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Groups with non-central dimension quotients

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Abstract

Let G be a group and let $D_n(G)$ and $\gamma_n(G)$ be its nth dimension and nth lower central subgroups. In an earlier paper we proved that $D_n(G)/\gamma_n(G)$ is abelian. Here we prove that $D_n(G)/\gamma_n(G)$ is not, in general, central in $G/\gamma_n(G)$. In fact, for any s there exists a group G and an integer n such that $D_n(G)/\gamma_n(G)$ is not contained in the sth upper central subgroups of $G/\gamma_n(G)$.

1. Introduction

Let G be a group and let $\Delta = \Delta(G)$ be the augmentation ideal of its group ring $\mathbb{Z}G$. We recall that by definition

$$D_n(G) = \{g \in G \mid g \equiv 1 \bmod \Delta^n\}$$

is the *nth dimension subgroup* of G. While $D_n(G)$ always contains the nth lower central subgroup $\gamma_n(G)$, it is known that there exist groups with $D_n(G) \neq \gamma_n(G)$ (see [5] for $n = 4$, and [2] for $n \geq 4$). The problem of identifying the subgroup $D_n(G)$ of G, or equivalently the subgroup $D_n(G)/\gamma_n(G)$ of $G/\gamma_n(G)$ is known as the *dimension subgroup problem*. By Sjogren's theorem [6] the *dimension quotient* $D_n(G)/\gamma_n(G)$ has finite exponent dividing a number e_n that does not depend on G. So if G can be generated by m elements then the nth dimension quotient is finite and its order is bounded in terms of m and n. In an earlier paper the authors have pointed out that $D_{n-1}(G)/\gamma_n(G)$ is contained in the intersection of all subgroups of $G/\gamma_n(G)$ which are maximal in the set

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of abelian normal subgroups of $G/\gamma_n(G)$ [4]. In particular $D_n(G)/\gamma_n(G)$ is abelian. It was natural to ask then whether the dimension quotient $D_n(G)/\gamma_n(G)$ is, in fact, central in $G/\gamma_n(G)$ (see [4, 7, problem 12.22]). For metabelian groups the answer is positive. If G is metabelian then even $D_{n-1}(G)/\gamma_n(G)$ is central [1, p. 85].. Here we prove that already for solvable groups of length 3 the answer is negative. In fact we prove a more precise result.

Theorem 1. *For any $s \geq 1$ there exists a group G, with nilpotent of class 3 commutator subgroup, such that for some n the dimension quotient $D_n(G)/\gamma_n(G)$ is not contained in the sth upper central subgroup of $G/\gamma_n(G)$.*

In particular, for $s = 1$ the theorem yields that $D_n(G)/\gamma_n(G)$ is not always central. Since $G/\gamma_4(G)$ is metabelian, $D_4(G)/\gamma_4(G)$ is central for any group G. The centrality of $D_5(G)/\gamma_5(G)$ was proved in [4]. Finding the least integer n for which there exists a group G with non-central quotient $D_n(G)/\gamma_n(G)$, we leave as an open question. It would also be of interest to know whether the dimension quotients are central in case the commutator subgroup of G is nilpotent of class 2.

2. Preliminaries and reductions to embedding theorems

In [3] the first author presented a family of groups with an unusual behavior of dimension subgroups. His result, in particular, yields the following theorem.

Theorem 2. *For any $s \geq 1$ there exists a group G and an integer n such that $\gamma_{n+1}(G) = 1$ but $D_{n+s}(G) \neq 1$.*

Because of such a gap (from $n + 1$ to $n + s$) it was tempting to conjecture that $[D_{n+1}(G), G] \neq 1$. However, a more careful analysis of the group G revealed that

$$\gamma_n(G) = D_n(G) = D_{n+1}(G) = \cdots = D_{n+s}(G).$$

So actually there was no gap between $D_n(G)$ and $D_{n+s}(G)$, and even $D_n(G)$ was central. We have found a very simple proof of Theorem 2 that is based on the following general embedding theorem.

