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On the characteristics of a class of Gaussian processes within the white noise space setting

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Abstract

Using the white noise space framework, we construct and study a class of Gaussian processes with stationary increments, which include as particular cases the Brownian and fractional Brownian motions. The derivative processes are computed using Hida's theory of stochastic distributions.

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1. Introduction

The theory of linear systems is a well developed field; see e.g. [36,22,30,5]. It is of importance to extend it to the stochastic context, in particular when randomness is allowed in the parameters of the system. In [7,8] the first and third authors initiated a new approach to the study of linear stochastic systems; see also the reprint [10]. These papers use Hida's white noise space theory (see [31–34]), and in particular the framework built by Elliot and van der Hoek for fractional Brownian motion (see [14,27]). The Wick product and the Kondratiev space of stochastic distributions are key tools in the arguments. Most of the results in [7,8,10] pertain to the discrete-time case. In order to develop the theory of stochastic continuous-time systems, and in particular to

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study linear system problems such as Kalman filtering, system identification and adaptive control, it is essential to build within the white noise space setting a wide family of stochastic processes, which drive or perturb the underlying system. The present paper addresses this question. As noted above, we plan to use the results developed here to study, within the white noise space setting, some classical system and control problems. In this context, it is worth mentioning [1] as a related study of linear systems driven by the fractional Brownian motion.

Here, using the white noise framework, we study zero-mean Gaussian processes with stationary increments, that is, with covariance functions of the form

$$K(t,s) = \int_{\mathbb{R}} \frac{e^{itu}-1}{u}\frac{e^{-isu}-1}{u} d\sigma(u), \tag{1.1}$$

where $d\sigma$ is a positive measure on $\mathbb{R}$ defined by an increasing right continuous function σ, such that

$$\int_{\mathbb{R}} \frac{d\sigma(u)}{u^2+1} < \infty. \tag{1.2}$$

The measure $d\nu(u) = d\sigma(u)/u^2$ is called the spectral measure (see [44, p. 25]) or the control measure. Setting

$$r(t) = -\int_{\mathbb{R}} \left\{ e^{itu} - 1 - \frac{itu}{u^2+1} \right\} \frac{d\sigma(u)}{u^2}, \tag{1.3}$$

(1.1) can be rewritten as

$$K_r(t,s) = r(t) + r(s)^* - r(t-s), \quad t,s \in \mathbb{R}. \tag{1.4}$$

Continuous functions r of a real variable, such that

$$r(-t) = r(t)^*, \quad t \in \mathbb{R},$$

and such that the kernel $K_r(t,s)$ is positive (in the sense of reproducing kernels) on the real line, have been characterized by von Neumann and Schoenberg for when r is real; see [47, Theorem 1, p. 229]. For the case of complex-valued functions, see [38,37], and [2, pp. 267–269] and the references therein. These are functions of the form

$$r(t) = r_0 + i\gamma t - \int_{\mathbb{R}} \left\{ e^{itu} - 1 - \frac{itu}{u^2+1} \right\} \frac{d\sigma(u)}{u^2}, \tag{1.5}$$

where $r_0 = r(0)$ and γ are real numbers. That the form (1.5) is sufficient to insure the positivity of the kernel $K_r(t,s)$ follows from the easily obtained formula

$$K_r(t,s) = \int_{\mathbb{R}} \frac{e^{itu}-1}{u}\frac{e^{-isu}-1}{u} d\sigma(u); \tag{1.6}$$

see for instance [39, Theorem 4, p. 115]. The idea of the proof of the converse is given in the next section. As we will recall in the sequel, such functions r have been investigated for a long time. Still, their applications in stochastic calculus appear to have been only partially developed. We mention in particular the recent work [41, p. 103]. In that work the notion of processes with covariance measure is introduced, and stochastic processes with covariance function of the form K_r are shown to belong to this class. We note however that the methods of [41] and of the present paper are completely different.

Note that

$$K_r(t,s) = K_{r-r(0)}(t,s),$$

and therefore one can always assume that $r(0) = 0$.

In the real-valued case, with $r(0) = 0$, the function r takes the form

$$r(t) = \int_{\mathbb{R}} \frac{1-\cos(tu)}{u^2} \mathrm{d}\sigma(u).$$

We will assume that $\mathrm{d}\sigma$ is absolutely continuous with respect the Lebesgue measure, and that its derivative, the spectral density, satisfies bounds of the form

$$m(u) \le \begin{cases} K\,|u|^{-b} & \text{if } |u| \le 1, \\ K|u|^{2N} & \text{if } |u| > 1, \end{cases} \tag{1.7}$$

where $b < 2$, $N \in \mathbb{N}_0$, and $0 < K < \infty$.

The fractional Brownian motion then corresponds to the choice

$$m(u) = \frac{1}{2\pi}|u|^{1-2H}, \quad H \in (0,1),$$

giving

$$r(t) = \frac{V_H}{2}|t|^{2H}, \quad \text{with } V_H = \frac{\Gamma(2-2H)\cos(\pi H)}{\pi(1-2H)H}, \tag{1.8}$$

and

$$K_r(t,s) \stackrel{\text{def.}}{=} k_H(t,s) = \frac{V_H}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}),$$

with Γ denoting the Gamma function, as can be seen using the formulas

$$\int_{\mathbb{R}} \frac{1-\cos(tu)}{u^{2H+1}} = -2|t|^{2H}\cos(\pi H)\Gamma(-2H)$$
$$\int_{\mathbb{R}} \frac{1-\cos(tu)}{u^2} = \pi|t|.$$

When furthermore $H = 1/2$, then $V_H = 1$, $r(t) = |t|/2$ and for $t, s \ge 0$, $K_r(t,s) = \min(t,s)$.

