}, and signature $(18, 2)$. The discriminant form of $II_{1,1}(2)$ is $\mathbb{Z}_2^2$ with norm $(a, b)^2/2 = -ab/2$ mod 1. This implies that the discriminant form of M has 527 nonzero elements of norm 0 mod 1 and 496 with norm $1/2$ mod 1. Using the above tables we find for γ of type 0 resp. type 1 in M'/M

	0_0	0	1
0	1	271	240
$\frac{1}{2}$	0	256	256

	0_0	0	1
0	1	255	256
$\frac{1}{2}$	0	272	240

Now we define the vector valued modular form F. Let

$$F(\tau) = \sum_{\gamma \in M'/M} f_\gamma(\tau)\, e^\gamma$$

with

$$\begin{aligned} f_\gamma(\tau) &= f(\tau)+g_0(\tau) && \text{if} \quad \gamma=0 \\ &= g_0(\tau) && \text{if} \quad -\gamma^2/2=0 \bmod 1 \\ &= g_1(\tau) && \text{if} \quad -\gamma^2/2=1/2 \bmod 1. \end{aligned}$$

Then F is a modular form of weight -8 and representation ρ_M. Clearly F transforms correctly under T. We show that F also transforms correctly under S. Let $\gamma=0$. Then

$$\begin{aligned} 2^5\tau^8 f_\gamma(S\tau) &= 2^5\tau^8 f(S\tau)+2^5\tau^8 g_0(S\tau) \\ &= f(\tau)+(2^9+2^4)\, g_0(\tau)+(2^9-2^4)\, g_1(\tau) \\ &= \sum_{\delta\in M'/M} f_\delta(\tau). \end{aligned}$$

For γ of type 0 we get

$$\begin{aligned} 2^5\tau^8 f_\gamma(S\tau) &= 2^5\tau^8 g_0(S\tau) \\ &= f(\tau)+2^4 g_0(\tau)-2^4 g_1(\tau) \\ &= \sum_{\delta\in M'/M} e((\gamma,\delta))\, f_\delta(\tau)\,. \end{aligned}$$

Finally for γ of type 1

$$\begin{aligned} 2^5\tau^8 f_\gamma(S\tau) &= 2^5\tau^8 g_1(S\tau) \\ &= f(\tau)-2^4 g_0(\tau)+2^4 g_1(\tau) \\ &= \sum_{\delta\in M'/M} e((\gamma,\delta))\, f_\delta(\tau)\,. \end{aligned}$$

Hence F is a vector valued modular form for M.

The singular theta correspondence now implies that there is a holomorphic automorphic form Ψ_M for $\mathrm{Aut}(M,F)$ of weight $16/2=8$. The zeros of Ψ_M are zeros of order 1 coming from divisors of norm 2 vectors of M and norm 1 vectors of M'.

The level of a primitive norm 0 vector of M divides 2. Near the corresponding cusp we can expand Ψ_M in an infinite product. We can also work out the Fourier expansion because Ψ_M has singular weight so that the nonzero Fourier coefficients correspond to norm 0 vectors.

At the level 1 cusp we decompose $M = L \oplus II_{1,1}$ where $L = \Lambda_2 \oplus II_{1,1}(2)$ and take z as primitive norm 0 vector in $II_{1,1}$. Then the product expansion of $\Psi_z(Z, F)$ is

$$e((\rho, Z)) \prod_{\lambda \in L^+} (1-e((\lambda, Z)))^{[f](-\lambda^2/2)} \prod_{\lambda \in L'^+} (1-e((\lambda, Z)))^{[f](-\lambda^2)}$$

$$= \sum_{w \in W} \det(w)\, e((w\rho, Z)) \prod_{n>0} (1-e((nw\rho, Z)))^8 (1-e((2nw\rho, Z)))^8,$$

where $\rho = (0, 0, 1/2)$ and W is the reflection group generated by the norm 1 vectors of L' and the norm 2 vectors of $L \subset L'$. Note that the vectors of L' have integral norms because L has level 2.

Next we write $M = L \oplus II_{1,1}(2)$ with $L = \Lambda_2 \oplus II_{1,1}$ and take z as primitive norm 0 vector in $II_{1,1}(2)$. We say that z has level 2 because $|M'/M| = 2^2\,|L'/L|$. At this cusp the product expansion of $\Psi_z(Z, F)$ is

$$e((\rho, Z)) \prod_{\lambda \in L^+} (1-e((\lambda, Z)))^{[f](-\lambda^2/2)} \prod_{\lambda \in 2L'^+} (1-e((\lambda, Z)))^{[f](-\lambda^2/4)}$$

$$= \sum_{w \in W} \det(w)\, e((w\rho, Z)) \prod_{n>0} (1-e((nw\rho, Z)))^8 (1-e((2nw\rho, Z)))^8,$$

where $\rho = (0, 0, 1)$ and W is the reflection group generated by the norm 2 vectors of L and the norm 4 vectors of $2L' \subset L$.

The expansion of Ψ_M at the level 2 cusp is the twisted denominator identity of the fake monster algebra corresponding to an automorphism of cycle shape $1^8 2^8$. This gives a new proof of this identity and shows that the denominator function of the corresponding generalized Kac–Moody algebra is an automorphic form of singular weight. The two expansions of Ψ_M look rather similar and are really the same. If we rescale the dual of $\Lambda_2 \oplus II_{1,1}(2)$ with a factor 2 then the expansion of Ψ_M at the level 1 cusp goes over into the expansion at the other cusp.

