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Stochastic incompleteness for graphs and weak Omori–Yau maximum principle ☆

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ABSTRACT

We prove an analogue of the weak Omori–Yau maximum principle and Khas'minskii's criterion for graphs in the general setting of Keller and Lenz. Our approach naturally gives the stability of stochastic incompleteness under certain surgeries of graphs. It allows to develop a unified approach to all known criteria of stochastic completeness/incompleteness, as well as to obtain new criteria.

0. Introduction

Recently, Wojciechowski [29–31] and Weber [28] independently studied the problem of stochastic completeness for the following Laplace operator (Weber calls it "physical Laplacian") on a graph (V, E) that is locally finite, connected, undirected and without loops and multi-edges:

$$\Delta f(x) = \sum_{y\in V,\ y\sim x} \big(f(x) - f(y)\big)$$

where V is the set of vertices and E is the set of edges, and $y \sim x$ means that $(x, y) \in E$. See also [15] for some remarks. Let μ be the counting measure on V. Essential self-adjointness of Δ on $L^2(V, \mu)$ has been shown by several authors independently, see [14,22,28,29]. The corresponding heat semigroup can be constructed as $P_t = \exp(-t\Delta)$ and extended from $L^2(V, \mu)$ to $L^\infty(V, \mu)$ (see [9, p. 49]). This semigroup determines a continuous time random walk on V, that is stochastically complete provided $P_t 1 = 1$, and incomplete otherwise. The latter can occur due to a very fast escape rate so that the random walk reaches infinity in finite time. This phenomenon in the setting of Brownian motions on manifolds was first observed by Azencott [1] (see also the survey [10]).

The study of continuous time Markov chains has a long history, see for example the work of Feller [7,8] and Reuter [27]. However, in the analytic study of random walks on graphs, the phenomenon of stochastic incompleteness has remained unnoticed until recently, perhaps because most attention was given to the normalized (combinatorial) Laplace operator

$$\tilde{\Delta} f(x) = \frac{1}{\deg x} \sum_{y\in V,\ y\sim x} \big(f(x) - f(y)\big),$$

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where $\deg x$ is the number of neighbors of x in the graph. The corresponding heat semigroup of $\tilde{\Delta}$ is always stochastically complete, which is a consequence of the boundedness of $\tilde{\Delta}$ in L^2. Following the classical approach [17] to the stochastic completeness in the framework of continuous spaces, Wojciechowski showed the equivalence of stochastic incompleteness and the existence of the so-called λ-(sub)harmonic functions. Wojciechowski used this equivalence to obtain many interesting sufficient conditions for stochastic completeness and incompleteness.

Weber [28] followed another approach via bounded solutions of the heat equation and discovered an interesting curvature-type criterion.

Keller and Lenz [18] have extended their work to a general setting, namely regular Dirichlet forms on discrete countable sets. In this setting the graphs are not necessarily locally finite and have general weight functions both for vertices and edges.

In this note we adopt an alternative approach to stochastic completeness on graphs. A cornerstone of this approach is Theorem 2.2, where we prove that the stochastic completeness is equivalent to a discrete analogue of the weak Omori–Yau maximum principle. The latter notion was introduced by Pigola, Rigoli, and Setti [26,25], where they proved the aforementioned equivalence in the setting of manifolds and gave many applications. For the original form of Omori–Yau maximum principle, see [24,32].

We use the weak Omori–Yau maximum principle and its consequence, a discrete Khas'minskii's criterion, to develop a unified approach to all known criteria of stochastic completeness/incompleteness, as well as to obtain new criteria. For example, in Theorem 5.4, we establish an improvement of the curvature-type criterion in [28]. Together with Lemma 2.3, the weak Omori–Yau maximum principle also easily gives stability results for stochastic incompleteness. For example, the subgraph of a stochastically incomplete graph, which consists of vertices with weighted degrees larger than some constant, is stochastically incomplete as well (Theorem 4.3). Due to special features of the graph case, some results are new and some are stronger than their manifold relatives. Part (3) of Lemma 2.3 and Theorem 2.9 have no analogues for manifolds to the author's knowledge. Our version of Khas'minskii's criterion Theorem 3.1 is stronger than a direct generalization of the manifold case (Theorem 5.1).

The paper is organized as follows. In Section 1, we introduce the framework of Keller and Lenz as our starting point. The weak Omori–Yau maximum principle for graphs is proved in Section 2 together with a useful Lemma 2.3. Then Khas'minskii's criterion is established in Section 3. Section 4 is devoted to the stability of stochastic incompleteness under certain surgeries of graphs. In Section 5, we concentrate on the special case of physical Laplacian and show how the weak Omori–Yau maximum principle and Khas'minskii's criterion are applied. In the last section, we present some open questions and further developments.

1. Foundations

We generally follow the framework set up in [18] except that we do not include killing terms here. Consider a triple (V, b, μ) where V is a discrete countably infinite set, μ is a measure on V with full support, and $b : V \times V \to [0, +\infty)$ satisfies:

(1) $b(x, x) = 0$;
(2) $b(x, y) = b(y, x)$;
(3) $\sum_{y \in V} b(x, y) < +\infty$.

The triple (V, b, μ) will be called a (weighted) graph, and sometimes we abuse the notation and denote a graph simply by V. We call the quantity

$$\mathrm{Deg}(x) := \frac{1}{\mu(x)} \sum_{y \in V} b(x, y)$$

the weighted degree of $x \in V$. For example, for the physical Laplacian, $\mu(x) \equiv 1$ and the weighted degree $\mathrm{Deg}(x)$ coincides with the usual degree $\deg x$. However, for the combinatorial Laplacian, $\mu(x) = \deg(x)$ and hence $\mathrm{Deg}(x) \equiv 1$.

The couple (V, μ) forms a measure space. Then the real function spaces $L^p(V, \mu)$, $0 < p < \infty$ are naturally defined as

$$\left\{ u : V \to \mathbb{R}\colon \sum_{x \in V} \mu(x) \left| u(x) \right|^p < \infty \right\}$$

and $L^\infty(V, \mu)$ is simply the space of bounded functions on V.

A *formal* Laplacian Δ:

$$\Delta u(x) = \frac{1}{\mu(x)} \sum_{y} b(x, y) \big(u(x) - u(y) \big)$$

is introduced on the domain

$$F = \left\{ u : V \to \mathbb{R}\colon \forall x \in V,\ \sum_y b(x, y)\left|u(y)\right| < \infty \right\}.$$

An obvious fact is that $L^\infty(V, \mu) \subseteq F$.

A quadratic form Q can be defined on the space of finitely supported functions $C_c(V)$ as

$$Q(u) = \frac{1}{2} \sum_{x,y\in V} b(x, y)\big(u(x) - u(y)\big)^2.$$

It is closable and its closure is a regular Dirichlet form which we also denote by Q. This is a nonlocal Dirichlet form in general. The semigroup P_t corresponding to the Dirichlet form Q on $L^2(V, \mu)$ can be extended to all $L^p(V, \mu)$, $p \in [1, \infty]$, and the associated generators are certain restrictions of the *formal* Laplacian Δ. We abuse the notation and denote all these operators by Δ. The explicit domains of these generators are irrelevant to the problem of stochastic completeness (for details see [18]). For the general theory of Dirichlet forms and semigroups, we refer to [9,21,3].