Theorem 3. *Let H be a nilpotent group of class c. Then for any n there exists a nilpotent group G of class at most nc and an embedding $\mu: H \to G$ such that $\mu(H) \subseteq \gamma_n(G)$.*

A proof of Theorem 3 will be given in the next section. Here we show how to deduce Theorem 2 from Theorem 3.

Lemma 4. *If $\varphi: H \to G$ is a homomorphism such that $\varphi(H) \subseteq \gamma_n(G)$ then $\varphi(D_m(H)) \subseteq D_{nm}(G)$.*

Proof. If $h \in D_m(G)$ then

$$h - 1 = \sum_\alpha n_\alpha(h_{\alpha 1} - 1) \cdots (h_{\alpha m} - 1) \quad (h_{\alpha\beta} \in H, n_\alpha \in \mathbb{Z}),$$

$$\varphi(h) - 1 = \sum_\alpha n_\alpha(\varphi(h_{\alpha 1} - 1)) \cdots (\varphi(h_{\alpha m} - 1)),$$

and, since $\varphi(h_{\alpha\beta}) \in \gamma_n(G)$ implies $\varphi(h_{\alpha\beta}) - 1 \in \Delta(G)^n$, we get $\varphi(h) - 1 \in \Delta(G)^{nm}$ as desired. □

Now let H be any counterexample to the dimension subgroup conjecture so that, for some $c, \gamma_{c+1}(H) = 1$ but $D_{c+1}(H) \neq 1$, and let $\mu: H \to G$ be an embedding that satisfies the conditions of Theorem 3. Then, by Lemma 4, $\mu(D_{c+1}(H)) \subseteq D_{n(c+1)}(G)$, hence $D_{n(c+1)}(G) \neq 1$ but $\gamma_{nc+1}(G) = 1$. Since the difference $n(c+1) - (nc+1) = n - 1$ can be arbitrary, Theorem 2 follows.

A similar trick allows us to deduce Theorem 1 from the following embedding theorem whose proof is also given in the next section.

Theorem 5. *Let H be a finitely generated nilpotent group of class c, let $1 \neq h_0 \in H$, and let $s \geq 1$. Then for some m (depending on h_0 and s) and any $n \geq m$ there exists a nilpotent group G of class at most nc and an embedding $\mu: H \to G$ such that $\mu(H) \subseteq \gamma_{n-m+1}(G)$ whereas $\mu(h_0) \notin Z_s(G)$, the sth upper central subgroup.*

To deduce Theorem 1 from Theorem 5 consider an arbitrary finitely generated group H such that $\gamma_{c+1}(H) = 1$ but $D_{c+1}(H) \neq 1$. Fix a natural number s and an element $h_0 \in D_{c+1}(H)$, $h_0 \neq 1$. Further choose m such that for any $n \geq m$ there exists an embedding $\mu: H \to G$ satisfying the conditions of Theorem 5. Then $\gamma_{nc+1}(G) = 1$ and (by Lemma 4)

$$D_{(n-m+1)(c+1)}(G) \supseteq \mu(D_{c+1}(H)).$$

For sufficiently large n the difference

$$(n - m + 1)(c + 1) - (nc + 1) = n - (m - 1)(c + 1) - 1$$

is non-negative, so $D_{nc+1}(G) \supseteq D_{(n-m+1)(c+1)}(G)$ and $\mu(h_0)$ is an element from $D_{nc+1}(G)$ that is not contained in $Z_s(G)$. It will be clear from the proof of Theorem 5 that if $\gamma_{c+1}(H) = 1$ then one can choose G so that $\gamma_{c+1}(G') = 1$. In the example due to Rips [5] $\gamma_4(H) = 1$ and $D_4(H) \neq 1$, so G can be chosen with $\gamma_4(G') = 1$.

Since $[D_k(G), G] \subseteq D_{k+1}(G)$, we also note that Theorem 1 is a generalization of Theorem 2.

