On the Algebra Associated with a Geometric Lattice

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Let $L$ be a geometric lattice. Following P. Orlik and L. Solomon, Combinatorics and topology of complements of hyperplanes, *Invent. Math.* 56 (1980), 167–189, we associate with $L$ a graded commutative algebra $A(L)$. In this paper we introduce a new invariant $\psi$ of the algebra $A(L)$ which suffices to distinguish algebras for which all other known invariants coincide. This result is applied to the study of arrangements of complex hyperplanes, with $L$ being the intersection lattice. In this case $A(L)$ is isomorphic to the cohomology algebra of the associated hyperplane complement. The goal is to find examples of arrangements with non-isomorphic lattices but homotopy equivalent complements. The invariant introduced here effectively narrows the list of candidates. Nevertheless, we exhibit combinatorially inequivalent arrangements for which all known invariants, including $\psi$, coincide.

I. INTRODUCTION

Let $A = \{H_1, \ldots, H_n\}$ be an arrangement of hyperplanes in $\mathbb{C}^d$. Let $M = \mathbb{C}^d - \bigcup \{H_i : 1 \leq i \leq n\}$. We call $M$ the complement of $A$. It was shown in [4] that the cohomology ring $H^*(M)$ is completely determined by the intersection lattice $L$. This result has prompted several conjectures concerning the relationship between $L$ and $M$. As one would expect, these all involve the dependence of other topological invariants of $M$ on the combinatorial structure of $L$. For example, the question has been raised whether the homotopy type of $M$ is determined by $L$. Conversely, one can ask whether there are combinatorially distinct arrangements (= non-isomorphic intersection lattices) which have homotopy equivalent complements. This is the problem addressed in this paper.

In [4] the determination of $H^*(M)$ from $L$ is accomplished by first defining an algebra $A(L)$ in terms of generators and relators which depend only on the lattice, and then showing this algebra to be isomorphic to $H^*(M)$. This allows us to cast the question posed above in terms of com-
binomial algebra. We look for distinct lattices $L_1$ and $L_2$ such that the algebras $A(L_1)$ and $A(L_2)$ are isomorphic, and then attempt to construct arrangements with intersection lattices $L_1$ and $L_2$. Because the definition of $A(L)$ is in terms of generators and relators, this becomes a highly non-trivial problem. One would then determine whether $M_1$ and $M_2$ are homotopy equivalent by examining the fundamental group (in our setting the spaces will be aspherical).

As a first step, the Poincaré polynomial of $A(L)$ may be computed in terms of the Möbius function of $L$. Thus we may limit our search for examples to those lattices for which these polynomials coincide. This will guarantee that the algebras are isomorphic as $\mathbb{Z}$-modules. In this case, there is another series of invariants of $A(L)$ which will sometimes distinguish the algebras [1]. These numbers arise in the rational homotopy theory of $M$, and are related to the fundamental group. In algebraic terms, they come out of a certain type of resolution of the algebra $A(L)$. At any rate, the main result of [2] implies that these invariants will also match up if the lattices come from fiber-type arrangements. In this case the lattices are supersolvable and the algebras are isomorphic as graded modules to the tensor product of free algebras determined by the exponents [5].

We are thus led to consider pairs of fiber-type arrangements with the same exponents. The smallest such arrangements are displayed in Example 3.1.

It is at this stage that the present result comes into play. We will define another invariant $\psi$ which suffices to distinguish many of these examples. This invariant places further restrictions on the multiplicities that can occur in the rank two part of the lattice. (The Poincaré polynomial already places some constraints on these multiplicities.) But there exist examples of combinatorially distinct fiber-type arrangements (with the same exponents) for which this new invariant will not distinguish the algebras. We construct two examples of this phenomenon in Section 3. In the first, the multiplicities which appear in rank two are different, but result in the same value for $\psi$. In the second example, the multiplicities in rank two actually coincide, but the lattices are not isomorphic. Whether the algebras are isomorphic in these cases remains an open question.