Now we turn to the general case. Let p be a prime such that $m = 24/(p+1)$ is an integer. The eta product

$$f(\tau) = 1/\eta(\tau)^m\, \eta(p\tau)^m = q^{-1} + m + \cdots$$

is a modular form of weight $-m$ for a level p subgroup of $SL_2(\mathbb{Z})$ with singularities at the cusps 0 and $i\infty$.

$$f(\tau/p) = 1/\eta(\tau/p)^m\, \eta(\tau)^m = q^{-1/p} + m + \cdots$$

can be written

$$f(\tau/p) = g_0(\tau) + g_1(\tau) + \cdots + g_{p-1}(\tau),$$

where the functions g_j have Fourier expansions of the form $\sum [g_j](n)\, q^n$ with $n \in \mathbb{Z} + j/p$. We will use the functions $f, g_0, ..., g_{p-1}$ to construct a vector valued modular form. The T-transformations of these functions are clear and their S-transformations can be calculated with Lemma 2.1.

Let

$$M = \Lambda_p \oplus II_{1,1}(p) \oplus II_{1,1}\ .$$

Then M is an even lattice of level p, determinant p^{m+2}, and signature $(2m+2, 2)$. We define

$$F(\tau) = \sum_{\gamma \in M'/M} f_\gamma(\tau)\, e^\gamma$$

with

$$\begin{aligned} f_\gamma(\tau) &= f(\tau) + g_0(\tau) && \text{if} \quad \gamma = 0 \\ &= g_j(\tau) && \text{if} \quad -\gamma^2/2 = j/p \bmod 1\ . \end{aligned}$$

Then we have

Proposition 3.1. *F is a modular form of weight $-m$ and representation ρ_M which is holomorphic on $\mathbb{H}$ and meromorphic at the cusps.*

From the singular theta correspondence we get

Theorem 3.2. *There is a holomorphic automorphic form Ψ_M for $\mathrm{Aut}(M, F)$ of singular weight m. The zeros of Ψ_M are zeros of order 1 coming from divisors of norm 2 vectors of M and norm $2/p$ vectors of M'. Ψ_M has the following expansions.*

At the level 1 cusp we decompose $M = L \oplus II_{1,1}$ with $L = \Lambda_p \oplus II_{1,1}(p)$ and take z as primitive norm 0 vector in $II_{1,1}$. Then the product expansion of $\Psi_z(Z, F)$ is

$$e((\rho, Z)) \prod_{\lambda \in L^+} (1 - e((\lambda, Z)))^{[f](-\lambda^2/2)} \prod_{\lambda \in L'^+} (1 - e((\lambda, Z)))^{[f](-p\lambda^2/2)}$$

$$= \sum_{w \in W} \det(w)\, e((w\rho, Z)) \prod_{n>0} (1 - e((nw\rho, Z)))^m\, (1 - e((pnw\rho, Z)))^m$$

where $\rho=(0,0,1/p)$ *in* L' *and* W *is the reflection group generated by the norm* $2/p$ *vectors of* L' *and the norm* 2 *vectors of* L.

Next we write $M=L\oplus II_{1,1}(p)$ *with* $L=\Lambda_p\oplus II_{1,1}$ *so that* $|M'/M|=p^2\,|L'/L|$ *and take* z *as primitive norm* 0 *vector in* $II_{1,1}(p)$. *Then we get the following level* p *expansion of* $\Psi_z(Z,F)$

$$e((\rho,Z))\prod_{\lambda\in L^+}(1-e((\lambda,Z)))^{[f](-\lambda^2/2)}\prod_{\lambda\in pL'^+}(1-e((\lambda,Z)))^{[f](-\lambda^2/2p)}$$

$$=\sum_{w\in W}\det(w)\,e((w\rho,Z))\prod_{n>0}(1-e((nw\rho,Z)))^m\,(1-e((pnw\rho,Z)))^m$$

with $\rho=(0,0,1)$ *and* W *the reflection group generated by the norm* 2 *vectors of* L *and the norm* $2p$ *vectors of* $pL'\subset L$.

The two expansions are identical upon rescaling the lattice $(\Lambda_p\oplus II_{1,1}(p))'$ *by* p.

This implies

COROLLARY 3.3. *The denominator function of the generalized Kac–Moody algebra obtained by twisting the fake monster algebra with an automorphism of cycle shape* $1^m p^m$ *defines a holomorphic automorphic form of singular weight.*

4. AUTOMORPHIC FORMS AND THE FAKE MONSTER SUPERALGEBRA

In this section we prove analogous results as in Section 3 for the fake monster superalgebra. The main difference is that the expansions of the automorphic forms at the level 1 and the level p cusp do not coincide so that we get new infinite product identities which are the denominator identities of generalized Kac–Moody superalgebras.