The following theorem about stochastic incompleteness is classical for the manifold case [10] and is proven independently by Wojciechowski and Weber for the graph case (physical Laplacian). Keller and Lenz [18,19] prove it in the general setting:

Theorem 1.1. *The following statements are equivalent*:

(1) *For some $t_0 > 0$, some $x_0 \in V$, $P_{t_0}1(x_0) < 1$.*
(2) *For every $\lambda > 0$, there exists a nonconstant, nonnegative, bounded function v on V such that $\Delta v + \lambda v = 0$. Such a function v is called a λ-harmonic function.*
(3) *For every* (*or, equivalently, for some*) *$\lambda > 0$, there exists a nonconstant, nonnegative, bounded function v on V such that $\Delta v + \lambda v \leqslant 0$. v is called a λ-subharmonic function.*
(4) *There exists a nonconstant, nonnegative, bounded solution to the Cauchy problem*

$$\begin{cases} \Delta u(x,t) + \dfrac{\partial u}{\partial t}(x,t) = 0, & \text{for all } x \in V,\ \text{all } t \geqslant 0, \\ u(\cdot, 0) = 0. \end{cases}$$

Remark 1.2. In the context of Markov chains, Feller [7,8] and Reuter [27] also cover parts of this result from the probabilistic point of view.

A graph is said to be stochastically incomplete if any one of these four conditions holds. Otherwise it is stochastically complete.

2. Weak Omori–Yau maximum principle

From now on, we will denote the supremum of a function f by f^*.

Definition 2.1. A weighted graph (V, b, μ) is said to satisfy the weak Omori–Yau maximum principle if for every nonnegative function f on V with $f^* = \sup_V f < +\infty$ and for every $\alpha > 0$,

$$\sup_{\Omega_\alpha} \Delta f \geqslant 0,$$

where

$$\Omega_\alpha = \left\{ x \in V\colon\ f(x) > f^* - \alpha \right\}.$$

It was first noticed by Pigola, Rigoli, and Setti [26] that in fact a smooth, connected, non-compact Riemannian manifold satisfies the weak Omori–Yau maximum principle if and only if the semigroup generated by the Laplace–Beltrami operator on it is stochastic complete. It is somewhat surprising that this also holds in the graph case although we are dealing with nonlocal operators here.

Theorem 2.2. *A weighted graph (V, b, μ) satisfies the weak Omori–Yau maximum principle if and only if it is stochastically complete.*

Proof. Assume that the weak Omori–Yau maximum principle holds but the graph is stochastically incomplete. Then there exists a bounded, nonnegative, nonconstant solution f of the equation $\Delta f + \lambda f = 0$ for some $\lambda > 0$. Choosing $\alpha = \frac{f^*}{2} > 0$, we have

$$\sup_{\Omega_\alpha} \Delta f = \sup_{\Omega_\alpha} -\lambda f \leqslant -\lambda \frac{f^*}{2} < 0$$

which is a contradiction.

Conversely, if V is stochastically complete but the weak maximum principle does not hold, there exists a nonnegative function f on V with $f^* < +\infty$ and some $\alpha > 0$ and $c > 0$ such that

$$\sup_{\Omega_\alpha} \Delta f < -2c.$$

Define

$$f_\alpha = \left(f + \alpha - f^*\right)_+,$$

which is obviously nonconstant, nonnegative and bounded. Setting $\lambda = \frac{c}{\alpha}$, we claim that

$$\Delta f_\alpha + \lambda f_\alpha \leqslant 0,$$

which implies stochastic incompleteness and leads to a contradiction.

For $x \in \Omega_\alpha^c$, $f_\alpha(x) = 0$, so the claim is trivially true.

For $x \in \Omega_\alpha$, we have

$$\lambda f_\alpha(x) \leqslant \lambda \alpha = c,$$

and

$$f_\alpha(x) - f_\alpha(y) = f(x) - f^* + \alpha - f_\alpha(y) \leqslant f(x) - f(y).$$

Hence

$$\begin{aligned}\Delta f_\alpha(x) + \lambda f_\alpha(x) &= \frac{1}{\mu(x)} \sum_y b(x,y)\big(f_\alpha(x) - f_\alpha(y)\big) + \lambda f_\alpha(x) \\ &\leqslant \frac{1}{\mu(x)} \sum_y b(x,y)\big(f(x) - f(y)\big) + c \\ &= \Delta f(x) + c \leqslant -c. \qquad \square\end{aligned}$$

Now we know that a graph is stochastically incomplete if and only if there exist a nonnegative function f on V with $f^* < +\infty$ and some $\alpha > 0$ and $c > 0$ such that

$$\sup_{\Omega_\alpha} \Delta f < -c.$$

The following lemma describes some elementary properties of a function f that violates the weak Omori–Yau maximum principle.

Lemma 2.3. *Suppose that a weighted graph (V, b, μ) is stochastically incomplete. Let f be a nonnegative function on V such that $f^* < +\infty$ and for some $\alpha > 0$ and $c > 0$,*

$$\sup_{\Omega_\alpha} \Delta f < -c.$$

Let $\alpha' = \min\{\alpha, c\}$. Then the following is true:

(1) *f cannot attain its supremum f^* on V, and in particular, is nonconstant;*
(2) $\sup_{\Omega_{\alpha'}} \Delta f < -\alpha'$;
(3) *for every $n \geqslant 1$, and every $x \in \Omega_{\frac{\alpha'}{n}}$,*

$$\mathrm{Deg}(x) = \frac{1}{\mu(x)} \sum_y b(x,y) > n.$$

In other words,

$$\Omega_{\frac{\alpha'}{n}} \subseteq \left\{x \in V\colon \mathrm{Deg}(x) > n\right\}.$$

Proof. (1) Suppose that there exists $x_0 \in V$ such that $f(x_0) = f^*$. In particular, $x_0 \in \Omega_\alpha$. We have that

$$\frac{1}{\mu(x_0)} \sum_{y \in V} b(x_0, y)\big(f(y) - f(x_0)\big) = -\Delta f(x_0) > c > 0.$$

Thus there exists $y \in V$ such that $f(y) > f(x_0)$, a contradiction.

(2) Since $\alpha' \leqslant \alpha$, we have $\Omega_{\alpha'} \subseteq \Omega_\alpha$. So

$$\sup_{\Omega_{\alpha'}} \Delta f \leqslant \sup_{\Omega_\alpha} \Delta f < -c \leqslant -\alpha'.$$

(3) For $x \in \Omega_{\frac{\alpha'}{n}}$, set

$$l = \frac{1}{\mu(x)} \sum_{y:\ f(y) > f(x)} b(x, y),$$

we have

$$\alpha' < -\Delta f(x) \leqslant \frac{1}{\mu(x)} \sum_{y:\ f(y) > f(x)} b(x, y)\big(f(y) - f(x)\big) \leqslant \frac{l\alpha'}{n}.$$

Therefore $l > n$ and, in particular, $\mathrm{Deg}(x) > n$ for all $x \in \Omega_{\frac{\alpha'}{n}}$. □

Remark 2.4. Part (3) of this lemma gives a control of "directions of increase" of the function violating the weak Omori–Yau maximum principle. An immediate consequence is that a stochastically incomplete graph necessarily has unbounded weighted degree. In particular, the semigroup corresponding to the combinatorial Laplacian on a graph is stochastically complete. This is a result of [5,6].