By a theorem of Kolmogorov there exists a Gaussian stochastic process $\{B_H(t)\}$ indexed by $\mathbb{R}$, which is called the fractional Brownian motion (with Hurst parameter $H \in (0,1)$), such that

$$k_H(t,s) = E(B_H(t)B_H(s)), \quad t,s \in \mathbb{R}.$$

Stochastic calculus for B_H has been developed for quite some time; see for instance [19,23,26, 12]. In a subsequent paper we show how most of the results of these works are extended to the case of general covariance functions of the form $K_r(t,s)$.

We mention also that functions r of the form (1.5) appear first in the work of Paul Lévy [43], in the result characterizing characteristic functions of infinitely divisible laws. More precisely, the characteristic function of a random variable is infinitely divisible if and only if it is of the form $\exp r(t)$, where $r(t)$ is of the form (1.5); see [45, Representation Theorem]. Similarly, $(Z_u)_{u\ge 0}$ is an infinitely divisible random process if and only if

$$E\left(\mathrm{e}^{\mathrm{i}tZ_u}\right) = \mathrm{e}^{-ur(t)},$$

where r is of the form (1.5), with $r(0) = 0, \gamma = 0$, and is called the *characteristic exponent of the Lévy process*. See [13, p. 12], [43, formula (9), p. 353]. This is the Lévy–Khintchine formula; see [43, formula (9), p. 353]. We also mention that positive kernels of the form K_r appear in the theory of Dirichlet spaces; see [20, p. 5–12]. These aspects of the theory of kernels of the form K_r will not be pursued in the present paper.

The paper consists of seven sections including the introduction, and its outline is as follows. In Section 2 we study and characterize the reproducing kernel Hilbert space associated with a kernel $K_r(t, s)$. In Section 3 we associate with certain kernels $K_r(t, s)$ an operator which will play a key role in the construction of the stochastic process with covariance $K_r(t, s)$. This paper uses Hida's white noise space theory, and in Section 4 we review the main features from white noise space theory which we will subsequently use. In Section 5 we recall the definition of the Wick product and of the Kondratiev space. Stochastic processes with covariance function $K_r(t, s)$ are built in Section 6, and their derivatives are studied in Section 7.

Some of the results presented here have been announced in the note [9]. The results of the present work are to be used in a subsequent paper where stochastic analysis associated with the processes considered here is developed.

Finally a word on notation. We denote the Fourier transform by

$$\hat{f}(u) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}ux} f(x)\mathrm{d}x. \tag{1.9}$$

The inverse transform is then given by

$$\overset{\vee}{f}(u) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}ux} f(x)\mathrm{d}x. \tag{1.10}$$

The same notation is used for the Fourier transform and inverse Fourier transform of distributions.

We set

$$\mathbb{N} = \{1, 2, 3, \ldots\} \quad \text{and} \quad \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

and denote by ℓ the set of sequences

$$(\alpha_1, \alpha_2, \ldots), \tag{1.11}$$

indexed by $\mathbb{N}$ with values in $\mathbb{N}_0$, for which only a finite number of elements $\alpha_j \neq 0$.

2. Some remarks on the kernels $K_r(t, s)$

As already mentioned, real-valued functions r for which the kernel $K_r(t, s)$ is positive on the real line (in the sense of reproducing kernels) were characterized by Schoenberg and von Neumann. For complex-valued functions, and by different methods, the following theorem has been given by Krein in 1944; see [39, Theorem 2, p. 256]. See also [2, Section 9, p. 268]. In Krein's result, the case $a = \infty$ is allowed, and then, $t, s \in \mathbb{R}$.

Theorem 2.1 ([39, Theorem 2, p. 256]). *The kernel $K_r(t, s) = r(t) + r(s)^* - r(t - s) - r(0)$ is positive for $t, s \in [-a, a]$ if and only if r is of the form (1.5):*

$$r(t) = r(0) + \mathrm{i}\gamma t - \int_{\mathbb{R}} \left\{ \mathrm{e}^{\mathrm{i}tu} - 1 - \frac{\mathrm{i}tu}{u^2 + 1} \right\} \frac{\mathrm{d}\sigma(u)}{u^2}, \quad t \in [-a, a].$$

For completeness, let us recall that Akhiezer's proof [2, pp. 268–270] goes along the following lines; one first shows that r satisfies an inequality of the form

$$|r(t)| \leq M(1+|t|^3) \tag{2.1}$$

for some positive number M (we recall the proof of this inequality in the sequel; see Lemma 2.4). One then shows that the function $H(z) = z^2 \int_0^\infty r(t)^* e^{itz} dt$, which, in view of (2.1), is analytic in the open upper half-plane, satisfies (see [2, (2), p. 268] and [40, p. 227])

$$\frac{H(z) - H(w)^*}{z - w^*} = zw^* \iint_{\mathbb{R}_+^2} K_r(t,s) e^{itz} e^{-isw^*} dt ds,$$

and in particular has a positive imaginary part in the open upper half-plane. To conclude the proof, one uses Herglotz's representation formula for analytic functions with a positive imaginary part in the open upper half-plane (see for instance [17, Theorem 4.7, p. 25], [3, Theorem 2, p. 220]).