4.1. *The Fake Monster Superalgebra*

The fake monster superalgebra [S1] is a supersymmetric generalized Kac–Moody superalgebra describing the physical states of a superstring moving on a torus. It can be constructed similar to the fake monster algebra as the cohomology group of a BRST-operator acting on the vertex algebra of a rational 18 dimensional lattice. The fake monster superalgebra has root lattice $II_{9,1}=E_8\oplus II_{1,1}$ with elements $\alpha=(r,m,n)$ and norm $\alpha^2=r^2-2mn$. The roots are the nonzero vectors α with $\alpha^2\leqslant 0$. The multiplicity of a root

α is given by $\mathrm{mult}_0(\alpha)=\mathrm{mult}_1(\alpha)=[f](-\alpha^2/2)$ where $f(\tau)=8\eta(2\tau)^8/\eta(\tau)^{16}$ $=8+128q+1152q^2+\cdots$ is a modular form for $\Gamma_0(2)$ of weight -4. The simple roots of the fake monster superalgebra are the norm 0 vectors in the closure of the positive cone of $II_{9,1}$. This implies that the Weyl group is trivial and the Weyl vector is 0. The denominator identity is given by

$$\prod_{\alpha\in II_{9,1}^+}\frac{(1-e^{\alpha})^{[f](-\alpha^2/2)}}{(1+e^{\alpha})^{[f](-\alpha^2/2)}}=1+\sum a(\lambda)\,e^{\lambda},$$

where $a(\lambda)$ is the coefficient of q^n in

$$\frac{\eta(\tau)^{16}}{\eta(2\tau)^8}=1-16q+112q^2-448q^3+1136q^4-\cdots$$

if λ is n times a primitive norm 0 vector in $II_{9,1}^+$ and 0 else.

Using the no-ghost theorem we can construct an action of $2.\mathrm{Aut}(E_8)$ on the fake monster superalgebra and calculate twisted denominator identities [S2]. Let p be a prime such that $m=8/(p+1)$ is integral. Then the twisted denominator identity corresponding to an automorphism of cycle shape $1^m p^m$ is

$$\prod_{\alpha\in L^+}\frac{(1-e^{\alpha})^{[f](-\alpha^2/2)}}{(1+e^{\alpha})^{[f](-\alpha^2/2)}}\prod_{\alpha\in pL'^+}\frac{(1-e^{\alpha})^{[f](-\alpha^2/2p)}}{(1+e^{\alpha})^{[f](-\alpha^2/2p)}}=1+\sum a(\lambda)\,e^{\lambda},$$

where $L=E_8^p\oplus II_{1,1}$ and $f(\tau)=m(\eta(2p\tau)\,\eta(2\tau))^m/(\eta(p\tau)\,\eta(\tau))^{2m}$. $a(\lambda)$ is the coefficient of q^n in $(\eta(p\tau)\,\eta(\tau))^{2m}/(\eta(2p\tau)\,\eta(2\tau))^m$ if λ is n times a primitive norm 0 vector in L^+ and 0 else. This is the untwisted denominator identity of a supersymmetric generalized Kac–Moody superalgebra whose simple roots are the norm 0 vectors in the closure of the positive cone of L. The even and the odd multiplicity of a simple root λ is $2m$ if λ is n times a primitive vector with p dividing n and m else. The root lattice of the algebra is L and the multiplicities of a root α are $\mathrm{mult}_0(\alpha)=\mathrm{mult}_1(\alpha)$ $=[f](-\alpha^2/2)$ if α is in L but not in pL' and $\mathrm{mult}_0(\alpha)=\mathrm{mult}_1(\alpha)=$ $[f](-\alpha^2/2)+[f](-\alpha^2/2p)$ if α is in $pL'\subset L$.

4.2. *Automorphic Forms*

Now we construct the automorphic forms whose expansions give the denominator identities of the twisted fake monster superalgebras and some new identities. In contrast to the bosonic case the components of the vector valued modular form depend not only on the norms of the elements in M'/M but also on their orders.

We start with the case $p=3$ as an example. Let

$$f(\tau)=2\frac{\eta(6\tau)^2\,\eta(2\tau)^2}{\eta(3\tau)^4\,\eta(\tau)^4}=2+8q+24q^2+72q^3+184q^4+\cdots$$

and

$$\gamma(\tau)=\frac{\eta(3\tau/2)^2\,\eta(\tau/2)^2}{\eta(3\tau)^4\,\eta(\tau)^4}=q^{-1/2}-2+3q^{1/2}-8q+15q^{3/2}-24q^2+\cdots .$$

Then f is a modular form for $\Gamma_0(6)$ of weight -2.

f and γ are related by supersymmetry (cf. [S2]) which means that the Fourier expansion of

$$\delta(\tau)=f(\tau)+\gamma(\tau)=q^{-1/2}+3q^{1/2}+15q^{3/2}+43q^{5/2}+\cdots$$

contains only half-integral powers of q. We write

$$f(\tau/3)=g_0(\tau)+g_2(\tau)+g_4(\tau)$$

with

$$\begin{aligned}
g_0(\tau)&=(f(\tau/3)+f((\tau+1)/3)+f((\tau+2)/3))/3\\
&=2+72q+984q^2+\cdots\\
g_2(\tau)&=(f(\tau/3)+\varepsilon^2 f((\tau+1)/3)+\varepsilon f((\tau+2)/3))/3\\
&=8q^{1/3}+184q^{4/3}+2112q^{7/3}+\cdots\\
g_4(\tau)&=(f(\tau/3)+\varepsilon f((\tau+1)/3)+\varepsilon^2 f((\tau+2)/3))/3\\
&=24q^{2/3}+432q^{5/3}+4344q^{8/3}+\cdots,
\end{aligned}$$