Stochastic incompleteness is a global property while the weighted degree function is a local quantity. We can define a "global weighted degree function" in an iterative way.

Lemma 2.5. *Fix a non-decreasing sequence $\Theta = \{a_k\}_{k \geqslant 0}$ of nonnegative real numbers. We use the convention that*

$$\sum_{y,\, y \in \emptyset} b(x, y) = 0.$$

For $x \in V$ and $k \in \mathbb{N}$, define

$$\mathrm{Deg}_{\Theta,0}(x) = \mathrm{Deg}(x),$$

and

$$\mathrm{Deg}_{\Theta,k+1}(x) = \frac{1}{\mu(x)} \sum_{y,\, \mathrm{Deg}_{\Theta,k}(y) > a_k} b(x, y).$$

Then for any $x \in V$, $\{\mathrm{Deg}_{\Theta,k}(x)\}_{k \geqslant 0}$ forms a non-increasing, nonnegative sequence. In particular,

$$\mathrm{Deg}_{\Theta,\infty}(x) = \lim_{k \to \infty} \mathrm{Deg}_{\Theta,k}(x)$$

exists for all $x \in V$.

Proof. The sequence $\{\mathrm{Deg}_{\Theta,k}(x)\}_{k \geqslant 0}$ obviously has nonnegative entries. We only need to prove that for any $k \geqslant 0$,

$$\mathrm{Deg}_{\Theta,k+1}(x) \leqslant \mathrm{Deg}_{\Theta,k}(x).$$

For $k = 0$, we have

$$\mathrm{Deg}_{\Theta,1}(x) = \frac{1}{\mu(x)} \sum_{y,\, \mathrm{Deg}(y) > a_0} b(x, y) \leqslant \frac{1}{\mu(x)} \sum_{y} b(x, y) = \mathrm{Deg}_{\Theta,0}(x).$$

Assume that the assertion holds for $k = n - 1 \geqslant 0$, that is

$$\mathrm{Deg}_{\Theta,n}(x) \leqslant \mathrm{Deg}_{\Theta,n-1}(x).$$

Since $a_n \geqslant a_{n-1}$, we see that for $k = n$,

$$\begin{aligned}\mathrm{Deg}_{\Theta,n}(x) &= \frac{1}{\mu(x)} \sum_{y,\mathrm{Deg}_{\Theta,n-1}(y)>a_{n-1}} b(x,y) \\ &\geqslant \frac{1}{\mu(x)} \sum_{y,\mathrm{Deg}_{\Theta,n-1}(y)>a_n} b(x,y) \\ &\geqslant \frac{1}{\mu(x)} \sum_{y,\mathrm{Deg}_{\Theta,n}(y)>a_n} b(x,y) \\ &= \mathrm{Deg}_{\Theta,n+1}(x).\end{aligned}$$

The assertion follows by induction. □

Definition 2.6. We call $\mathrm{Deg}_{\Theta,\infty}(x)$ the global weighted degree of x with respect to the sequence Θ. For the special case when $a_k \equiv n$, we denote $\mathrm{Deg}_{\Theta,\infty}(x)$ by $\mathrm{Deg}_{n,\infty}(x)$ and call it the global weighted degree of x with parameter n.

Remark 2.7. Note that unlike the weighted degree $\mathrm{Deg}(x)$, the global weighted degree of x contains information of points that may not be neighbors of x.

Lemma 2.8. *For $m > n \geqslant 1$, $k \in \mathbb{N}$, the following holds for any $x \in V$,*

$$\mathrm{Deg}_{n,k}(x) \geqslant \mathrm{Deg}_{m,k}(x).$$

In particular, for any $x \in V$,

$$\mathrm{Deg}_{n,\infty}(x) \geqslant \mathrm{Deg}_{m,\infty}(x).$$

Proof. The first assertion can be proven by an induction procedure similar to the proof of Lemma 2.5. The $k = 0$ case is obvious as

$$\mathrm{Deg}_{n,0}(x) = \mathrm{Deg}(x) = \mathrm{Deg}_{m,0}(x).$$

Assume that for all $x \in V$,

$$\mathrm{Deg}_{n,k}(x) \geqslant \mathrm{Deg}_{m,k}(x).$$

Then we have

$$\begin{aligned}\mathrm{Deg}_{n,k+1}(x) &= \frac{1}{\mu(x)} \sum_{y,\mathrm{Deg}_{n,k}(y)>n} b(x,y) \\ &\geqslant \frac{1}{\mu(x)} \sum_{y,\mathrm{Deg}_{m,k}(y)>n} b(x,y) \\ &\geqslant \frac{1}{\mu(x)} \sum_{y,\mathrm{Deg}_{m,k}(y)>m} b(x,y) \\ &= \mathrm{Deg}_{m,k+1}(x).\end{aligned}$$

This completes the induction. The last assertion follows by taking the limit $k \to \infty$ in

$$\mathrm{Deg}_{n,k}(x) \geqslant \mathrm{Deg}_{m,k}(x). \qquad \square$$

The notion of the global weighted degree function allows us to improve Lemma 2.3 as follows.

Theorem 2.9. *Suppose that a weighted graph (V, b, μ) is stochastically incomplete. Let f be a nonnegative function on V such that $f^* < +\infty$ and for some $\alpha > 0$,*

$$\sup_{\Omega_\alpha} \Delta f < -\alpha.$$

Then for any $n \geqslant 1$,

$$\Omega_{\frac{\alpha}{n}} \subseteq \{x \in V\colon \operatorname{Deg}_{n,\infty}(x) > n\}.$$

As a consequence, (V, b, μ) has unbounded global weighted degree for any parameter $n \geqslant 1$.

Proof. In the proof of part (3) of Lemma 2.3, we already showed that for $x \in \Omega_{\frac{\alpha}{n}}$,

$$l = \frac{1}{\mu(x)} \sum_{y\colon f(y) > f(x)} b(x, y) > n.$$

We claim that for all $x \in \Omega_{\frac{\alpha}{n}}$, $k \in \mathbb{N}$,

$$n < l \leqslant \operatorname{Deg}_{n,k}(x).$$

Assuming the claim, we see that for any $x \in \Omega_{\frac{\alpha}{n}}$,

$$n < l \leqslant \operatorname{Deg}_{n,\infty}(x).$$

Hence

$$\Omega_{\frac{\alpha}{n}} \subseteq \{x \in V\colon \operatorname{Deg}_{n,\infty}(x) > n\}.$$

Now we complete the proof of the claim. For all $x \in \Omega_{\frac{\alpha}{n}}$,

$$\operatorname{Deg}_{n,0}(x) = \frac{1}{\mu(x)} \sum_y b(x, y) \geqslant \frac{1}{\mu(x)} \sum_{y\colon f(y) > f(x)} b(x, y) = l.$$

Assume that the claim is true for k. In other words, for all $x \in \Omega_{\frac{\alpha}{n}}$,

$$\operatorname{Deg}_{n,k}(x) \geqslant l > n.$$

Note that if $f(y) > f(x)$ for $x \in \Omega_{\frac{\alpha}{n}}$, y is necessarily in $\Omega_{\frac{\alpha}{n}}$ and consequently,

$$\operatorname{Deg}_{n,k}(y) \geqslant l > n.$$

Thus we have

$$\operatorname{Deg}_{n,k+1}(x) = \frac{1}{\mu(x)} \sum_{y, \operatorname{Deg}_{n,k}(y) > n} b(x, y) \geqslant \frac{1}{\mu(x)} \sum_{y\colon f(y) > f(x)} b(x, y) = l$$

for any $x \in \Omega_{\frac{\alpha}{n}}$. The claim follows by induction.