Formula (1.6) allows us to characterize the reproducing kernel Hilbert space associated with K_r in terms of de Branges spaces. See Theorem 2.3. We first make the following remarks: let χ_t denote the function of a real variable u

$$\chi_t(u) = \frac{e^{itu} - 1}{iu}, \quad t \in \mathbb{R},$$

and let $\mathscr{M}_T$ denote the closed linear span in $\mathbf{L}_2(d\sigma)$ of the χ_t for $|t| \leq T$. Assume that $\mathscr{M}_T \neq \mathbf{L}_2(d\sigma)$. Then, $\mathscr{M}_T$ is a reproducing kernel Hilbert space with reproducing kernel of the form

$$\frac{A(T,\lambda)A(T,\omega)^* - B(T,\lambda)B(T,\omega)^*}{-i(\lambda - \omega^*)}$$

where $A(T,\lambda)$, $B(T,\lambda)$ are entire functions of finite exponential type. See [16,25] and [6, Theorem 3.1, p. 600].

Remark 2.2. The spaces $\mathscr{M}_T$ were introduced by de Branges and play a key role in prediction theory. See [16,25,24]. When $d\sigma(u) = du$ we have

$$\mathscr{M}_T = \mathbf{H}_2 \ominus e_T \mathbf{H}_2,$$

where $\mathbf{H}_2$ denotes the Hardy space of the open half-plane, and where $e_T(z) = e^{izT}$, $B(T,\cdot) = 1$ and $A(T,\cdot) = \chi_T$, with the reproducing kernel given by

$$\frac{1 - e_T(\lambda) e_T(\omega^*)^*}{-i(\lambda - \omega^*)}.$$

Let $\mathscr{S}(\mathbb{R})$ denote the Schwartz space of rapidly decreasing functions, and let $\mathscr{S}'(\mathbb{R})$ denote the topological dual of $\mathscr{S}(\mathbb{R})$, that is, the space of tempered distributions. In view of the next two results, we recall the following: condition (1.2) insures that the measure $d\sigma$ has a Fourier transform $\widehat{d\sigma}$ which is a tempered distribution. Furthermore, this Fourier transform induces a distribution $\widehat{d\sigma}(t-s)$ on the Schwartz space $\mathscr{S}(\mathbb{R}^2)$ of functions of two real variables via the formula

$$\langle \widehat{d\sigma}(t-s), \phi(t,s) \rangle = \int_{\mathbb{R}} \left(\iint_{\mathbb{R}^2} e^{-i(t-s)u} \phi(t,s) dt ds \right) d\sigma(u). \tag{2.2}$$

When $\int_{\mathbb{R}} \mathrm{d}\sigma(u) < \infty$, we have that

$$\begin{aligned}\langle \widehat{\mathrm{d}\sigma}(t-s), \phi(t,s)\rangle &= \iint_{\mathbb{R}^2} \left\{ \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}(t-s)u} \mathrm{d}\sigma(u) \right\} \phi(t,s) \mathrm{d}t \mathrm{d}s \\ &= \iint_{\mathbb{R}^2} \widehat{\mathrm{d}\sigma}(t-s)\phi(t,s)\mathrm{d}t\mathrm{d}s.\end{aligned}$$

Theorem 2.3. *Let $T < \infty$. The reproducing kernel Hilbert space $\mathcal{H}_T(K_r)$ associated with $K_r(t,s)$ for $t, s \in [-T, T]$ consists of functions of the form*

$$F(t) = \int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{i}tu} - 1}{iu} f(u) \mathrm{d}\sigma(u), \quad t \in [-T, T],\ f \in \mathscr{M}_T \tag{2.3}$$

with norm

$$\|F\|_{\mathcal{H}_T(K_r)} = \|f\|_{\mathbf{L}_2(\mathrm{d}\sigma)}. \tag{2.4}$$

Moreover, by extending $F(t)$ to the real line via formula (2.3), F defines a tempered distribution and

$$F'(t) = 2\pi \ \overset{\vee}{f\mathrm{d}\sigma}\ (t)$$

in the sense of distributions, and where $\overset{\vee}{f\mathrm{d}\sigma}$ denotes the inverse Fourier transform of the tempered distribution defined by $f\mathrm{d}\sigma$.

Proof. The first claim follows from the representation of the kernel K_r as an inner product, and we focus on the second claim. We note that the function $F(t)$ extends to a continuous function on $\mathbb{R}$. Indeed, for every $t, h \in \mathbb{R}$ we have

$$\left| \frac{\mathrm{e}^{\mathrm{i}(t+h)u} - \mathrm{e}^{\mathrm{i}tu}}{iu} \right| = \left| \frac{\mathrm{e}^{\mathrm{i}hu} - 1}{iu} \right| \leq \begin{cases} |h| & \text{if } |u| \leq 1, \\ \dfrac{2}{|u|} & \text{if } |u| > 1. \end{cases} \tag{2.5}$$

Using these inequalities, we utilize the dominated convergence theorem to conclude that

$$\lim_{h \to 0} F(t+h) = F(t), \quad t \in \mathbb{R}.$$

Furthermore, (2.5) leads to the bound

$$|F(t)| \leq |t| \int_{-1}^{1} |f(u)| \mathrm{d}\sigma(u) + 2 \int_{|u| \geq 1} \left| \frac{f(u)}{u} \right| \mathrm{d}\sigma(u).$$

We recall that σ is assumed right continuous. When it has a jump at 0, we define

$$F(t) = tf(0)(\sigma(0) - \sigma(0_-)) + \int_{\mathbb{R}} f(u)\chi_s(u)\mathrm{d}\sigma_1(u),$$

where $\mathrm{d}\sigma_1$ has no jump at 0. The function F is in particular slowly growing, and therefore defines a tempered distribution (see [49, Théorème VI, p. 239], [11, Section 4, p. 110]).