3. Proofs of Theorems 3 and 5

Let F be the free nilpotent group of class c with free generators x_α ($\alpha \in I$). Consider n isomorphic copies $F^{(i)}$ of F ($i = 1, \dots, n$). We shall denote by $x_\alpha^{(i)}$ the free generators of $F^{(i)}$. Further, let F_n be the nilpotent (of class c) product of the groups $F^{(i)}$. Thus F_n is free nilpotent on $x_\alpha^{(i)}$ ($\alpha \in I$; $i = 1, \dots, n$). Evidently the map

$$x_\alpha^{(n)} \longrightarrow x_\alpha^{(n)}, \quad x_\alpha^{(i)} \longrightarrow x_\alpha^{(i)} x_\alpha^{(i+1)} \qquad (i < n)$$

can be extended to an automorphism of F_n. Let (x) be infinite cyclic. Define a semidirect product $\Phi_n = F_n \leftthreetimes \langle x \rangle$ assuming that conjugation by x induces on F_n the above automorphism. Hence

$$[x_\alpha^{(n)}, x] = 1, \quad [x_\alpha^{(i)}, x] = x_\alpha^{(i+1)} \qquad (i < n). \tag{1}$$

Let $m \le n$. Sending elements $x_\alpha^{(i)} \in \Phi_m$ to elements $x_\alpha^{(n-m+i)} \in \Phi_n$ we get an embedding

$$\mu_{mn} : \Phi_m \longrightarrow \Phi_n.$$

It is also clear that putting in Φ_n additional relations

$$x_\alpha^{(i)} = 1 \quad (i = m+1, \dots, n)$$

we get an epimorphism

$$\sigma_{nm} : \Phi_n \longrightarrow \Phi_m.$$

Note that Φ_n is nilpotent of class nc. To see this, let $\Phi_{n,1} = \Phi_n$ and let $\Phi_{n,k}$ ($k > 1$) be the subgroup of Φ_n generated by commutators

$$[x_{\alpha_1}^{(i_1)}, \dots, x_{\alpha_t}^{(i_t)}]$$

with $i_1 + \cdots + i_t \ge k$ (the arrangement of brackets is arbitrary). Then $\Phi_{n,k}$ ($1 \le k \le nc + 1$) is a central series of length nc. It is easily seen that there exist non-trivial commutators of weight nc, so Φ_n has class exactly nc.

Now consider a nilpotent group H of class c given as a factor group $H = F/R$ ($R \lhd F$). Denote by $R^{(k)}$ the isomorphic copy of R in $F^{(k)}$ ($k \le n$), and let $R_n^{(k)}$ be its normal closure in Φ_n. Then by definition

$$G_n^{(k)} = \Phi_n / R_n^{(k)}.$$

The maps μ_{mn} and σ_{nm} induce homomorphisms

$$\mu_{mn}^{(k)} : G_m^{(k)} \longrightarrow G_n^{(n-m+k)}, \; \sigma_{nm}^{(k)} : G_n^{(k)} \longrightarrow G_m^{(k)} \qquad (k \le m \le n).$$

Define also a homomorphism

$$\mu_n^{(k)} : H \longrightarrow G_n^{(k)}$$

to be the composition of the isomorphism $F/R \to F^{(k)}/R^{(k)}$ and the natural map $F^{(k)}/R^{(k)} \to \Phi_n/R_n^{(k)}$.

Lemma 6. *The maps $\mu_{mn}^{(k)}$ and $\mu_n^{(k)}$ are monomorphisms.*

Proof. We start with $\mu_{mn}^{(k)}$. The injection $\mu_{mn}: \Phi_m \to \Phi_n$ maps $R^{(k)}$ isomorphically onto $R^{(r)}$ where $r = n - m + k$. Let $R_n^{(r)}$ be the normal closure of $R^{(r)}$ in Φ_n and let $R_m^{(r)}$ be its normal closure in $\mu_{mn}(\Phi_m)$. Then $\mu_{mn}^{(k)}$ is injective if and only if $R_n^{(r)} \cap \mu_{mn}(F_m) = R_m^{(r)}$. Consider also the subgroup $\bar{R}_m^{(r)}$,*the normal closure of* $R_m^{(r)}$ in F_n. Since there exists a retraction $\pi: F_n \to \mu_{mn}(F_m)$, we have $\bar{R}_m^{(r)} \cap \mu_{mn}(F_m) = R_m^{(r)}$, so it suffices to prove that $\bar{R}_m^{(r)} = R_m^{(r)}$ or, equivalently, that $\bar{R}_m^{(r)} \lhd \Phi_n$. This, however, is a special case of the following elementary lemma.