The ideas which lead to the invariant $\psi$ were introduced in [1] from the perspective of rational homotopy theory. At that point, $\psi$ was defined in terms of the lattice (cf. Theorem 2.1) and was not known to be an algebraic invariant. Here we define $\psi$ algebraically and show that this definition is equivalent to the previous combinatorial definition of [1]. The present work is written without reference to minimal models or rational homotopy theory, and is independent of [1]. The construction of this paper may indeed have applications to other purely algebraic ring isomorphism questions.
Let $L$ be a geometric lattice, and let $A = A(L)$ be the graded commutative algebra associated to $L$ as defined in [4]. Let $E$ be the free exterior algebra generated by $A^1$, and let $\pi: E \to A$ be the canonical map. Then $\pi$ is onto, and the relation ideal $I = \ker(\pi)$ has a nice description in terms of the lattice [4]. We consider the map $d: E^1 \otimes I^2 \to E^3$ defined by multiplication in $E$. Let $V = \ker(d)$, and let $W$ be the subspace spanned by decomposable elements of $V$. Then $\psi = \psi(A) = \dim(W)$ and $\phi = \phi(A) = \dim(V)$ are invariants of $A$, which we call “local” and “global” for reasons which will become apparent later.

The invariant $\phi$ appears in the rational homotopy theory of $M$—namely, $\phi = \rank(G^2/G^3)$, where $G$ is the fundamental group of $M$ and $G = G^0 \supseteq G^1 \supseteq G^2 \cdots$ is the lower central series of $G$. This number is tedious to compute in general, but is given by a known function of the betti numbers $\dim(A^i)$, $i \geq 0$, in the case of fiber-type arrangements [2]. On the other hand, the local invariant $\psi$ is equal to

$$\sum_{m \geq 3} 2 \left( \begin{array}{c} m \\ 3 \end{array} \right) c_m,$$

where $c_m$ is the number of rank two lattice elements of multiplicity $m$.

By counting (unordered) pairs of hyperplanes, one obtains a second identity relating the $c_m$ to algebraic invariants of $A$. Specifically,

$$\sum_{m \geq 2} \left( \begin{array}{c} m \\ 2 \end{array} \right) c_m = \left( \begin{array}{c} n \\ 2 \end{array} \right),$$

where $n$ is the first betti number ($= \text{the number of atoms in } L$). This leads to the result in rank 3 that among fiber-type arrangements with the same exponents, the “trivial” product arrangement is characterized by the equation $\phi - \psi = 0$.

Here is an outline of the paper. In Section 2 we define the algebra $A(L)$, the invariants $\phi$ and $\psi$, and establish the identity relating $\psi$ to the rank two part of the lattice. Section 3 consists of applications to the topology of hyperplane complements. Here we prove that $\phi - \psi = 0$ determines the product arrangements among the class of rank 3 fiber-type arrangements. Then we give examples of arrangements with the same Poincaré polynomial for which $\psi$ is also the same. Among these examples are fiber-type arrangements for which the multiplicities in rank two (i.e., the $c_m$, $m \geq 2$) coincide, but the lattices are not isomorphic. It is still not known whether the algebras $A(L)$ are isomorphic for the Examples 3.3, 3.5, or 3.6.
II. Local and Global Invariants of $\mathcal{A}(L)$

Let $L$ be a geometric lattice. Thus $L$ is an atomic lattice with rank function $r$ which satisfies the semi-modular law

$$r(X \land Y) + r(X \lor Y) \leq r(X) + r(Y).$$

Let $A = \{a_1, \ldots, a_n\}$ be the set of atoms of $L$. Let $E = \wedge (e_1, \ldots, e_n)$ be the free exterior algebra (with rational coefficients) on generators $e_i$ corresponding to the atoms of $L$. If $J = \{i_1, \ldots, i_p\} \subseteq \{1, \ldots, n\}$, write $e_J = e_{i_1} \cdots e_{i_p}$ and

$$\hat{c} e_J = \sum_{k=1}^p (-1)^k e_{i_1} \cdots \hat{e}_{i_k} \cdots e_{i_p}.$$

We say $J$ is dependent if $r(a_{i_1} \lor \cdots \lor a_{i_p}) < p$. Let $I$ be the ideal generated by $\{\hat{c} e_J | J$ is dependent $\}$. Then $I$ is an homogeneous ideal, so $E/I$ inherits a grading from $E$. The associative, graded commutative algebra $E/I$ is denoted by $A(L)$. Let $\pi : E \to A(L)$ denote the canonical projection.