Let $\varphi \in \mathscr{S}(\mathbb{R})$. The integral

$$\int_{\mathbb{R}} f(u)\varphi(u)\mathrm{d}\sigma(u) = \int_{\mathbb{R}} \frac{f(u)}{u + \mathrm{i}} \left((u + \mathrm{i})\varphi(u)\right) \mathrm{d}\sigma(u)$$

exists, since $(u+\mathrm{i})\varphi(u)$ is bounded and since $1/(u+\mathrm{i}) \in \mathbf{L}_2(\mathrm{d}\sigma)$. Thus,

$$\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\left|\frac{\mathrm{e}^{\mathrm{i}tu}-1}{iu}f(u)\varphi'(t)\right|\mathrm{d}\sigma(u)\right)\mathrm{d}t \le \int_{|u|\le 1}|f(u)|\,\mathrm{d}\sigma(u)\int_{\mathbb{R}}\left|t\varphi'(t)\right|\mathrm{d}t + 2\int_{|u|>1}\left|\frac{f(u)}{u}\right|\mathrm{d}\sigma(u)\int_{\mathbb{R}}\left|\varphi'(t)\right|\mathrm{d}t < \infty.$$

Using Fubini's theorem, we have

$$\int_{\mathbb{R}}\left\{\int_{\mathbb{R}}\frac{\mathrm{e}^{\mathrm{i}tu}-1}{iu}f(u)\mathrm{d}\sigma(u)\right\}\varphi'(t)\mathrm{d}t = \int_{\mathbb{R}}\left\{\int_{\mathbb{R}}\frac{\mathrm{e}^{\mathrm{i}tu}-1}{iu}\varphi'(t)\mathrm{d}t\right\}f(u)\mathrm{d}\sigma(u),$$

and by integration by parts, we obtain

$$\begin{aligned}\int_{\mathbb{R}}\left\{\int_{\mathbb{R}}\frac{\mathrm{e}^{\mathrm{i}tu}-1}{iu}\varphi'(t)\mathrm{d}t\right\}f(u)\mathrm{d}\sigma(u) &= -\int_{\mathbb{R}}\left\{\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}tu}\varphi(t)\mathrm{d}t\right\}f(u)\mathrm{d}\sigma(u)\\ &= -2\pi\int_{\mathbb{R}}\overset{\vee}{\varphi}(u)f(u)\mathrm{d}\sigma(u).\end{aligned}$$

Thus,

$$\int_{\mathbb{R}}F(t)\varphi'(t)\mathrm{d}t = -2\pi\int_{\mathbb{R}}\overset{\vee}{\varphi}(u)f(u)\mathrm{d}\sigma(u),$$

and therefore we obtain on one hand,

$$\langle F,\varphi'\rangle = -2\pi\left\langle f\mathrm{d}\sigma,\overset{\vee}{\varphi}\right\rangle = -2\pi\left\langle \overset{\vee}{f\mathrm{d}\sigma},\varphi\right\rangle,$$

while on the other hand,

$$\langle F,\varphi'\rangle = -\langle F',\varphi\rangle.$$

Thus, $F' = 2\pi\ \overset{\vee}{f\mathrm{d}\sigma}$. □

In preparation for the proof of Theorem 2.5 we now prove inequality (2.1).

Lemma 2.4. *Assume the kernel $K_r(t,s)$ to be positive in $\mathbb{R}$. Then (2.1) is in force, that is*

$$|r(t)| \le M(1+|t|^3)$$

for some positive number M.

Proof. We follow the arguments in [2, pp. 264–265], with slight modifications. We first note that we may assume that $r(0)=0$. The positivity of the kernel $K_r(t,s)$ implies that the matrix

$$\begin{pmatrix} K_r(t,t) & K_r(t,-t)\\ K_r(-t,t) & K_r(-t,-t)\end{pmatrix}$$

has a non-negative determinant. Therefore,

$$|2r(t)-r(2t)| \le |2\mathrm{Re}\ r(t)|,$$

and thus

$$|r(2t)| \le 4|r(t)|. \tag{2.6}$$

Let

$$R(t) = \frac{|r(t)|}{1 + |t|^3}.$$

Then, (2.6) implies that

$$R(2t) \le \frac{4(1 + |t|^3)}{1 + 8\,|t|^3} R(t). \tag{2.7}$$

Let $T_0 \in \mathbb{R}_+$ be such that

$$|t| \ge T_0 \Longrightarrow \frac{4(1 + |t|^3)}{1 + 8\,|t|^3} \le 1.$$

It follows from (2.7) that $R(t)$ is bounded in $\mathbb{R}$ by an expression of the form $M(1+|t|^3)$ for some $M > 0$. In fact, since r is continuous, one may take