Lemma 7. *Let $G = A \setminus B$ be a semidirect product of groups A and B, and let Y be a B-invariant subset of A. Then the normal closure $\bar{Y}$ of Y in A is also normal in G.*

Indeed, the elements $a^{-1}ya$ $(a \in A, y \in Y)$ generate $\bar{Y}$ as a subgroup. But the set of such elements is B-invariant because if $b \in B$ then

$$b^{-1}(a^{-1}ya)b = (b^{-1}ab)^{-1}(b^{-1}yb)(b^{-1}ab) = c^{-1}zc,$$

where $c \in A$, $z \in Y$.

Passing to $\mu_n^{(k)}$ we consider the diagram

$$\begin{array}{ccc} H & \xrightarrow{\mu_n^{(k)}} & G_n^{(k)} \\ \downarrow{\scriptstyle \mu_1^{(1)}} & & \downarrow{\scriptstyle \sigma_{nk}^{(k)}} \\ G_1^{(1)} & \xrightarrow[\mu_{1k}^{(1)}]{} & G_k^{(k)} \end{array}$$

The map $\mu_1^{(1)}$ is surely injective because $G_1^{(1)} \simeq H \times \langle x \rangle$, and we proved already that $\mu_{1k}^{(1)}$ is injective. It follows from the diagram that so is $\mu_n^{(k)}$. This finishes the proof of Lemma 6. □

In particular we have the embedding $\mu_n^{(n)}: H \to G_n^{(n)}$. It is evident from the definitions that $\mu_n^{(n)}(H) \subseteq \gamma_n(G_n^{(n)})$. Since $G_n^{(n)}$ is nilpotent of class at most nc it proves Theorem 3.

To prove Theorem 5 we use the embedding

$$\mu_n^{(n-m+1)}: H \longrightarrow G_n^{(n-m+1)}.$$

Again $G_n^{(n-m+1)}$ has class at most nc. Besides, the image of H is contained in $\gamma_{n-m+1}(G_n^{(n-m+1)})$. We are going to show that for a given $s \geq 1$ and

$h_0 \in H$ ($h_0 \neq 1$) there exists m such that $\mu_n^{(n-m+1)}(h_0) \notin Z_s(G_n^{(n-m+1)})$ ($n \geq m$). Consider the diagram

$$\begin{array}{ccc} & \overset{\mu_m^{(1)}}{\nearrow} & G_m^{(1)} \\ H & & \downarrow \mu_{mn}^{(1)} \\ & \underset{\mu_n^{(n-m+1)}}{\searrow} & G_n^{(n-m+1)} \end{array}$$

By Lemma 6 all the three maps are injective, so it is sufficient to prove that if m is sufficiently large then $\mu_m^{(1)}(h_0) \notin Z_s(G_m^{(1)})$.

Suppose first that H is a finite p-group for some prime p. Identify H with its image in $G_m^{(1)}$ and use the usual notation for Engel commutator

$$[h_0, sx] = [h_0, \underbrace{x, \ldots, x}_{s}].$$

To verify that $h_0 \notin Z_s(G_m^{(1)})$ we shall find a homomorphism of $G_m^{(1)}$ in a wreath product such that the image of this commutator is non-trivial.

We recall that the wreath product $A \operatorname{wr} B$ of groups A and B is the semidirect product $K \leftthreetimes B$ where K (the base group) is a discrete direct product of groups $A(b)$ ($b \in B$) isomorphic to A (under the isomorphism $a \to a(b)$) and B acts on K by multiplication:

$$b_2^{-1} a(b_1) b_2 = a(b_1 b_2) \quad (b_1, b_2 \in B).$$

Now fix t such that $p^t > s$ and let $\langle y \rangle$ be cyclic of order p^t. Then the wreath product $W = H \operatorname{wr} \langle y \rangle$ is a finite p-group. It follows that W is nilpotent of class, say c'. We claim that if $m \geq c'$ then the map

$$x \longrightarrow y, \qquad h \longrightarrow h(1) \quad (h \in H)$$

can be extended to a homomorphism $G_m^{(1)} \to W$. Indeed, let h_α be the images of $x_\alpha \in F$ in $H = F/R$. Define elements $h_\alpha^{(i)} \in W$ ($i = 1, \ldots, m$) by induction:

$$h_\alpha^{(1)} = h_\alpha(1), \qquad h_\alpha^{(i+1)} = [h_\alpha^{(i)}, y] \quad (1 \leq i < m).$$