Remark. Observe that the combinatorial generators $\pi(e_i)$ are not a priori determined by the algebra structure. If these generators could be identified, one could reconstruct the lattice $L$ from the algebra $A(L)$.

In what follows, we abbreviate $A(L)$ to $A$. Note that $I^1 = 0$, since any pair of distinct atoms $a_i$ and $a_j$ satisfies $r(a_i \lor a_j) = 2$. Thus, $A^1$ is isomorphic to $E^1$. Henceforth, we identify $E$ with $\wedge (A^1)$.

Let $d : E^1 \otimes I^2 \to E^3$ be the map defined by multiplication in $E$. Note that $d$ depends only on the isomorphism type of the algebra $A$, since $I^2 = \ker (E^2 = \wedge^2 (A^1) \to A^2)$. Let $V = \ker (d)$. Let $W$ be the subspace spanned by elements of the form $e \otimes r$ in $V$. (Elements of this form are called decomposable.) Set $\phi = \phi(A) = \dim(V)$ and $\psi = \psi(A) = \dim(W)$.

Let $X \in L$ with $r(X) = 2$, and let $L_X = \{Y \in L | Y \leq X\}$. Let $A_X = A(L_X)$. Note that $A_X = E_X / I_X$ where $E_X \subseteq E$ and $I_X \subseteq I$. Define $V_X = \ker (E_X \otimes I_X^2 \to E_X^2)$ as above. We have inclusions $V_X \to V$ for each $X$, inducing a map

$$i : \bigoplus_{r(X) = 2} V_X \to V.$$

Theorem 2.1. The map $i$ is an isomorphism onto $W$.

The present section is devoted to the proof of 2.1. The proof also allows the calculation of $\psi = \dim(W)$ in terms of the multiplicities (i.e., Möbius numbers) in the rank two part of the lattice (Corollary 2.10).
Theorem 2.1 shows that $W$ consists of local relations (i.e., relations supported locally near rank two lattice elements) among the elements of $I^2$, with coefficients from $E^1$. On the other hand, $V$ consists of all such relations. Hence the local-global terminology of the introduction.

Theorem 2.1 will be proven in stages. First we show that $i$ is injective. Next we show that each $V_X$ is spanned by decomposable elements. Finally we show that $W$ is contained in the image of $i$.

**Lemma 2.2.** $i$ is injective.

**Proof.** $I^2$ has a basis consisting of elements $e_{ijk}$ where $X = a_i \lor a_j \lor a_k$ has rank 2. Thus $I = \bigoplus_{X \in I, r(X) = 2} I^2_X$.

It follows that $i|V_X$ is injective for each $X$, and

$$
(E^1_X \otimes I^2_X) \cap \bigoplus_{Y \neq X} (E^1_Y \otimes I^2_Y) \subseteq (E^1_X \otimes I^2_X) \cap \bigoplus_{Y \neq X} (E^1_Y \otimes I^2_Y)
$$

$$
= E^1 \otimes \left( I^2_X \cap \bigoplus_{Y \neq X} I^2_Y \right)
$$

$$
= 0.
$$

Thus $i(V_X) \cap i(\bigoplus_{Y \neq X} V_Y) = 0$, and $i$ is injective. 

Lemma 2.2 is a special case (with a simpler proof) of [1, Lemma 3.11].

Now we assume that the lattice $L$ has rank two. In this case, we show that $V$ is spanned by decomposable elements. As a consequence, we obtain one inclusion $i(\bigoplus V_X) \subseteq W$.