$$M = \max_{t \in [0, T_0]} |r(t)|. \quad \square$$

Theorem 2.5. *It holds that*

$$\frac{\partial^2}{\partial t \partial s} K_r(t, s) = r''(t - s) = \widehat{\mathrm{d}\sigma}(s - t)$$

in the sense of distributions. Furthermore, for $\varphi \in \mathscr{S}(\mathbb{R})$, $\overset{\vee}{\mathrm{d}\sigma} * \varphi$ is a function and it holds that for $\varphi \in \mathscr{S}(\mathbb{R})$

$$\left\langle \mathrm{d}\sigma \widehat{\varphi}, (\widehat{\varphi})^* \right\rangle_{\mathscr{S}'(\mathbb{R}), \mathscr{S}(\mathbb{R})} = \int_{\mathbb{R}} \varphi(-u)^* (\overset{\vee}{\mathrm{d}\sigma} * \varphi)(u) \mathrm{d}u. \tag{2.8}$$

Before the proof, we make the following observation. We note that the right hand side of (2.8) can formally be rewritten as

$$\int_{\mathbb{R}} \varphi(-u)^* \left(\int_{\mathbb{R}} \overset{\vee}{\mathrm{d}\sigma}\ (u - v) \varphi(v) \mathrm{d}v \right) \mathrm{d}u$$

where in general

$$\int_{\mathbb{R}} \overset{\vee}{\mathrm{d}\sigma}\ (u - v) \varphi(v) \mathrm{d}v$$

is not a real integral, but an abuse of notation.

In the proof of the theorem, use is made of properties of the space $\mathcal{O}_M$ of multiplication operators in $\mathscr{S}(\mathbb{R})$ (also called C^∞ functions slowly decreasing at infinity; see [50, p. 275]), and of the space $\mathcal{O}'_C$ of distributions rapidly decreasing at infinity (see [50, p. 315]). Recall [50, Theorem 30.3, p. 318] that the Fourier transform is one-to-one from $\mathcal{O}_M$ onto $\mathcal{O}'_C$ and from $\mathcal{O}'_C$ onto $\mathcal{O}_M$.

Proof of Theorem 2.5. Since $\mathrm{d}\sigma$ defines a distribution in $\mathscr{S}'(\mathbb{R})$, we have that $\overset{\vee}{\mathrm{d}\sigma} \in \mathscr{S}'(\mathbb{R})$; see for instance [50, Theorem 25.6, p. 276]. Let $\varphi \in \mathscr{S}(\mathbb{R})$. The convolution $\overset{\vee}{\mathrm{d}\sigma} * \varphi$ is a function, and belongs to $\mathcal{O}_M$; see [49, p. 248]. So, by [50, Theorem 30.3, p. 318]

$$\widehat{(\overset{\vee}{\mathrm{d}\sigma} * \varphi)} \in \mathcal{O}'_C. \tag{2.9}$$

We now compute (2.9) using [50, Theorem 30.4, p. 319] with (in the notation of that book) $S = \overset{\vee}{\mathrm{d}\sigma} \in \mathscr{S}'(\mathbb{R})$ and $T = \varphi \in \mathscr{S}(\mathbb{R}) \subset \mathcal{O}'_C$ (see [50, Example 30.1, p. 315] for the latter inclusion), to obtain

$$\widehat{(\overset{\vee}{\mathrm{d}\sigma} * \varphi)} = \mathrm{d}\sigma\widehat{\varphi},$$

which is a measure. Using the fact that $\mathcal{O}'_C \subset \mathscr{S}'(\mathbb{R})$ (see [50, p. 318], [49]), we have

$$\left\langle \mathrm{d}\sigma\widehat{\varphi}, \widehat{\varphi}^* \right\rangle_{\mathscr{S}'(\mathbb{R}),\mathscr{S}(\mathbb{R})} = \int_{\mathbb{R}} |\widehat{\varphi}|^2 \,\mathrm{d}\sigma. \tag{2.10}$$

But for $\psi \in \mathscr{S}'(\mathbb{R})$ and $\varphi \in \mathscr{S}(\mathbb{R})$ we have

$$\left\langle \widehat{\psi}, \varphi \right\rangle_{\mathscr{S}'(\mathbb{R}),\mathscr{S}(\mathbb{R})} = \langle \psi, \widehat{\varphi} \rangle_{\mathscr{S}'(\mathbb{R}),\mathscr{S}(\mathbb{R})}.$$

Let $\phi \in \mathscr{S}(\mathbb{R})$. Then the function

$$\psi : u \mapsto \varphi(-u)^*$$

is also in $\mathscr{S}(\mathbb{R})$ and we have

$$\begin{aligned}
\int_{\mathbb{R}} \phi(-u)^*(\overset{\vee}{\mathrm{d}\sigma} * \varphi)(u)\mathrm{d}u &= \left\langle \overset{\vee}{\mathrm{d}\sigma} * \varphi, \psi \right\rangle_{\mathscr{S}'(\mathbb{R}),\mathscr{S}(\mathbb{R})} \\
&= \left\langle \mathrm{d}\sigma\widehat{\varphi}, \widehat{\psi} \right\rangle_{\mathscr{S}'(\mathbb{R}),\mathscr{S}(\mathbb{R})} \\
&= \left\langle \mathrm{d}\sigma\widehat{\varphi}, \widehat{\varphi}^* \right\rangle_{\mathscr{S}'(\mathbb{R}),\mathscr{S}(\mathbb{R})},
\end{aligned}$$

since $\widehat{\psi} = (\widehat{\varphi})^*$. Thus, (2.10) can be rewritten as

$$\left\langle \mathrm{d}\sigma\widehat{\varphi}, \widehat{\varphi}^* \right\rangle_{\mathscr{S}'(\mathbb{R}),\mathscr{S}(\mathbb{R})} = \int_{\mathbb{R}} \varphi(-u)^*(\overset{\vee}{\mathrm{d}\sigma} * \varphi)(u)\mathrm{d}u.$$