By Lemma 2.8, we see that for $m > n \geqslant 1$,

$$\Omega_{\frac{\alpha}{m}} \subseteq \{x \in V\colon \operatorname{Deg}_{m,\infty}(x) > m\} \subseteq \{x \in V\colon \operatorname{Deg}_{n,\infty}(x) > m\}.$$

The set $\Omega_{\frac{\alpha}{m}}$ is nonempty for any $m > n$, so that the function $\operatorname{Deg}_{n,\infty}(x)$ is necessarily unbounded for any $n \geqslant 1$. □

3. Khas'minskii's criterion

Now we are ready to prove the following analogue of Khas'minskii's criterion for stochastic completeness.

Theorem 3.1. *Assume that the weighted degree function* $\operatorname{Deg}(x)$ *is unbounded for the weighted graph (V, b, μ). If there exists a nonnegative function $\gamma \in F$ on V such that*

$$\gamma(x) \to +\infty \quad \text{as } \operatorname{Deg}(x) \to +\infty \tag{3.1}$$

and

$$\Delta\gamma(x) + \lambda\gamma(x) \geqslant 0 \quad \text{outside a set } A \text{ of bounded weighted degree} \tag{3.2}$$

for some $\lambda > 0$, then V is stochastically complete.

Proof. We only need prove that V satisfies the weak Omori–Yau maximum principle. If not, there exists a nonnegative function f on V with $f^* < +\infty$ and some $\alpha > 0$ such that

$$\sup_{\Omega_\alpha} \Delta f < -\alpha.$$

Let

$$M = \sup\{\mathrm{Deg}(x)\colon x \in A\} < +\infty.$$

By Lemma 2.3, changing α if necessary, we can assume that $\mathrm{Deg}(x) > M$ for all $x \in \Omega_\alpha$.

Let

$$u = f - c\gamma,$$

where the parameter $c > 0$ will be chosen later.

Since $f^* < +\infty$ and

$$\gamma(x) \to +\infty \quad \text{as } \mathrm{Deg}(x) \to +\infty,$$

there exists $N(c) > M$ such that

$$\sup_{\{x\in V\colon \mathrm{Deg}(x)<N(c)\}} u(x) = u^* := \sup_V u(x) < +\infty.$$

Let $0 < \eta < \min(\frac{\alpha}{2}, \frac{\alpha}{2\lambda})$. We can choose $\bar{x}$ such that

$$f(\bar{x}) > f^* - \frac{\eta}{2}.$$

Choose $c = c(\eta, \bar{x}) > 0$ small enough to insure that $c\gamma(\bar{x}) < \frac{\eta}{2}$.

For $n \in \mathbb{N}$, we can choose x_n with $\mathrm{Deg}(x_n) < N(c)$ such that

$$u(x_n) + \frac{1}{n} > u^* \geqslant u(\bar{x}) = f(\bar{x}) - c\gamma(\bar{x}).$$

We have

$$f(x_n) + \frac{1}{n} > f(x_n) - c\gamma(x_n) + \frac{1}{n} > f(\bar{x}) - c\gamma(\bar{x}) > f^* - \eta,$$

and

$$c\gamma(x_n) < f(x_n) - f^* + \eta + \frac{1}{n} < \eta + \frac{1}{n}.$$

So for every index $n > \frac{2}{\eta}$, note that by definition $\eta < \frac{\alpha}{2}$ and $\eta < \frac{\alpha}{2\lambda}$,

$$f(x_n) > f^* - \frac{3}{2}\eta > f^* - \alpha,$$
$$c\lambda\gamma(x_n) < \frac{3}{2}\lambda\eta < \frac{3}{4}\alpha.$$

In particular, for every index $n > \frac{2}{\eta}$, $x_n \in \Omega_\alpha$. It follows that for all $n > \frac{2}{\eta}$,

$$\Delta\gamma(x_n) + \lambda\gamma(x_n) \geqslant 0,$$

and

$$\Delta f(x_n) < -\alpha.$$

Then

$$\begin{aligned}\Delta(f - c\gamma)(x_n) &= \Delta f(x_n) - c\Delta\gamma(x_n)\\ &< -\alpha + c\lambda\gamma(x_n) < -\alpha/4.\end{aligned} \tag{3.3}$$

On the other hand, we have

$$\begin{aligned}\Delta(f-c\gamma)(x_n)&=\Delta u(x_n)\\&=\frac{1}{\mu(x_n)}\sum_y b(x_n,y)\big(u(x_n)-u(y)\big)\\&\geqslant\frac{1}{\mu(x_n)}\sum_y b(x_n,y)\big(u(x_n)-u^*\big)\\&>\frac{1}{\mu(x_n)}\sum_y b(x_n,y)\left(-\frac{1}{n}\right)\\&=-\frac{\operatorname{Deg}(x_n)}{n}>-\frac{N(c)}{n}.\end{aligned}$$

Choosing sufficiently large n, we obtain a contradiction to (3.3). □

Remark 3.2. Note that unlike in the case of manifolds we do not require that the exceptional set A is compact.

A convenient version of Khas'minskii's criterion on manifolds is given in [25]. We give the discrete analogue here.

Theorem 3.3. *Let (V,b,μ) be a weighted graph. If there exists a nonnegative function $\sigma\in F$ on V with*

$$\sigma(x)\to+\infty\quad\text{as }\operatorname{Deg}(x)\to+\infty$$

satisfying

$$\Delta\sigma(x)+f\big(\sigma(x)\big)\geqslant 0\quad\text{outside a set } A \text{ of bounded weighted degree}$$

for some positive, increasing function $f\in C^1([0,+\infty))$ with

$$\int_0^{+\infty}\frac{dr}{f(r)}=+\infty,$$

then V is stochastically complete.

Proof. Let

$$\phi(r)=\exp\left(\int_0^r\frac{ds}{f(s)+s}\right),$$

we have $\phi(r)\to+\infty$ as $r\to+\infty$ (cf. Lemma 3.4 below).

The function $\phi(r)$ is increasing and concave since:

(1) $\phi'(r)=\frac{\phi(r)}{f(r)+r}>0$;
(2) $\phi''(r)=-\frac{\phi(r)f'(r)}{(f(r)+r)^2}\leqslant 0$.