**Proposition 2.3.** Suppose $L$ has rank two. Then $\dim(V) = 2(\binom{m}{3})$, where $m = \#\text{atom}(L)$.

**Proof.** It follows easily from $r(L) = 2$ that $I^2$ generates $I$, and that $I^n = E^n$ for all $n \geq 3$. (Use the identity $e_j e_j = \pm e_j$ for $j \in J$.) Then $d: E^1 \otimes I^2 \to E^3$ is surjective. According to [4], $\dim(E^2/I^2) = \dim(A^2) - \mu(L) = m - 1$, so $\dim(I^3) = (\binom{m}{2}) - (m - 1) = (\binom{m}{3})$. Thus $\dim(V) = \dim(\ker(d)) = \dim(E^1 \otimes I^2) - \dim(E^3) = m(\binom{m-1}{2}) - (\binom{m}{3}) = 2(\binom{m}{3})$. 

Proposition 2.3 is just another manifestation of Witt's formula for the rank of the third factor in the lower central series of a free group or free Lie algebra (cf. [2, Thm. 4.1]).

Now suppose $\{a_i, a_j, a_k\} \subseteq \text{atom}(L)$ with $i < j < k$ and $r(a_i \lor a_j \lor a_k) = 2$. Then $e_{ijk} = e_{ik} - e_{ik} + e_{ij} = (e_j - e_i)(e_k - e_j) \in I^2$, and both $(e_j - e_i) \otimes e_{ijk}$ and $(e_k - e_j) \otimes e_{ijk}$ are elements of $V = \ker(d)$, and are decomposable. If $L$ has rank two, there are precisely $2(\binom{m}{3})$ of these elements, where $m = \#\text{atom}(L)$. We will show that these elements are linearly independent. To do this we construct a nice basis for $E^1 \otimes I^2$. 

Set \( x_i = e_i - e_1 \) for \( 2 \leq i \leq m \), and set \( r_{ij} = \hat{e}_{ij} = x_i x_j \) for \( 2 \leq i < j \leq m \). Clearly the \( x_i \) are linearly independent elements of \( E^1 \).

**Lemma 2.4.** If \( 2 \leq i < j < k \leq m \), then \( \hat{e}_{ijk} = r_{jk} - r_{ik} + r_{ij} \).

**Proof.** Since \( \hat{e} = 0 \), we have
\[
0 = \hat{e} \hat{e}_{ijk} = \hat{e}_{ijk} - r_{jk} + r_{ik} - r_{ij}.
\]

**Lemma 2.5.** If \( r(I) = 2 \), then the set \( \{ r_{ij} \mid 2 \leq i < j \leq m \} \) forms a basis for \( I^2 \).

**Proof.** By 2.4, the \( r_{ij} \) span \( I^2 \). From the proof of 2.3, we have \( \dim(I^2) = \binom{m}{2} \), and the assertion follows by comparing dimensions.

**Proposition 2.6.** The set \( S = \{(e_i - e_j) \otimes \hat{e}_{ijk} \mid 1 \leq i < j < k \leq m\} \cup \{(e_k - e_i) \otimes \hat{e}_{ijk} \mid 1 \leq i < j < k \leq m\} \) is linearly independent.

**Proof.** Suppose we have a dependence relation among the elements of \( S \). Using 2.4, such a relation may be written
\[
\sum_{2 \leq i < j < k \leq m} (x_{ij} x_i + \beta_{ij} x_j) \otimes r_{ij} + \left( \gamma_{ijk} (x_j - x_i) + \delta_{ijk} (x_k - x_i) \right) \otimes (r_{jk} - r_{ik} + r_{ij}) = 0.
\]

Expanding the left hand side, we see that the coefficient of \( x_k \otimes r_{ij} \) is precisely \( \delta_{ijk} \), for \( i < j < k \). Similarly, the coefficient of \( x_j \otimes r_{ik} \) is \( -\gamma_{ijk} \). By 2.5 and the independence of the \( x_i \), we must have \( \delta_{ijk} = 0 = \gamma_{ijk} \) for all \( i < j < k \). Then
\[
\sum \alpha_{ij} x_i \otimes r_{ij} + \beta_{ij} x_j \otimes r_{ij} = 0,
\]
which implies \( \alpha_{ij} = 0 = \beta_{ij} \) for all \( i < j \), for the same reason.