Now let $\phi \in \mathscr{S}(\mathbb{R}^2)$: we have

$$\begin{aligned}
\left\langle \frac{\partial^2}{\partial t \partial s} K_r, \phi \right\rangle &= \left\langle K_r, \frac{\partial^2}{\partial t \partial s}\phi \right\rangle \\
&= \iint_{\mathbb{R}^2} K_r(t,s) \frac{\partial^2}{\partial t \partial s}\phi(t,s)\mathrm{d}t\mathrm{d}s \\
&= \iint_{\mathbb{R}^2} r(t) \frac{\partial^2}{\partial t \partial s}\phi(t,s)\mathrm{d}t\mathrm{d}s + \iint_{\mathbb{R}^2} r(s)^* \frac{\partial^2}{\partial t \partial s}\phi(t,s)\mathrm{d}t\mathrm{d}s \\
&\quad - \iint_{\mathbb{R}^2} r(t-s) \frac{\partial^2}{\partial t \partial s}\phi(t,s)\mathrm{d}t\mathrm{d}s.
\end{aligned}$$

Since r satisfies inequality (2.1), the above integrals make sense; the first and the second integrals on the right hand side are identically zero since ϕ is a Schwartz function. Thus, since

$$\begin{aligned}
\left\langle r''(t-s), \phi(t,s) \right\rangle &= -\left\langle r(t-s), \frac{\partial^2}{\partial t \partial s}\phi \right\rangle \\
&= -\iint_{\mathbb{R}^2} r(t-s) \frac{\partial^2}{\partial t \partial s}\phi(t,s)\mathrm{d}t\mathrm{d}s,
\end{aligned}$$

it follows that

$$\left\langle \frac{\partial^2}{\partial t \partial s} K_r, \phi \right\rangle = \left\langle r''(t-s), \phi \right\rangle.$$

Using (1.5), that is

$$r(t-s) = i\gamma(t-s) - \int_{\mathbb{R}} \left\{ \mathrm{e}^{\mathrm{i}(t-s)u} - 1 - \frac{i(t-s)u}{u^2+1} \right\} \frac{\mathrm{d}\sigma(u)}{u^2},$$

we get

$$\begin{aligned} \iint_{\mathbb{R}^2} r(t-s) \frac{\partial^2}{\partial t \partial s} \phi(t,s) \mathrm{d}t \mathrm{d}s &= \iint_{\mathbb{R}^2} i\gamma(t-s) \frac{\partial^2}{\partial t \partial s} \phi(t,s) \mathrm{d}t \mathrm{d}s \\ &- \iint_{\mathbb{R}^2} \left\{ \int_{\mathbb{R}} \left\{ \mathrm{e}^{\mathrm{i}(t-s)u} - 1 - \frac{i(t-s)u}{u^2+1} \right\} \frac{\mathrm{d}\sigma(u)}{u^2} \right\} \frac{\partial^2}{\partial t \partial s} \phi(t,s) \mathrm{d}t \mathrm{d}s. \end{aligned}$$

The first integral on the right hand side vanishes since ϕ is a Schwartz function. The function

$$K(t,s,u) = \begin{cases} \left\{ \mathrm{e}^{\mathrm{i}(t-s)} - 1 - \dfrac{i(t-s)u}{u^2+1} \right\} \dfrac{1}{u^2}, & \text{if } u \neq 0, \\ -\dfrac{(t-s)^2}{2}, & \text{if } u = 0, \end{cases}$$

is continuous since

$$\lim_{u \to 0} \left\{ \mathrm{e}^{\mathrm{i}(t-s)} - 1 - \frac{i(t-s)u}{u^2+1} \right\} \frac{1}{u^2} = -\frac{(t-s)^2}{2}.$$

Moreover, we have the bounds

$$|K(t,s,u)| \leq \begin{cases} \dfrac{(t-s)^2 + |t-s|}{u^2+1} & \text{if } |u| < 1, \\ \dfrac{4 + |t-s|}{u^2+1} & \text{if } |u| \geq 1. \end{cases}$$