Therefore for $r,s\geqslant 0$ we have

$$\phi(r)-\phi(s)\geqslant\phi'(r)(r-s).$$

Thus

$$\begin{aligned}\Delta\phi\big(\sigma(x)\big)&=\frac{1}{\mu(x)}\sum_{y\in V}b(x,y)\big(\phi\big(\sigma(x)\big)-\phi\big(\sigma(y)\big)\big)\\&\geqslant\phi'\big(\sigma(x)\big)\frac{1}{\mu(x)}\sum_{y\in V}b(x,y)\big(\sigma(x)-\sigma(y)\big)\\&=\phi'\big(\sigma(x)\big)\Delta\sigma(x),\end{aligned}\tag{3.4}$$

which also shows that $\phi(\sigma(x)) \in F$. Now, consider $\gamma(x) = \phi(\sigma(x))$, then

$$\gamma(x) \to +\infty \quad \text{as } \mathrm{Deg}(x) \to +\infty.$$

On the complement of A we have

$$\begin{aligned}
\Delta\gamma(x) + \gamma(x) &= \Delta\phi\big(\sigma(x)\big) + \phi\big(\sigma(x)\big) \\
&\geqslant \phi'\big(\sigma(x)\big)\Delta\sigma(x) + \phi\big(\sigma(x)\big) \\
&= \phi'\big(\sigma(x)\big)\left(\Delta\sigma(x) + \frac{\phi(\sigma(x))}{\phi'(\sigma(x))}\right) \\
&= \phi'\big(\sigma(x)\big)\big(\Delta\sigma(x) + f\big(\sigma(x)\big) + \sigma(x)\big) \\
&\geqslant \phi'\big(\sigma(x)\big)\big(\Delta\sigma(x) + f\big(\sigma(x)\big)\big) \geqslant 0.
\end{aligned} \tag{3.5}$$

Theorem 3.1 applied to $\gamma(x)$ with $\lambda = 1$ implies stochastic completeness. □

In the previous proof, we have made use of the following elementary fact.

Lemma 3.4. *Let $f \in C^1([0, +\infty))$ be a positive, increasing function. Assume further that*

$$\int_0^{+\infty} \frac{dr}{f(r)} = +\infty.$$

Then

$$\int_0^{+\infty} \frac{dr}{f(r) + r} = +\infty.$$

For the sake of completeness, we give a proof here.

Proof. Note that the integral is only improper at $+\infty$ since f is positive and increasing on $[0, +\infty)$. Assume that

$$\int_0^{+\infty} \frac{dr}{f(r) + r} < +\infty.$$

For all $x > 0$, we have

$$0 < \frac{x}{2} \cdot \frac{1}{f(x) + x} \leqslant \int_{\frac{x}{2}}^{x} \frac{dr}{f(r) + r} \leqslant \int_{\frac{x}{2}}^{+\infty} \frac{dr}{f(r) + r}.$$

The third integral necessarily goes to 0 as x approaches $+\infty$. Thus there exists $r_0 > 0$ such that for any $r > r_0$,

$$\frac{r}{f(r) + r} \leqslant \frac{1}{2}.$$

It follows that $f(r) \geqslant r$ for all $r > r_0$. But then

$$\int_{r_0}^{+\infty} \frac{dr}{f(r) + r} \geqslant \int_{r_0}^{+\infty} \frac{dr}{2f(r)} = +\infty.$$

A contradiction. □

4. Stability results

In this section we show that after certain surgeries, a stochastically incomplete graph will remain stochastically incomplete. The weak Omori–Yau maximum principle allows us to pass from the stability of existence of certain functions to the stability of stochastic incompleteness. Roughly speaking, part (3) of Lemma 2.3 implies that a perturbation of bounded weighted degree does not affect the stochastically incompleteness. This intuition is made explicit by the following theorems.

Theorem 4.1. *Let (V, b, μ) be a weighted graph and $W \subseteq V$. $(W, b|_{W\times W}, \mu|_W)$ forms a subgraph. Assume that W is stochastically incomplete. If one of the following two conditions holds, V is also stochastically incomplete*:

(1) *For some $n \geqslant 1$*, $\sup\{\mathrm{Deg}_W(x)\colon x \in W,\ \exists y \in V\backslash W,\ b(x,y) > 0\} < n$;
(2) *There exists $n \geqslant 1$, such that $\forall x \in W$*,

$$\frac{1}{\mu(x)} \sum_{y \in V\backslash W} b(x,y) < n.$$

Proof. (1) Since W is stochastically incomplete there exists a nonnegative function f on W and $\alpha > 0$ such that

$$\sup_{\Omega_\alpha^W} \Delta^W f < -\alpha.$$

Here

$$\Omega_\alpha^W = \{x \in W\colon f(x) > f^* - \alpha\},$$

and

$$\Delta^W f(x) = \frac{1}{\mu(x)} \sum_{y \in W} b(x,y)\big(f(x) - f(y)\big)$$

for $x \in W$.

Define a function u on V by

$$u(x) = \begin{cases} (f(x) + \frac{\alpha}{n} - f^*)_+, & x \in W, \qquad (4.6)\\ 0, & x \in V\backslash W. \qquad (4.6') \end{cases}$$

We see that $u^* = \frac{\alpha}{n}$ and

$$\Omega_{\frac{\alpha}{n}}^V = \{x \in V\colon u(x) > 0\} = \left\{x \in W\colon f(x) > f^* - \frac{\alpha}{n}\right\} \subseteq \{x \in W\colon \mathrm{Deg}_W(x) > n\}$$

by (3) of Lemma 2.3.

Thus for $x \in \Omega_{\frac{\alpha}{n}}^V$, $y \in V\backslash W$, we have $b(x,y) = 0$. Hence for every $x \in \Omega_{\frac{\alpha}{n}}^V$

$$\begin{aligned} \Delta^V u(x) &= \frac{1}{\mu(x)} \sum_{y \in V} b(x,y)\big(u(x) - u(y)\big) \\ &= \frac{1}{\mu(x)} \sum_{y \in W} b(x,y)\big(u(x) - u(y)\big) \\ &\leqslant \frac{1}{\mu(x)} \sum_{y \in W} b(x,y)\big(f(x) - f(y)\big) \\ &= \Delta^W f(x) < -\alpha. \end{aligned} \qquad (4.7)$$

The stochastic incompleteness of V then follows from Theorem 2.2.

(2) As in (1), there is a nonnegative function f on W and $\alpha > 0$ such that

$$\sup_{\Omega_\alpha^W} \Delta^W f < -\alpha$$

since W is stochastically incomplete by assumption.