**Proposition 2.7.** If \( L \) has rank two, then \( V = W \), i.e., \( V \) is spanned by decomposable elements.

**Proof.** This follows immediately from 2.6 and 2.3, since the set \( S \) of 2.6 consists of \( 2(\binom{m}{2}) \) decomposable elements of \( V \).

This proposition yields the following corollary for \( L \) of arbitrary rank, completing the second step in the proof of 2.1.

**Corollary 2.8.** The image of \( i: \bigoplus_{X \in I \setminus \{I\}} \mathbb{Z} V_X \to V \) is contained in \( W \).
It remains only to show that $W$ is contained in the image of $i$. For this it suffices to show that any decomposable element of $V$ is contained in $\bigoplus V_X$.

**Proposition 2.9.** Suppose $s = e \otimes r$ satisfies $ds = 0$. Then

$$s = \sum_{X \in L} s_X,$$

where $s_X \in V_X$ for each $X$.

**Proof.** Let $\{\partial e_J | J \in J\}$ be a basis for $I^2$. Write $e = \sum \lambda_i e_i$ and $r = \sum \mu_J \partial e_J$, summing over $1 \leq i \leq n$ and $J \in J$, respectively. For $J \in J$, $J = \{j, k, l\}$, set $X_J = a_j \vee a_k \vee a_l$, and note that $r(X_J) = 2$. Fix $J_0 = \{j, k, l\}$ with $j < k < l$. First we show that $\lambda_i \mu_{J_0} \neq 0$ implies that $a_i < X_{J_0}$. Suppose not. Then $a_i \not< X_{J_0}$, which implies that $e_{ik}$ and $e_{il}$ do not occur in any $\partial e_J$, and $e_{kl}$ does not occur in any $\partial e_J$ besides $\partial e_{J_0}$. Thus, the coefficient of $e_{ikl}$ in

$$ds = \sum \lambda_i \mu_J e_i \partial e_J$$

is precisely $\lambda_i \mu_{J_0}$. Hence $ds = 0$ implies $\lambda_i \mu_{J_0} = 0$. This proves the claim.

Now set

$$r_X = \sum_{J \in J} \mu_J \partial e_J,$$

so that $r = \sum r_X$. Set $s_X = e \otimes r_X = \sum_{X_J \subset X} \lambda_i \mu_J e_i \otimes \partial e_J$. By the claim above, we have $s_X \in E_X^1 \otimes I_X^2$. From this we get $ds_X \in E_X^3$. Since $\bigoplus E_X^3 \subseteq E^3$, we conclude from $0 = ds = \sum ds_X$ that $ds_X = 0$ for all $X$. Thus $s_X \in V_X$, and we are done. \(\square\)

The main result 2.1 is now a consequence of 2.2, 2.8, and 2.9.

Using 2.1 and 2.3, we can compute the dimension $\psi$ of $W$, a numerical invariant of the algebra $A(L)$. For $X \in L$ with $r(X) = 2$, define the multiplicity of $X$ $m(X) = \#\text{atom}(L_X)$, the number of atoms of $L$ covered by $X$. Let $c_m$ be the number of rank 2 lattice elements of multiplicity $m$.

**Corollary 2.10.** The dimension of $W$ is

$$\psi = \sum_{X \in L} \frac{m(X)}{3} = \sum_{m \geq 3} \frac{m}{3} c_m.$$
III. EXAMPLES FROM THE THEORY OF ARRANGEMENTS

In this section we apply the results of Section 2 to the topology of hyperplane complements. Let $A = \{H_1, \ldots, H_n\}$ be a collection of hyperplanes in $C^l$, and let $M = C^l - \bigcup_{i=1}^n H_i$. $A$ is called an arrangement, and $M$ the complement of $A$. Let $L$ be the set of intersections of subcollections of $A$, ordered by reverse inclusion. Then $L$ is a geometric lattice, with smallest element $C^l$ (as the empty intersection) and rank function $r(X) = \text{codim}(X)$. The join of a pair of elements is their intersection. $L$ is called the intersection lattice of $A$. The main result of [4] states that the algebra $A(L)$ constructed in the previous section is isomorphic to the cohomology algebra $H^*(M)$. This fact motivates the first of the two fundamental questions:

**Question 1.** Do arrangements with isomorphic intersection lattices have homotopy equivalent complements?

**Question 2.** Are there (central) arrangements which have non-isomorphic intersection lattices but homotopy equivalent complements?