Therefore,

$$\begin{aligned} &\iint_{\mathbb{R}^2} \int_{\mathbb{R}} \left| K(t,s,u) \frac{\partial^2}{\partial t \partial s} \phi(t,s) \right| \mathrm{d}\sigma(u) \mathrm{d}t \mathrm{d}s \\ &\leq \iint_{\mathbb{R}^2} \left| \frac{\partial^2}{\partial t \partial s} \phi(t,s) \right| \left\{ \int_{\mathbb{R}} |K(t,s,u)| \, \mathrm{d}\sigma(u) \right\} \mathrm{d}t \mathrm{d}s \\ &\leq \iint_{\mathbb{R}^2} \left| \frac{\partial^2}{\partial t \partial s} \phi(t,s) \right| \left\{ \int_{|u|<1} \frac{(t-s)^2 + |t-s|}{u^2+1} \mathrm{d}\sigma(u) \right\} \mathrm{d}t \mathrm{d}s \\ &\quad + \iint_{\mathbb{R}^2} \left| \frac{\partial^2}{\partial t \partial s} \phi(t,s) \right| \left\{ \int_{|u| \geq 1} \frac{4 + |t-s|}{u^2+1} \mathrm{d}\sigma(u) \right\} \mathrm{d}t \mathrm{d}s \\ &< K \iint_{\mathbb{R}^2} \left| \frac{\partial^2}{\partial t \partial s} \phi(t,s) (|t-s|+2)^2 \right| \mathrm{d}t \mathrm{d}s < \infty \end{aligned}$$

where

$$K = \int_{\mathbb{R}} \frac{\mathrm{d}\sigma(u)}{u^2+1} < \infty.$$

By Fubini's theorem and integration by parts we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2}\left\{\int_{\mathbb{R}}\left\{\mathrm{e}^{\mathrm{i}(t-s)u}-1-\frac{i(t-s)u}{u^2+1}\right\}\frac{\mathrm{d}\sigma(u)}{u^2}\right\}\frac{\partial^2}{\partial t\partial s}\phi(t,s)\mathrm{d}t\mathrm{d}s\\
&=\int_{\mathbb{R}}\left\{\int_{\mathbb{R}^2}\left\{\mathrm{e}^{\mathrm{i}(t-s)u}-1-\frac{i(t-s)u}{u^2+1}\right\}\frac{1}{u^2}\frac{\partial^2}{\partial t\partial s}\phi(t,s)\mathrm{d}t\mathrm{d}s\right\}\mathrm{d}\sigma(u)\\
&=\int_{\mathbb{R}}\left\{\int_{\mathbb{R}^2}\mathrm{e}^{\mathrm{i}(t-s)u}\phi(t,s)\mathrm{d}t\mathrm{d}s\right\}\mathrm{d}\sigma(u)\\
&=\int_{\mathbb{R}^2}\left\{\int_{\mathbb{R}}\mathrm{e}^{-\mathrm{i}(s-t)u}\mathrm{d}\sigma(u)\right\}\phi(t,s)\mathrm{d}t\mathrm{d}s,
\end{aligned}$$

and by (2.2) we conclude that

$$\left\langle r''(t-s),\phi(t,s)\right\rangle=\left\langle\widehat{\mathrm{d}\sigma}(s-t),\phi(t,s)\right\rangle.$$

Thus,

$$\frac{\partial^2}{\partial t\partial s}K_r(t,s)=r''(t-s)=\widehat{\mathrm{d}\sigma}(s-t).\quad\square$$

3. The operator T_m

We now focus on the case $\mathrm{d}\sigma(u)=m(u)\mathrm{d}u$ in (1.5), where m is a positive and measurable function such that

$$\int_{\mathbb{R}}\frac{m(u)\mathrm{d}u}{u^2+1}<\infty. \tag{3.1}$$

We define an (unbounded in general) operator T_m by

$$\widehat{T_m f}(u)\overset{\text{def.}}{=}\sqrt{m(u)}\widehat{f}(u), \tag{3.2}$$

where $\widehat{f}$ denotes the Fourier transform of f; see (1.9). The domain of T_m,

$$\operatorname{dom}(T_m)\overset{\text{def.}}{=}\left\{f\in\mathbf{L}_2(\mathbb{R}):\int_{\mathbb{R}}m(u)\left|\widehat{f}(u)\right|^2\mathrm{d}u<\infty\right\}, \tag{3.3}$$

contains in particular the Schwartz space $\mathscr{S}(\mathbb{R})$ since m satisfies (3.1) and since the Fourier transform maps $\mathscr{S}(\mathbb{R})$ into itself.

When m is summable, the integral in (3.3) can be rewritten as a double integral as explained in the previous section:

$$\int_{\mathbb{R}}m(u)|\widehat{f}(u)|^2\mathrm{d}u=\iint_{\mathbb{R}^2}f(t)f(s)^*\widehat{m}(t-s)\mathrm{d}t\mathrm{d}s. \tag{3.4}$$

When

$$m(u)=\frac{1}{2\pi}|u|^{1-2H}, \tag{3.5}$$

the operator T_m reduces, up to a multiplicative constant, to the operator M_H defined in [27, (2.10), p. 304] and in [14, Definition 3.1, p. 354], and the function $r(t)$ in (1.5) is given by (1.8).