where $\varepsilon=e^{2\pi i/3}$ and

$$\delta(\tau/3)=\lambda_1(\tau)+\lambda_3(\tau)+\lambda_5(\tau)$$

with

$$\lambda_1(\tau) = (\delta(\tau/3) + \varepsilon^2\delta((\tau+2)/3) + \varepsilon\delta((\tau+4)/3))/3$$
$$= 3q^{1/6} + 114q^{7/6} + 1437q^{13/6} + \cdots$$
$$\lambda_3(\tau) = (\delta(\tau/3) + \delta((\tau+2)/3)) + \delta((\tau+4)/3))/3$$
$$= 15q^{3/6} + 285q^{9/6} + 3051q^{15/6} + \cdots$$
$$\lambda_5(\tau) = (\delta(\tau/3) + \varepsilon\delta((\tau+2)/3) + \varepsilon^2\delta((\tau+4)/3))/3$$
$$= q^{-1/6} + 43q^{5/6} + 662q^{11/6} + \cdots .$$

f and δ generate an 8 dimensional representation of $SL_2(\mathbb{Z})$ of weight -2. With respect to the basis $\{f, g_0, g_2, g_4, \delta, \lambda_1, \lambda_3, \lambda_5\}$ the T-matrix and S-matrix are given by

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\varepsilon^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon \end{bmatrix}$$

and

$$S = \frac{1}{18}\begin{bmatrix} 0 & 27 & 27 & 27 & 0 & -27 & -27 & -27 \\ 1 & -6 & 3 & 3 & -1 & -3 & 6 & -3 \\ 1 & 3 & 3 & -6 & -1 & 6 & -3 & -3 \\ 1 & 3 & -6 & 3 & -1 & -3 & -3 & 6 \\ 0 & -81 & -81 & -81 & 0 & -27 & -27 & -27 \\ -3 & -9 & 18 & -9 & -1 & -3 & -3 & 6 \\ -3 & 18 & -9 & -9 & -1 & -3 & 6 & -3 \\ -3 & -9 & -9 & 18 & -1 & 6 & -3 & -3 \end{bmatrix}.$$

The S-matrix satisfies $S^2 = 1$.

We define the even lattice

$$M = E_8^3 \oplus II_{1,1}(6) \oplus II_{1,1}$$

of level 6, signature $(6, 2)$, and determinant 324. The discriminant form of M can be described as follows. The lattice E_8^3 is $A_2 \oplus A_2$ and has discriminant form $\mathbb{Z}_3^2$ with norm $(n, m)^2/2 = (n^2+m^2)/3 \bmod 1$. The elements in $II_{1,1}(6)'/II_{1,1}(6) = \mathbb{Z}_6^2$ have norm $(a, b)^2/2 = -ab/6 \bmod 1$.

Now let

$$F(\tau) = \sum_{\gamma \in M'/M} f_\gamma(\tau)\, e^\gamma$$

with

$$\begin{aligned}
f_\gamma(\tau) &= f(\tau)+g_0(\tau) && \text{if } \gamma = 0\\
&= -f(\tau)-g_0(\tau) && \text{if } -\gamma^2/2 = 0 \text{ and } \gamma \text{ has order 2}\\
&= \delta(\tau)+\lambda_3(\tau) && \text{if } -\gamma^2/2 = 1/2 \text{ and } \gamma \text{ has order 2}\\
&= \lambda_j(\tau) && \text{if } -\gamma^2/2 = j/6 \text{ where } j \text{ is odd and } \gamma \text{ has order 6}\\
&= g_j(\tau) && \text{if } -\gamma^2/2 = j/6 \text{ where } j \text{ is even and } \gamma \text{ has order 3}\\
&= -g_j(\tau) && \text{if } -\gamma^2/2 = j/6 \text{ where } j \text{ is even and } \gamma \text{ has order 6.}
\end{aligned}$$

As usual $\gamma^2/2$ is taken mod 1. Then F is a modular form of weight -2 and representation ρ_M.

The singular theta correspondence implies that there is a holomorphic automorphic form Ψ_M for $\mathrm{Aut}(M, F)$ of weight $4/2 = 2$. The zeros of Ψ_M are zeros of order 1 coming from divisors of norm $1/3$ vectors in M' and from divisors α in M' of norm 1 with $2\alpha \in M$.

The level of a primitive norm 0 vector in M divides 6. We can work out the Fourier expansion at the corresponding cusp using that Ψ_M has singular weight.

We start with the expansion at the level 1 cusp. We decompose $M = L \oplus II_{1,1}$ where $L = E_8^3 \oplus II_{1,1}(6)$ and take z as primitive norm 0 vector in $II_{1,1}$. Then the product expansion of $\Psi_z(Z, F)$ is

$$e((\rho, Z)) \prod_{\substack{\alpha \in L'^+\\ 2\alpha \in L}} (1-e((\alpha, Z)))^{\pm[f+\delta](-\alpha^2/2)} \prod_{\alpha \in L'^+} (1-e((\alpha, Z)))^{\pm[f+\delta](-3\alpha^2/2)}$$

$$= \sum_{w \in W} \det(w)\, e((w\rho, Z)) \prod_{n>0} (1-e((nw\rho, Z)))^{(-1)^n 2} (1-e((3nw\rho, Z)))^{(-1)^n 2},$$

where the sign in the exponent of the first product is $-$ if α^2 is even and the image of α has even order in L'/L, i.e., order 2, and $+$ else and the sign in the exponent of the second product is $-$ if $3\alpha^2/2$ is integral and α has even order and $+$ in the other cases. Note that $\alpha \in L'$ with $2\alpha \in L$ implies $\alpha^2 \in \mathbb{Z}$

because L is even and has level 6. The Weyl vector is $\rho=(0,0,1/6)$ and the Weyl group W is generated by the α in L' with $\alpha^2=1/3$ and the α in L' of norm 1 with $2\alpha\in L$. We remark that the roots of L' are the vectors α of norm $1/3, 2/3, 1$, and 2 with resp. $6\alpha\in L$, $3\alpha\in L$, $2\alpha\in L$ and $\alpha\in L$. This implies that W has infinite index in the full reflection group of L'.