Define a function u on V by

$$u(x)=\begin{cases}(f(x)+\frac{\alpha}{2n}-f^*)_+, & x\in W, \quad (4.8)\\ 0, & x\in V\backslash W. \quad (4.8')\end{cases}$$

We see that $u^*=\frac{\alpha}{2n}$ and

$$\Omega^V_{\frac{\alpha}{2n}}=\big\{x\in V\colon\ u(x)>0\big\}=\left\{x\in W\colon\ f(x)>f^*-\frac{\alpha}{2n}\right\}.$$

So for $x\in\Omega^V_{\frac{\alpha}{2n}}$

$$\begin{aligned}\Delta^V u(x)&=\frac{1}{\mu(x)}\sum_{y\in V}b(x,y)\big(u(x)-u(y)\big)\\ &=\frac{1}{\mu(x)}\sum_{y\in W}b(x,y)\big(u(x)-u(y)\big)+\frac{1}{\mu(x)}\sum_{y\in V\backslash W}b(x,y)\big(u(x)-u(y)\big)\\ &\leqslant\frac{1}{\mu(x)}\sum_{y\in W}b(x,y)\big(f(x)-f(y)\big)+\frac{1}{\mu(x)}\sum_{y\in V\backslash W}b(x,y)\frac{\alpha}{2n}\\ &\leqslant\Delta^W f(x)+\frac{\alpha}{2}<-\frac{\alpha}{2}.\end{aligned}\tag{4.9}$$

The stochastic incompleteness of V then follows from Theorem 2.2. □

Remark 4.2. Part (2) of Theorem 4.1 was first proved by Keller and Lenz [18]. Our proof here is more elementary.

In Theorem 4.1 we derive stochastic incompleteness of graphs from that of subgraphs. The weak Omori–Yau maximum principle allows also to obtain implications in the opposite direction, as in the next statement.

Theorem 4.3. *Let (V,b,μ) be a stochastically incomplete weighted graph and $n\geqslant 1$. The subgraph*

$$W=\big\{x\in V\colon\ \mathrm{Deg}(x)>n\big\}$$

with weights $(b|_{W\times W},\mu|_W)$ is stochastically incomplete as well.

Proof. There exists a nonnegative function f on V and $\alpha>0$ such that

$$\sup_{\Omega^V_\alpha}\Delta^V f<-\alpha.$$

We will show that $f|_W$ is a function violating the weak Omori–Yau maximum principle.

From Lemma 2.3, we see that

$$\sup_W f=\sup_V f,$$

and

$$\Omega^W_{\frac{\alpha}{n}}=\Omega^V_{\frac{\alpha}{n}}.$$

We claim that for any $x\in\Omega^W_{\frac{\alpha}{n}}$,

$$\Delta^W f(x)\leqslant\Delta^V f(x)<-\alpha.$$

In fact, for $x\in\Omega^W_{\frac{\alpha}{n}}$, $y\in V\backslash W$, we claim that $f(y)\leqslant f(x)$. If not

$$f(y)>f(x)>f^*-\frac{\alpha}{n},$$

so that $y\in\Omega^W_{\frac{\alpha}{n}}\subseteq W$, a contradiction.

Then for any $x \in \Omega^W_{\frac{\alpha}{n}}$, we obtain

$$-\alpha > \Delta^V f(x) = \frac{1}{\mu(x)} \sum_{y \in V} b(x,y)\big(f(x) - f(y)\big)$$
$$= \frac{1}{\mu(x)} \sum_{y \in W} b(x,y)\big(f(x) - f(y)\big) + \frac{1}{\mu(x)} \sum_{y \in V \setminus W} b(x,y)\big(f(x) - f(y)\big)$$
$$\geqslant \frac{1}{\mu(x)} \sum_{y \in W} b(x,y)\big(f(x) - f(y)\big) = \Delta^W f(x).$$

The stochastic incompleteness of V then follows from Theorem 2.2. □

5. Applications to the physical Laplacian

In this section, we apply the weak Omori–Yau maximum principle and Khas'minskii's criterion to the physical Laplacian on an (un-weighted) graph. We assume that (V, E) is a locally finite, connected infinite graph without loops and multi-edges where V is the set of vertices and E is the set of edges. This corresponds to the special case that $b(x, y) \in \{0, 1\}$, $\mu(x) \equiv 1$ and $b(x, y) = 1 \Leftrightarrow (x, y) \in E$.

As before, we use V to denote the graph if no confusion arises. We write $y \sim x$ if there is an edge connecting x and y. In this case, we call the vertices x and y neighbors. Then the weighted degree function

$$\mathrm{Deg}(x) = \sum_{y \in V} b(x,y) = \#\{y \in V\colon\ y \sim x\},$$

is exactly the number of neighbors of x in V, i.e. $\deg(x)$.

Let d be the graph metric on V, that is, for any two vertices $x, y \in V$, $d(x, y)$ is the smallest number of edges in a chain of edges connecting x and y. We fix a point $x^* \in V$ as a root of the graph and define

$$r(x) = d\big(x, x^*\big).$$

A key feature of the graph metric is that if $x \sim y$, then

$$\big|r(x) - r(y)\big| \leqslant 1.$$

We use further the notations

$$S_R = \big\{y \in V\colon\ r(y) = R\big\},$$
$$B_R = \bigcup_{n=0}^{R} S_n = \big\{y \in V\colon\ r(y) \leqslant R\big\},$$
$$m_\pm(x) = \#\big\{y\colon\ y \sim x,\ r(y) = r(x) \pm 1\big\},$$
$$K_\pm(r) = \max_{x \in S_r} m_\pm(x),$$

and

$$k_\pm(r) = \min_{x \in S_r} m_\pm(x).$$

The *formal* Laplacian in this case is

$$\Delta f(x) = \sum_{y, y \sim x} \big(f(x) - f(y)\big). \tag{5.10}$$

Here f can now be an arbitrary function on V because of the local finiteness. For example,

$$\Delta r(x) = m_-(x) - m_+(x). \tag{5.11}$$

The machinery of weak Omori–Yau maximum principle and Khas'minskii's criterion can be applied in two ways.

(1) Choose a series $\sum_{n=0}^{\infty} a_n$ with nonnegative terms, and define the function

$$f(x) = \sum_{n=0}^{r(x)} a_n$$

which then can be used in the weak Omori–Yau maximum principle and Khas'minskii's criterion. Choosing the series appropriately we obtain sufficient conditions for stochastic completeness and incompleteness.

(2) Alternatively, one can determine "natural" values of a_n by solving certain difference equations or inequalities.

Before going into details we would like to point out that for a locally finite graph of unbounded degree, $\deg(x) \to +\infty$ implies $r(x) \to +\infty$. Thus Theorem 3.1 can be restated in a weaker form:

Theorem 5.1. *Let (V, E) be a locally finite, connected infinite graph without loops and multi-edges. Assume that the degree function $\deg(x)$ is unbounded. If there exists a nonnegative function γ on V with*

$$\gamma(x) \to +\infty \quad \text{as } r(x) \to +\infty$$

satisfying

$$\Delta\gamma(x) + \lambda\gamma(x) \geqslant 0 \quad \text{outside a finite set } A$$

for some $\lambda > 0$, then V is stochastically complete.

Remark 5.2. Wojciechowski and Keller [20] also obtained independently this form of Khas'minskii's criterion using a different method.

5.1. Criteria for stochastic completeness

In what follows, $\sum_{n=0}^{\infty} a_n$ is a series with nonnegative terms.