The first of these is the focus of much current research in the theory of arrangements; neither question has been resolved. The construction of Section 2 provides one further tool for examining Question 2.

Let us look at some examples. Suppose $A_1$ and $A_2$ are fiber-type arrangements with exponents $1 = d_1$, $d_2$, and $d_3$ (refer to [3] for definition and relevant properties). Then $H^p(M_1)$ is isomorphic to $H^p(M_2)$ for each $p$. Furthermore, $M_1$ and $M_2$ are both aspherical spaces, and the graded groups defined by the lower central series of the fundamental group are also isomorphic [2]. In particular the invariant $\phi = \dim(V)$ of Section 2 will be the same for $L_1$ and $L_2$. Until now it was not known whether $H^*(M_1)$ and $H^*(M_2)$ are necessarily isomorphic as rings.

**Example 3.1.** Let $A_1$ consist of planes in $C^3$ with defining forms $x$, $(x+z)$, $(x-z)$, $(y+z)$, $(y-z)$, and $z$. Let $A_2$ be given by $x$, $(x-z)$, $y$, $(y-z)$, $(x-y)$, and $z$. Then each $A_i$ is fiber-type with exponents 1, 2, and 3. One can see the geometry of these arrangements in the projective image of the real part of $A_i$, sketched below.

![A1 and A2](image-url)
We see that $A_1$ has $c_3 = 1$ and $c_4 = 1$, while $A_2$ has $c_3 = 4$. (All other $c_m = 0$ for $m > 3$.) Thus $\psi = 10$ for $A_1$ while $\psi = 8$ for $A_2$. Therefore $H^*(M_1)$ is not isomorphic to $H^*(M_2)$.

This is an instance of a more general phenomenon.

A product arrangement in $\mathbb{C}^3$ is an arrangement which (in some coordinate system) consists of two subarrangements $A_v$ and $A_h$ such that

(i) the plane $z = 0$ is in $A_v$;
(ii) all planes in $A_v$ have defining forms independent of $y$; and
(iii) all planes in $A_h$ have defining forms independent of $x$.

Product arrangements are fiber-type with exponents $1$, $d_2 = \#A_v$ and $d_3 = \#A_h$. It follows from Witt's formula that $\phi - \psi = 0$ for product arrangements (cf. [1, 2]). Arrangement $A_1$ of Example 3.1 is a product arrangement with exponents 1, 2, and 3.

**PROPOSITION 3.2.** If $A_1$ is a product arrangement, and $A_2$ is a fiber-type arrangement with $L_2$ not isomorphic to $L_1$, then $H^*(M_1)$ is not isomorphic to $H^*(M_2)$.

The proof of 3.2 is somewhat involved, so we merely provide the outline, and leave the details to the reader. First observe that fiber-type arrangements in $\mathbb{C}^3$ also consist of vertical and horizontal subarrangements $A_v$ and $A_h$ (cf. [5]), where $A_h$ satisfies

(iii)' for each $X \in L(A_h)$ with $r(X) = 2$, there is an $H \in A_v$ such that $X \subseteq H$.

A fiber-type arrangement is a product if and only if $A_h$ has rank two, that is, the planes in $A_h$ have a line in common. Also, the multiplicities $c_m$ of $A$ are related to the multiplicities $\check{c}_m$ of $A_h$ by

$$c_m = \begin{cases} \check{c}_m + 1 & \text{if } m \neq \#A_v \\ \check{c}_m + 1 & \text{if } m = \#A_v. \end{cases}$$

Using this, one checks that the invariant $\psi$ of the two arrangements will match up if and only if

$$\sum_{m \geq 3} \binom{m}{3} \check{c}_m = \binom{d_3}{3}.$$

Next one establishes the identity

$$\sum_{m \geq 2} \binom{m}{2} \check{c}_m = \binom{d_2}{2}.$$
by counting unordered pairs of hyperplanes. The proof is completed by showing that the two identities above uniquely determine the $c_m$, $m \geq 2$.