This identity is the denominator identity of a generalized Kac–Moody superalgebra with the following simple roots. The real simple roots are the simple roots of the reflection group W, i.e., the roots α satisfying $(\rho,\alpha)=-\alpha^2/2$. The imaginary simple roots are the positive multiples $n\rho$ of the Weyl vector with multiplicity $(-1)^n 4$ if 3 divides n and $(-1)^n 2$ else. Here we use the convention that odd roots have negative multiplicity. Note that there are no odd real simple roots. The root lattice of this algebra is L'. A root α is odd if and only if $3\alpha^2/2$ is integral and α has even order. The multiplicity of an even root $\alpha\in L'$ is $\text{mult}(\alpha)=[f+\delta](-3\alpha^2/2)$ if $2\alpha\notin L$ and $\text{mult}(\alpha)=[f+\delta](-\alpha^2/2)+[f+\delta](-3\alpha^2/2)$ if $2\alpha\in L$. Up to a sign the same formula holds for the odd roots.

Next we write $M=L\oplus II_{1,1}(6)$ with $L=E_8^3\oplus II_{1,1}$ and take z as primitive norm 0 vector in $II_{1,1}(6)$. Then $|M'/M|=6^2\,|L'/L|$ so that z has level 6. The product expansion of $\Psi_z(Z,F)$ is

$$\prod_{\alpha\in L^+}\frac{(1-e((\alpha,Z)))^{[f](-\alpha^2/2)}}{(1+e((\alpha,Z)))^{[f](-\alpha^2/2)}}\prod_{\alpha\in 3L'^+}\frac{(1-e((\alpha,Z)))^{[f](-\alpha^2/6)}}{(1+e((\alpha,Z)))^{[f](-\alpha^2/6)}}$$
$$=1+\sum a(\lambda)\,e((\lambda,Z)),$$

where $a(\lambda)$ is the coefficient of q^n in

$$\frac{\eta(3\tau)^4\,\eta(\tau)^4}{\eta(6\tau)^2\,\eta(2\tau)^2}=1-4q+4q^2-4q^3+20q^4-24q^5+4q^6-\cdots$$

if λ is n times a primitive norm 0 vector in L^+ and 0 else.

This is the twisted denominator identity of the fake monster superalgebra corresponding to an automorphism of cycle shape $1^2 3^2$.

We can calculate a level 2 expansion by taking the primitive norm 0 vector $z=((1,0),(2,0))$ in $II_{1,1}\oplus II_{1,1}(6)$. Then $L=(M\cap z^\perp)/\mathbb{Z}z=E_8^3\oplus II_{1,1}(3)$. We find the product expansion

$$\prod_{\alpha\in L^+}\frac{(1-e((\alpha,Z)))^{[f](-\alpha^2/2)}}{(1+e((\alpha,Z)))^{[f](-\alpha^2/2)}}\prod_{\alpha\in L'^+}\frac{(1-e((\alpha,Z)))^{[f](-3\alpha^2/2)}}{(1+e((\alpha,Z)))^{[f](-3\alpha^2/2)}}$$
$$=1+\sum a(\lambda)\,e((\lambda,Z))$$

where $a(\lambda)$ is the coefficient of q^n in

$$\frac{\eta(3\tau)^4\,\eta(\tau)^4}{\eta(6\tau)^2\,\eta(2\tau)^2} = 1-4q+4q^2-4q^3+20q^4-24q^5+4q^6-\cdots$$

if λ is n times a primitive norm 0 vector in L'^+ and 0 else.

When we rescale the lattice $L' = E_8^3(1/3) \oplus II_{1,1}(1/3)$ by 3 we obtain the expansion at the level 6 cusp.

The general case is as follows. Let p be a prime such that $m = 8/(p+1)$ is an integer. Define

$$f(\tau) = m\,\frac{\eta(2p\tau)^m\,\eta(2\tau)^m}{\eta(p\tau)^{2m}\,\eta(\tau)^{2m}} = m+2m^2q+\cdots$$

and

$$\gamma(\tau) = \frac{\eta(p\tau/2)^m\,\eta(\tau/2)^m}{\eta(p\tau)^{2m}\,\eta(\tau)^{2m}} = q^{-1/2}-m+\cdots\,.$$

f is a modular form of level $2p$ and weight $-m$.

A supersymmetry relation implies that $\delta(\tau) = f(\tau)+\gamma(\tau)$ contains only half-integral powers of q.

We write

$$f(\tau/p) = g_0(\tau)+g_2(\tau)+\cdots+g_{2p-2}(\tau),$$

where the functions g_j have Fourier expansions of the form $\sum\,[g_j](n)\,q^n$ with $n \in \mathbb{Z}+j/p$ and similarly for

$$\delta(\tau/p) = \lambda_1(\tau)+\lambda_3(\tau)+\cdots+\lambda_{2p-1}(\tau)\,.$$

We will use the functions $f, g_0, \ldots, g_{2p-2}$ and $\delta, \lambda_1, \ldots, \lambda_{2p-1}$ to construct a vector valued modular form. Their transformations under T are clear and their S-transformations follow from Lemma 2.1.