Theorem 5.3. *Let (V, E) be a locally finite, connected infinite graph without loops and multi-edges. If $\sum_{n=0}^{\infty} a_n = +\infty$ and for some $\lambda > 0$, the following inequality*

$$m_+(x)a_{r(x)+1} - m_-(x)a_{r(x)} \leqslant \lambda \sum_{n=0}^{r(x)} a_n$$

holds outside a finite set, then V is stochastically complete.

Proof. Let $\gamma(x) = \sum_0^{r(x)} a_n$, then

$$\Delta\gamma(x) + \lambda\gamma(x) = m_-(x)a_{r(x)} - m_+(x)a_{r(x)+1} + \lambda \sum_0^{r(x)} a_n \geqslant 0$$

outside a finite set and $\gamma(x) \to +\infty$ as $r(x) \to +\infty$. By Theorem 5.1, V is stochastically complete. □

Theorem 5.3 already gives some nontrivial results through some obvious choices of a_n. One natural choice is $a_n \equiv 1$. Then a sufficient condition for stochastic completeness is

$$m_+(x) - m_-(x) \leqslant \lambda r(x)$$

outside a finite set for some $\lambda > 0$. This improves the curvature type criterion of Weber [28] where the sufficient condition is

$$m_+(x) - m_-(x) \leqslant C$$

for some constant $C > 0$.

One can improve this result by choosing divergent series with smaller terms. We do this via Theorem 3.3.

Theorem 5.4. *Let (V, E) be a locally finite, connected infinite graph without loops and multi-edges. If for some positive, increasing function $f \in C^1([0, +\infty))$ with $\int_0^{+\infty} \frac{dr}{f(r)} = +\infty$,*

$$m_+(x) - m_-(x) \leqslant f\bigl(r(x)\bigr)$$

outside a finite set, then V is stochastically complete.

Proof. We only need to take $\sigma(x) = r(x)$ in Theorem 3.3. □

The following result was first obtained by Wojciechowski [30]. We give a shorter proof, based on Theorem 5.1.

Theorem 5.5. *Let (V, E) be a locally finite, connected infinite graph without loops and multi-edges. If $\sum_{r=0}^{\infty} \frac{1}{K_+(r)} = +\infty$, then V is stochastically complete.*

Proof. Let

$$\gamma(x) = \sum_{r=0}^{r(x)-1} \frac{1}{K_+(r)}$$

for $r(x) > 0$, and $\gamma(x^*) = 0$. We then have that

$$\gamma(x) \to +\infty \quad \text{as } r(x) \to +\infty,$$

and outside a finite set

$$\Delta\gamma(x) + \gamma(x) = m_-(x)\frac{1}{K_+(r(x)-1)} - m_+(x)\frac{1}{K_+(r(x))} + \gamma(x) \geqslant \gamma(x) - 1 \geqslant 0.$$

The assertion follows from Theorem 5.1. □

5.2. Criteria for stochastic incompleteness

Similarly, using test series to define functions that violate the weak Omori–Yau maximum principle, we obtain a curvature type criterion for stochastic incompleteness:

Theorem 5.6. *Let (V, E) be a locally finite, connected infinite graph without loops and multi-edges. If $\sum_{l=0}^{\infty} a_l < +\infty$, $a_l \geqslant 0$ and for some $n \in \mathbb{N}$, $c > 0$, the inequality*

$$m_+(x)a_{r(x)+1} - m_-(x)a_{r(x)} > c$$

holds for $r(x) > n$, then V is stochastically incomplete.

Proof. Let

$$f(x) = \sum_{l=0}^{r(x)} a_l.$$

Then

$$f^* = \sum_{r=0}^{\infty} a_r < +\infty.$$

Let $\alpha = \sum_{l=n+1}^{\infty} a_l$. Then $f(x) > f^* - \alpha$ implies $r(x) > n$. So in this case,

$$-\Delta f(x) = m_+(x)a_{r(x)+1} - m_-(x)a_{r(x)} > c.$$

By Theorem 2.2, V is stochastically incomplete. □

Theorem 2.2 can also be used to derive the following result about stochastic incompleteness obtained by Wojciechowski [31].

Theorem 5.7. *Let (V, E) be a locally finite, connected infinite graph without loops and multi-edges. If*

$$\sum_{r=1}^{\infty} \max_{x \in S_r} \frac{m_-(x)}{m_+(x)} < +\infty, \tag{5.12}$$

then V is stochastically incomplete.

Proof. Denote $\max_{x\in S_r} \frac{m_-(x)}{m_+(x)}$ by $\eta(r)$. Let

$$f(x) = \sum_{r=1}^{r(x)-1} \eta(r)$$

for $r(x) \geqslant 2$, and $f(x) = 0$ elsewhere. Then

$$f^* = \sup f(x) = \sum_{r=1}^{\infty} \eta(r) < +\infty.$$

Choose $r_0 > 2$ sufficiently large so that

$$0 < \alpha = \sum_{r=r_0-1}^{\infty} \eta(r) < \frac{1}{2}.$$

Then

$$\begin{aligned} \Omega_\alpha &= \bigl\{x \in V:\ f(x) > f^* - \alpha\bigr\} \\ &= \left\{x \in V:\ f(x) > \sum_{r=1}^{r_0-2} \eta(r)\right\} \\ &= \left\{x \in V:\ \sum_{r=1}^{r(x)-1} \eta(r) > \sum_{r=1}^{r_0-2} \eta(r)\right\} \\ &= \bigl\{x \in V:\ r(x) > r_0 - 1\bigr\} \\ &= B^c_{r_0-1}. \end{aligned}$$

But for $x \in B^c_{r_0-1}$, we have $r(x) - 1 \geqslant r_0 - 1$ and thus

$$\eta\bigl(r(x) - 1\bigr) < \alpha < \frac{1}{2}.$$

Hence we obtain that for $x \in \Omega_\alpha$,

$$\begin{aligned} \Delta f(x) &= m_-(x)\eta\bigl(r(x) - 1\bigr) - m_+(x)\eta\bigl(r(x)\bigr) \\ &\leqslant \frac{1}{2} m_-(x) - m_-(x) \\ &\leqslant -\frac{1}{2} m_-(x) \leqslant -\frac{1}{2}. \end{aligned}$$

By Theorem 2.2 V is stochastically incomplete. □

Remark 5.8. Theorem 5.7 first appeared in a slightly weaker form as Theorem 3.4 in [31]. There stochastic incompleteness is established under the condition

$$\sum_{r=1}^{\infty} \frac{K_-(r)}{k_+(r)} < +\infty$$

instead of (5.12).

5.3. The symmetric case

Definition 5.9. Let (V, E) be a locally finite, connected infinite graph without loops and multi-edges. The graph V is called weakly symmetric if it satisfies

$$m_+(x) = g_+\bigl(r(x)\bigr), \qquad m_-(x) = g_-\bigl(r(x)\bigr)$$

with functions $g_+(r), g_-(r) : \mathbb{N} \to \mathbb{N}$.

For graphs that are weakly symmetric, Wojciechowski [30] proved the following criterion. Here we present a proof based on the weak Omori–Yau maximum principle.