The situation is not so nice when neither arrangement is a product. I am grateful to Steve Wilson for his help in constructing the next example.

**Example 3.3.** Consider the pair of arrangements pictured below.

These arrangements both have Poincaré polynomial $1 + 11t + 34t^2 + 24t^3 = (1 + t)(1 + 4t)(1 + 6t)$. $A_1$ has multiplicities $c_2 = 13$, $c_3 = 2$, and $c_4 = 6$, while $A_2$ has multiplicities $c_2 = 9$, $c_3 = 8$, $c_4 = 2$, and $c_5 = 1$ (all other $c_m$ are zero). Checking 2.10 we see that the invariant $\psi = 52$ for each of these two arrangements. Using some results from Section 3 of [1], one can determine that the global invariant $\phi$ is equal to 77 for each of these arrangements (both are parallel, hence 2-determined arrangements, so that $\phi$ is given by the LCS formula as a function of the betti numbers). However, these examples are not fiber-type, so it is possible that the higher order invariants which appear in the minimal model will distinguish the cohomology rings. This has not been checked. In any case, these examples may be used to construct fiber-type arrangements with the same exponents for which $\psi$ will still coincide. To do this one uses the arrangements above as the horizontal subarrangements, obtaining a pair of fiber-type arrangements with 23 hyperplanes and exponents 1, 11, and 11.

In attempting to construct the example above, it became clear to us that forcing the Poincaré polynomials and local invariants to match up imposes strong restrictions on the multiplicities. Suppose, for instance, that $c_m = 0$ for all $m \geq 5$. Then the equations

\[
\sum_{m \geq 2} \binom{m}{\gamma} c_m = \binom{n}{\gamma} \quad (n = \dim A^1)
\]

\[
\sum_{m \geq 2} (m - 1) c_m = \dim(A^2)
\]

\[
\sum_{m \geq 3} 2 \binom{m}{3} c_m = \psi(A)
\]

determine $c_2$, $c_4$, and $c_5$ uniquely. This establishes the following result.
**Proposition 3.4.** Let $L$ be a geometric lattice satisfying $m(X) \leq 4$ for all $X \in L$ of rank 2. Then the number $c_m$ of those $X$ with $m(X) = m$ is determined by the algebra $A(L)$, for each $m$.

It seems there are more general conditions under which the conclusion of 3.4 holds.

There are also examples in rank 3 where the multiplicities match up, though the lattices are not isomorphic.

**Example 3.5.** The pair of arrangements pictured below each have Poincare polynomial $1 + 6t + 13t^2 + 8t^3 = (1 + t)(1 + 5t + 8t^2)$ and multiplicities $c_2 = 9$ and $c_3 = 2$. Thus $\psi = 4$ for both arrangements. A routine computation shows that $\phi = 4$ also for both arrangements. It is not known whether the higher order invariants of the minimal model agree for these arrangements, since they are not fiber-type.

The intersection lattices of these arrangements are not isomorphic, since both multiple intersection lines in $A_2$ are contained in a single hyperplane, which is not the case in $A_1$.

**Example 3.6.** Here is a pair of fiber-type arrangements with exponents 1, 4, and 4 for which the multiplicities match up. Thus all known invariants of the cohomology algebra coincide.
The intersection lattices of these arrangements are not isomorphic because the unique line of multiplicity 5 lies in a plane which contains no lines of multiplicity 3 in $A_1$, but not in $A_3$.

For any of the last three pairs of arrangements, it would be interesting to know whether the cohomology rings are isomorphic, and, if so, whether the complements are homotopy equivalent.

*Note added in proof.* We have recently shown that the pair of arrangements in 3.5 have homotopy equivalent complements.

**REFERENCES**