The quotient $E_8^{p\prime}/E_8^p = \mathbb{Z}_p^m$ is a vector space over $\mathbb{Z}_p$ with an orthogonal basis $\{\gamma_1, \ldots, \gamma_m\}$ satisfying $\gamma_j^2/2 = 1/p \bmod 1$. Define the even lattice

$$M = E_8^p \oplus II_{1,1}(2p) \oplus II_{1,1}$$

of level $2p$, determinant 2^2p^{m+2} and signature $(2m+2, 2)$ and let

$$F(\tau) = \sum_{\gamma \in M'/M} f_\gamma(\tau)\, e^\gamma$$

with

$$\begin{aligned} f_\gamma(\tau) &= f(\tau)+g_0(\tau) && \text{if } \gamma=0 \\ &= -f(\tau)-g_0(\tau) && \text{if } -\gamma^2/2=0 \text{ and } \gamma \text{ has order } 2 \\ &= \delta(\tau)+\lambda_p(\tau) && \text{if } -\gamma^2/2=1/2 \text{ and } \gamma \text{ has order } 2 \\ &= \lambda_j(\tau) && \text{if } -\gamma^2/2=j/2p \text{ with } j \text{ odd and } \gamma \text{ of order } 2p \\ &= g_j(\tau) && \text{if } -\gamma^2/2=j/2p \text{ with } j \text{ even and } \gamma \text{ of order } p \\ &= -g_j(\tau) && \text{if } -\gamma^2/2=j/2p \text{ with } j \text{ even and } \gamma \text{ of order } 2p. \end{aligned}$$

Then we have

PROPOSITION 4.1. *F is a modular form of weight $-m$ and representation ρ_M which is holomorphic on $\mathbb{H}$ and meromorphic at the cusps.*

From the singular theta correspondence we get

THEOREM 4.2. *There is a holomorphic automorphic form Ψ_M for $\mathrm{Aut}(M,F)$ of weight m. The zeros of Ψ_M are zeros of order 1 coming from divisors of norm $1/p$ vectors in M' and from divisors α in M' with norm 1 and $2\alpha\in M$. Ψ_M has singular weight so that the only nonzero Fourier coefficients of Ψ_M correspond to norm 0 vectors.*

At the level 1 cusp we decompose $M=L\oplus II_{1,1}$ with $L=E_8^p\oplus II_{1,1}(2p)$ and take z as primitive norm 0 vector in $II_{1,1}$. Then the product expansion of $\Psi_z(Z,F)$ is

$$e((\rho,Z))\prod_{\substack{\alpha\in L'^+\\ 2\alpha\in L}}(1-e((\alpha,Z)))^{\pm[f+\delta](-\alpha^2/2)}\prod_{\alpha\in L'^+}(1-e((\alpha,Z)))^{\pm[f+\delta](-p\alpha^2/2)}$$

$$=\sum_{w\in W}\det(w)\,e((w\rho,Z))\prod_{n>0}(1-e((nw\rho,Z)))^{(-1)^n m}\,(1-e((pnw\rho,Z)))^{(-1)^n m},$$

where the sign in the exponent of the first product is $-$ if α^2 is even and the image of α has even order in L'/L, i.e. order 2, and $+$ else and the sign in the exponent of the second product is $-$ if $p\alpha^2/2$ is integral and α has even order and $+$ in the other cases. The Weyl vector is $\rho=(0,0,1/2p)$ and the Weyl group W is generated by the α in L' with $\alpha^2=1/p$ and the α in L' of norm 1 with $2\alpha\in L$.

Next we write $M = L \oplus II_{1,1}(2p)$ *with* $L = E_8^p \oplus II_{1,1}$ *so that* $|M'/M| = 2^2p^2|L'/L|$ *and take z as primitive norm* 0 *vector in* $II_{1,1}(2p)$. *Then we get the following level* $2p$ *expansion of* $\Psi_z(Z, F)$

$$\prod_{\alpha \in L^+} \frac{(1-e((\alpha, Z)))^{[f](-\alpha^2/2)}}{(1+e((\alpha, Z)))^{[f](-\alpha^2/2)}} \prod_{\alpha \in pL'^+} \frac{(1-e((\alpha, Z)))^{[f](-\alpha^2/2p)}}{(1+e((\alpha, Z)))^{[f](-\alpha^2/2p)}}$$
$$= 1 + \sum a(\lambda)\, e((\lambda, Z)),$$

where $a(\lambda)$ *is the coefficient of* q^n *in*

$$\frac{\eta(p\tau)^{2m}\,\eta(\tau)^{2m}}{\eta(2p\tau)^m\,\eta(2\tau)^m} = 1 - 2mq + \cdots$$

if λ *is n times a primitive norm* 0 *vector in* L^+ *and* 0 *else.*

The expansion of Ψ_M at the level $2p$ cusp shows

COROLLARY 4.3. *The denominator function of the generalized Kac–Moody superalgebra obtained by twisting the fake monster superalgebra with an element of cycle shape* $1^m p^m$ *defines a holomorphic automorphic form of singular weight.*