Theorem 5.10. *A weakly symmetric graph V is stochastically complete if and only if*

$$\sum_{r=0}^{\infty} \frac{V(r)}{g_+(r)S(r)} = +\infty$$

where $S(r) = \#S_r$ and $V(r) = \#B_r$.

Proof. Since

$$m_+(x) = g_+\big(r(x)\big), \qquad m_-(x) = g_-\big(r(x)\big),$$

we see that

$$g_-(r)S(r) = g_+(r-1)S(r-1).$$

Let

$$\gamma(x) = \sum_{r=0}^{r(x)-1} \frac{V(r)}{g_+(r)S(r)}$$

for $r(x) > 0$, and $\gamma(x^*) = 0$. We have

$$\begin{aligned}\Delta\gamma(x) &= g_-\big(r(x)\big)\frac{V(r(x)-1)}{g_+(r(x)-1)S(r(x)-1)} - g_+\big(r(x)\big)\frac{V(r(x))}{g_+(r(x))S(r(x))} \\ &= \frac{V(r(x)-1)}{S(r(x))} - \frac{V(r(x))}{S(r(x))} = -1 \end{aligned} \tag{5.13}$$

for $r(x) \geqslant 1$.

If $\gamma(x) \to +\infty$ as $r(x) \to +\infty$, then

$$\Delta\gamma(x) + \gamma(x) = \gamma(x) - 1 \geqslant 0$$

outside a finite set. The stochastic completeness then follows from Theorem 5.1.

For the other implication suppose that $\gamma^* = \sup \gamma(x) < +\infty$. Letting $\alpha = \gamma^*$, we see that on $\Omega_\alpha = B_0^c$, by the calculation (5.13) above,

$$\Delta\gamma(x) = -1.$$

The stochastic incompleteness then follows from Theorem 2.2. □

Remark 5.11. As pointed out by Wojciechowski [30], it is interesting to notice that for a weakly symmetric graph, the edges between points on the same sphere play no role in stochastic completeness. See also [20] for further studies of weakly symmetric graphs.

6. Further remarks

(1) A rich source for ideas behind the study of stochastic completeness of the physical Laplacian is the literature about the Riemannian manifold case. However, due to the fact that the Dirichlet form on a graph is nonlocal, there are some essential differences in our case. For example, as shown by Wojciechowski [30], there exist stochastically incomplete graphs with polynomial volume growth which never happens in the manifold case. His examples of stochastically incomplete graphs that satisfy

$$\mu(B_r) \leqslant Cr^{3+\varepsilon}, \quad C, \varepsilon > 0,$$

are presented in the next remark.

It is then interesting to ask what is the smallest possible volume growth for stochastically incomplete graphs. It is natural to conjecture that for the physical Laplacian on graphs, the condition

$$\mu(B_r) \leqslant Cr^3, \quad C > 0,$$

implies stochastic completeness. This is proven in a forthcoming paper of Grigor'yan, Huang and Masamune [12]. Note that for geodesically complete Riemannian manifolds, the almost sharp condition [4,11,13,16]

$$\mu(B_r) \leqslant \exp Cr^2, \quad C > 0,$$

implies stochastic completeness.

On the other hand, there exist stochastically complete graphs with arbitrarily large volume growth. For example, take a set of vertices $\{0, 1, 2, \dots, n, \dots\}$ with edges $n \sim n+1$. For each vertex n, we associate a distinct finite set V_n and add extra edges between n and points in V_n. The resulting graph V is then a tree whose volume growth can be chosen to be arbitrarily large. One can observe that every vertex in V has at most two neighbors which have degree larger than 1. It is then of bounded global weighted degree with parameter 1 and hence is stochastically complete by Theorem 2.9. The stochastic completeness of V can be shown via Theorem 4.3 as well.

(2) Let $S(r)$ be given with $S(0)=1$. By connecting every vertex in S_r to every vertex in S_{r+1} we get a spherically symmetric graph G_S. Then by Theorem 5.10, G_S is stochastically incomplete if and only if

$$\sum_{r=0}^{\infty} \frac{\sum_{i=0}^{r} S(i)}{S(r+1)S(r)} < +\infty$$

since $m_\pm(x) = S(r(x) \pm 1)$. Taking $S(r) = [(r+1)^{2+\varepsilon}]$, $\varepsilon > 0$ where $[c]$ is the integer part of c, we see that G_S is stochastically incomplete whereas

$$\mu(B_r) \leqslant Cr^{3+\varepsilon}$$

for some $C > 0$.

This construction of Wojciechowski [30], at the same time gives a counterexample to the converse to Theorem 5.7. The graph G_S with $S(r) = (r+1)^3$ satisfies

$$\sum_{r=1}^{\infty} \max_{x \in S_r} \frac{m_-(x)}{m_+(x)} = \sum_{r=1}^{\infty} \frac{(r-1)^3}{(r+1)^3} = +\infty,$$

but is stochastically incomplete.

(3) Can we conclude that V is stochastically incomplete if

$$m_+(x) - m_-(x) \geqslant f\bigl(r(x)\bigr),$$

where $f(r) > 0$ and $\sum_{r=0}^{\infty} \frac{1}{f(r)} < +\infty$? This may be a useful complement to Theorem 5.4.

(4) We conjecture that the converse to Theorem 3.1 is true. Namely, if a weighted graph (V, b, μ) is stochastically complete, then there should exist a function $\gamma(x) \in F$ on V satisfying the conditions (3.1), (3.2). This is motivated by Nakai's result [23] that the converse to Khas'minskii's criterion [17] for parabolic Riemannian manifolds is true.

(5) For a subset A of V, we define its (outer) boundary to be

$$\partial A = \bigl\{x\colon\ x \in A^c,\ \text{and}\ \exists y \in A,\ \text{s.t.}\ x \sim y\bigr\}$$

and its closure to be $\bar{A} = A \cup \partial A$. Wojciechowski and Keller [20] proposed the following conjecture.

Conjecture 6.1. *If for some fixed point $x^* \in V$ as root,*

$$\sum_{r=0}^{\infty} \frac{\#B_r}{\#\partial B_r} = +\infty, \tag{6.14}$$

then (V, E) is stochastically complete.

This is an analogue of a conjecture for the stochastic completeness of manifold proposed by Grigor'yan in [10]. However, recently Bär and Bessa [2] constructed a counterexample to Grigor'yan's conjecture. Their idea can also be applied to the physical Laplacian as follows.

Take a stochastically complete tree T with root x_1, for example, a binary tree. Then T has exponential volume growth with respect to graph distance. Choose a stochastically incomplete graph with only polynomial volume growth, for example, the graph G_S in the previous remark with the root denoted by x_2. Now we make a single extra edge between x_1 and x_2 resulting in a new graph V. Since the gluing happens at only one point at G_S, the graph V is stochastically incomplete by Theorem 4.1. However, for any fixed point $x^* \in V$ as a root, the quantities $\#B_r$ and $\#\partial B_r$ are always of the order 2^n. So we know that V satisfies (6.14) while it is stochastically incomplete. This example is simpler than the example of [2] in the manifold case, thanks to special features of the discrete setting.

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