The subgroup generated by elements $h_\alpha^{(i)}$ is nilpotent of class at most c, hence the map $x_\alpha^{(i)} \to h_\alpha^{(i)}$ can be extended to a homomorphism of F_m to the base group of the wreath product W. The semidirect product $\Phi_m = F_m \leftthreetimes \langle x \rangle$ is defined by relations (1) (substitute m for n). But the same relations are valid for elements y, $h_\alpha^{(i)}$ in W. For $i < m$ it follows directly from the definitions and $[h_\alpha^{(m)}, y] = 1$ just because

$$[h_\alpha^{(m)}, y] \in \gamma_{m+1}(W) \subseteq \gamma_{c'+1}(W) = 1.$$

It follows that the map $x \to y$, $x_\alpha^{(i)} \to h_\alpha^{(i)}(1)$ can be extended to a homomorphism $\Phi_m \to W$. Since $R^{(1)}$, the isomorphic copy of R in $F^{(1)}$, is contained in the kernel, we can define the induced map $G_m^{(1)} \to W$. To finish the proof of Theorem 5 in case H is a finite p-group it suffices to show $[h_0(1), sy] \neq 1$. Put $K_0 = 1$ and for $1 \leq i \leq p^t$ let K_i be the subgroup of W generated by elements $h(y^k)$ with $k < i$ ($i = 1, \ldots, p^t$). If

$a \in K_i \backslash K_{i-1}$ $(0 < i < p^t)$ then $y^{-1}ay \in K_{i+1} \backslash K_i$ and hence $[a, y] \in K_{i+1} \backslash K_i$. Since $h_0(1) \in K_1 \backslash K_0$ and $s < p^t$ it follows by induction that $[h_0(1), sy] \in K_{s+1} \backslash K_s$. In particular this element is not trivial.

Now let H be an arbitrary finitely generated nilpotent group, $h_0 \in H$, $h_0 \neq 1$. There exists a prime number p and an epimorphism $\varphi: H \to \bar{H}$ on a finite p-group $\bar{H}$ such that $\varphi(h_0) \neq 1$. If $H = F/R$ then $\bar{H} = F/\bar{R}$ where $\bar{R}$ is the preimage of $\operatorname{Ker} \varphi$ in F. Using presentations $H = F/R$ and $\bar{H} = F/\bar{R}$ we can construct the groups $G_m^{(1)}$, $\bar{G}_m^{(1)}$ and embeddings $\mu_m^{(1)}: H \to G_m^{(1)}$, $\bar{\mu}_m^{(1)}: \bar{H} \to \bar{G}_m^{(1)}$. The induced epimorphism $\varphi_m^{(1)}: G_m^{(1)} \to \bar{G}_m^{(1)}$ makes the following diagram commutative:

$$\begin{array}{ccc} H & \xrightarrow{\ \varphi\ } & \bar{H} \\ \Big\downarrow{\scriptstyle \mu_m^{(1)}} & & \Big\downarrow{\scriptstyle \bar{\mu}_m^{(1)}} \\ G_m^{(1)} & \xrightarrow[\ \varphi_m^{(1)}\ ]{} & \bar{G}_m^{(1)} \end{array}$$

As we already know, if m is sufficiently large then $\bar{\mu}_m^{(1)}(\varphi(h_0)) \notin Z_s(\bar{G}_m^{(1)})$, or equivalently $\varphi_m^{(1)}(\mu_m^{(1)}(h_0)) \notin Z_s(\bar{G}_m^{(1)})$. Since $\varphi_m^{(1)}$ is an epimorphism, $\mu_m^{(1)}(h_0) \notin Z_s(G_m^{(1)})$. This completes the proof of Theorem 5.

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