The expansion of Ψ_M at the other cusp implies

COROLLARY 4.4. *There is a generalized Kac–Moody superalgebra with the following properties. The root lattice is the dual* L' *of the lattice* $L = E_8^p \oplus II_{1,1}(2p)$. *The Weyl group* W *is the group generated by the reflections in the norm* $1/p$ *vectors of* L' *and the norm* 1 *vectors* α *in* L' *with* $2\alpha \in L$. *The Weyl vector is* $\rho = (0, 0, 1/2p)$. *The real simple roots are the simple roots of* W, *i.e., the roots satisfying* $(\rho, \alpha) = -\alpha^2/2$. *The imaginary simple roots are the positive multiples* $n\rho$ *of the Weyl vector with multiplicity* $(-1)^n\, 2m$ *if p divides n and* $(-1)^n\, m$ *else. A root* α *is odd if and only if* $p\alpha^2/2$ *is integral and* α *has even order. The multiplicity of an even root* $\alpha \in L'$ *is* $\mathrm{mult}(\alpha) = [f+\delta](-p\alpha^2/2)$ *if* $2\alpha \notin L$ *and* $\mathrm{mult}(\alpha) = [f+\delta](-\alpha^2/2) + [f+\delta](-p\alpha^2/2)$ *if* $2\alpha \in L$. *For odd roots the same formula holds with opposite signs. The denominator identity is given by the expansion of* Ψ_M *at the level* 1 *cusp.*

5. HYPERBOLIC REFLECTION GROUPS

In this section we describe the reflection groups of the root lattices of the twisted fake monster algebras. In the bosonic case we get some information about these groups from the singular theta correspondence. In the super case we work out their fundamental domains using Vinberg's algorithm.

5.1. *Lorentzian Lattices*

Let L be a Lorentzian lattice of dimension n. There are 2 cones of negative norm vectors in $L \otimes \mathbb{R}$. The vectors of norm -1 in one of these cones form a copy of the $n-1$ dimensional hyperbolic space H. The automorphism group $\mathrm{Aut}(L)$ of L is the direct product of $\mathbb{Z}_2$ and the subgroup $\mathrm{Aut}(L)^+$ fixing the 2 cones of negative norm vectors. The reflection group W of L is the subgroup of $\mathrm{Aut}(L)^+$ generated by reflections in the roots of L. W acts on $L \otimes \mathbb{R}$ and by restriction on H. The reflection hyperspaces divide H into Weyl chambers. We choose one Weyl chamber D and call it the fundamental Weyl chamber. Then $\mathrm{Aut}(L)^+$ is the semidirect product $\mathrm{Aut}(L)^+ = \Gamma.W$ where Γ is the subgroup of $\mathrm{Aut}(L)^+$ fixing D. W is called arithmetic if Γ is finite. The roots corresponding to the faces of D form a set of simple roots of L. The reflections in these roots generate W. The angles between the simple roots and thus the defining relations of W are usually described through the Dynkin diagram of L.

Vinberg [V] describes the following algorithm for finding a set of simple roots of L.

Choose a vector w in L with $w^2 \leqslant 0$. The roots orthogonal to w form a root system which is finite if $w^2 < 0$ and affine else. Choose a fundamental Weyl chamber C for this root system. Then there is unique fundamental Weyl chamber D of W containing w and contained in C, and its simple roots can be found as follows.

All the simple roots of C are simple roots of D. Order the roots α which have negative inner product (α, w) with respect to the distance $-(\alpha, w)/\sqrt{\alpha^2}$ of their hyperplanes from w.

Take a root α as a simple root for D if and only if it has inner product at most 0 with all the simple roots we already have. It is sufficient to check this for the simple roots whose hyperplanes are strictly closer to w than the hyperplane of α.

If at some point the roots we have already found by this algorithm span $L \otimes \mathbb{R}$ and contain at least one Dynkin diagram that is spherical of rank $n-1$ or affine of rank $n-2$, and every spherical diagram of rank $n-2$ in such a diagram is contained in a second such diagram, then these roots form a complete set of simple roots for D (cf., for example, Theorem 1.4 in [B1]).

5.2. *Reflection Groups*

First we consider the root lattices of the fake monster algebras. They are similar to the lattice $II_{25,1} = \Lambda \oplus II_{1,1}$. Let $L = \Lambda_p \oplus II_{1,1}$. The roots of L are the norm 2 vectors in L and the norm $2p$ vectors in pL'. The reflection group of L is also the Weyl group of the fake monster algebra with root lattice L. Let $M = L \oplus II_{1,1}(p)$ and $f_{M+\delta}(\tau)$ be the components of the modular form F with representation ρ_M given in Section 3.2. We define

$$F_L(\tau) = \sum_{\gamma \in L'/L} f_{L+\gamma}(\tau)\, e^{\gamma}$$

with components

$$f_{L+\gamma}(\tau) = \sum_{\substack{\delta \in M'/M \\ \delta | K = \gamma}} f_{M+\delta}(\tau),$$

where z is a primitive norm 0 vector in $II_{1,1,}(p)$ and $K = M \cap z^{\perp}$. It is easy to see that F_L is a modular form of type ρ_L and weight $-m$ (cf. Theorem 5.3 in [B3]). The components $f_{L+\gamma}(\tau)$ can be described explicitly as

$$f_{L+\gamma}(\tau) = \theta_{\Lambda_p^{\perp} + \gamma^{\perp}}(\tau)/\Delta(\tau),$$

where $\Lambda_p^{\perp}$ is the orthogonal complement of Λ_p in Λ and $\gamma + \gamma^{\perp} \in \Lambda$. The singular theta correspondence associates an automorphic form to F_L whose singularities are exactly at the reflection hyperplanes of W. Theorem 2.3 implies

PROPOSITION 5.1. *The norm* 0 *vector* $\rho = (0, 0, 1)$ *is a Weyl vector for* L *and the simple roots of* L *are the roots* α *satisfying* $(\rho, \alpha) = -\alpha^2/2$. *Furthermore the quotient* $\mathrm{Aut}(L)^+/W$ *contains a free abelian subgroup of finite index.*

The lattice Λ_p has no roots so that $\mathrm{Aut}(L)^+/W$ is actually equal to the group of affine automorphisms of Λ_p by Theorem 3.3 of [B1].

Now we consider the root lattices of the fake monster superalgebras. We will see that they are similar to the lattice $II_{9,1} = E_8 \oplus II_{1,1}$. Here we apply Vinberg's algorithm rather than Theorem 2.3 because the latter would not give much information on the reflection groups.

The lattice E_8^3 is isomorphic to $A_2 \oplus A_2$. Let $v_1 = (1, -1, 0)$, $v_2 = (0, 1, -1)$, and $v_3 = (-1, 0, 1)$ be 3 roots of A_2. We choose the vector $w = (0, 0, 1, 1)$ in $L = A_2 \oplus A_2 \oplus II_{1,1}$ with norm $w^2 = -2$ and apply Vinberg's algorithm to determine the simple roots of L. We find the following complete set of simple roots

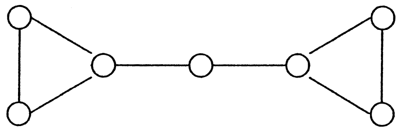
$$\alpha_1 = (v_1, 0, 0, 0), \qquad \alpha_5 = (0, 0, 1, -1)$$

$$\alpha_2 = (v_2, 0, 0, 0), \qquad \alpha_6 = (v_3, 0, 0, 1)$$

$$\alpha_3 = (0, v_1, 0, 0), \qquad \alpha_7 = (0, v_3, 0, 1)$$

$$\alpha_4 = (0, v_2, 0, 0).$$

Proposition 5.2. *The Lorentzian lattice $L = E_8^3 \oplus II_{1,1}$ has 7 simple roots and Dynkin diagram*



The reflection group of L is arithmetic and $\Gamma = \mathbb{Z}_2^2$.

The vector $\rho = (v_3, v_3, 3, 4)$ of norm $\rho^2 = -20$ satisfies $(\rho, \alpha_i) = -\alpha_i^2/2$ for all simple roots. The only other roots satisfying this relation are $(x, 0, 0, 0)$ and $(0, x, 0, 0)$ with $x = (1, 1, -2)$ or $x = (2, -1, -1)$. But these roots are not simple.

The root lattice of the other fake monster superalgebra is the dual of $E_8^3 \oplus II_{1,1}(6)$. We rescale this lattice by $2p = 6$ to obtain the even lattice $L = E_8^3(2) \oplus II_{1,1}$. This lattice has level 6 and roots of norm 2, 4, 6, and 12 in L, $L \cap 2L'$, $L \cap 3L'$, and $6L'$. As above we find

Proposition 5.3. *L has* 11 *simple roots and $\Gamma = \mathbb{Z}_2^2$.*

We remark that L does not have a Weyl vector.

E_8^7 is the 2 dimensional lattice with elements (m_1, m_2) where either m_1 and m_2 are in $\mathbb{Z}$ and $m_1 + m_2$ is even or m_1 and m_2 are in $\mathbb{Z} + 1/2$ and $m_1 + m_2$ is odd, and norm $(m_1, m_2)^2 = m_1^2 + 7m_2^2$. The roots of E_8^7 are the norm 2 vectors $\pm(1/2, 1/2)$ and the norm 14 vectors $\pm(7/2, -1/2)$. We define the vectors $v_1 = (1/2, 1/2)$, $v_2 = (7/2, -1/2)$, and $x = (-2, 0)$.

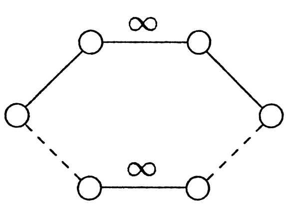
Let $L = E_8^7 \oplus II_{1,1}$. Choosing $w = (0, 1, 1)$ we find the following simple roots

$$\alpha_1 = (v_1, 0, 0), \qquad \alpha_4 = (-v_1, 0, 1)$$

$$\alpha_2 = (0, 1, -1), \qquad \alpha_5 = (x, 1, 1)$$

$$\alpha_3 = (v_2, 0, 0), \qquad \alpha_6 = (-v_2, 0, 7).$$

Proposition 5.4. *The Lorentzian lattice L has* 6 *simple roots with Dynkin diagram*

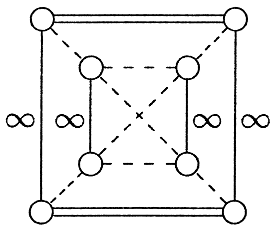


The reflection group of L has index 2 *in* $\mathrm{Aut}(L)^+$.

The vector $\rho = (x, 2, 3)$ of norm -8 satisfies $(\rho, \alpha_i) = -\alpha_i^2/2$ for all simple roots. There are no other roots with this property.

When we rescale the root lattice of the other fake monster superalgebra by 14 we obtain the lattice $L = E_8^7(2) \oplus II_{1,1}$ of level 14. This lattice has roots of norm 2, 4, 14 and 28 in L, $L \cap 2L'$, $L \cap 7L'$ and $14L'$. We have

PROPOSITION 5.5. *The Lorentzian lattice L has* 8 *simple roots and a Weyl vector of norm* -4. *The Dynkin diagram of L is*



The reflection group of L is arithmetic and has index 2 *in* $\mathrm{Aut}(L)^+$.

We remark that L has 2 simple roots of each possible root length.

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