We note that the set (3.3) has been introduced in [48, Theorem 3.1, p. 258] for m of the form (3.5). Multiplying (3.5) by

$$\frac{2\pi H(1-2H)}{\Gamma(2-2H)\cos(\pi H)},$$

that is, considering

$$m(u) = \frac{H(1-2H)}{\Gamma(2-2H)\cos(\pi H)}\,|u|^{1-2H}$$

and using [28, Formula 12, p. 170], that is, in the sense of distributions,

$$\int_{\mathbb{R}} |x|^{\lambda} \mathrm{e}^{\mathrm{i}ux}\mathrm{d}x = -2\Gamma(1+\lambda)\sin\left(\frac{\pi\lambda}{2}\right)|u|^{-\lambda-1}, \quad \lambda \notin \mathbb{Z},$$

leads to (with $\lambda = 1-2H$)

$$\begin{aligned}\widehat{m}(u) &= -\frac{H(1-2H)\Gamma(2-2H)\sin\left(\frac{\pi(1-2H)}{2}\right)}{\Gamma(2-2H)\cos(\pi H)}\,|u|^{2H-2}\\ &= 2H(2H-1)\,|u|^{2H-2}.\end{aligned}$$

See [23, (2.1), p. 584]. The norm $|f|_{\phi}^{2}$ defined in [23, (2.2), p. 584] is then equal to (3.4).

More generally, the operator T_m can be defined for m which do not satisfy (3.1). One could for instance assume only that

$$\int_{\mathbb{R}} \frac{m(u)\mathrm{d}u}{(u^2+1)^{N+1}} < \infty \tag{3.6}$$

for some $N \in \mathbb{N}_0$. Then, the Schwartz space is still in the domain of T_m, but, in general, one cannot define an associated function r via (1.5). We will go back to this condition in the sequel; see (1.7). A key property of T_m proved in the sequel uses specifically (3.1); see Lemma 3.1. We note that (3.6) means that the measure $m(u)\mathrm{d}u$ is *slowly increasing*, and can be seen as the spectral measure of a generalized stochastic process with N-th stationary increments; see [29, Théorème 1, p. 257]. We mention that generalized stochastic process with N-th stationary increments are studied in particular in [46]. The point of view there is that of infinite dimensional analysis and is different from that of white noise space analysis. In particular, one considers Wick polynomials, built from the Hermite polynomials (see [46, pp. 8–10]) rather than functions defined in terms of the Hermite functions. We also refer the reader to [18] for more on this approach.

The approach developed in the present paper can be used to develop a stochastic calculus for processes with N-th stationary increments. This question will not be pursued here.

The operator T_m plays a central role in this work. We now study its main properties:

Lemma 3.1. *Assume that the spectral density m satisfies (3.1). Then, for every $t \in \mathbb{R}$, the function*

$$I_t \stackrel{\text{def.}}{=} \begin{cases} 1_{[0,t]}, & \text{if } t > 0,\\ 1_{[t,0]}, & \text{if } t < 0,\end{cases}$$

belongs to the domain of T_m.

Proof. We consider the case $t > 0$. The case where $t < 0$ is treated in a similar way. We have

$$\int_{\mathbb{R}} m(u)|\widehat{1_{[0,t]}}(u)|^2 \mathrm{d}u = \int_{-\infty}^{-1} m(u) \left| \frac{\mathrm{e}^{-\mathrm{i}tu} - 1}{-\mathrm{i}u} \right|^2 \mathrm{d}u + \int_{-1}^{1} m(u) \left| \frac{\mathrm{e}^{-\mathrm{i}tu} - 1}{-\mathrm{i}u} \right|^2 \mathrm{d}u + \int_{1}^{\infty} m(u) \left| \frac{\mathrm{e}^{-\mathrm{i}tu} - 1}{-\mathrm{i}u} \right|^2 \mathrm{d}u.$$

The first and last integrals converge in view of (3.1), and the second is trivially convergent. □

Theorem 3.2. *Assume that the spectral density m satisfies (3.1). The operator T_m is self-adjoint and closed. It is bounded if and only if m is bounded.*

Proof. For f and g in the domain of T_m we have

$$\langle f, T_m g\rangle_{\mathbf{L}_2(\mathbb{R})} = \langle T_m f, g\rangle_{\mathbf{L}_2(\mathbb{R})} .$$

Thus, $T_m \subset T_m^*$ and the operator T_m is hermitian. We show that it is self-adjoint: let $g \in \mathrm{dom}\,(T_m^*)$. The map $f \to \langle T_m f, g\rangle_{\mathbf{L}_2(\mathbb{R})}$ is continuous and so is the map

$$f \to \left\langle \widehat{T_m f}, \widehat{g} \right\rangle_{\mathbf{L}_2(\mathbb{R})} = \left\langle \sqrt{m}\widehat{f}, \widehat{g} \right\rangle_{\mathbf{L}_2(\mathbb{R})} ,$$

and the map

$$\widehat{f} \to \left\langle \widehat{f}, \sqrt{m}\widehat{g} \right\rangle_{\mathbf{L}_2(\mathbb{R})}$$

is also continuous. By Riesz representation theorem, $\sqrt{m}\widehat{g} \in \mathbf{L}_2(\mathbb{R})$; hence $g \in \mathrm{dom}\,(T_m)$ and we get $T_m^* \subset T_m$.

We now show that T_m is closed: let $f_n \to f$ and $T_m f_n \to g$. We have $\widehat{f_n} \to \widehat{f}$. Thus $T_m f_n \to g$ leads to $\widehat{T_m f_n} \to \widehat{g}$, and thus $\sqrt{m}\widehat{f_n} \to \widehat{g}$. By [21, Théorème 2.3, p. 95] there exists a subsequence n_k such that $\widehat{f_{n_k}} \to \widehat{f}$ pointwise, a.e., and so $\widehat{T_m f_{n_k}} \to \widehat{g}$ pointwise, a.e., and we have $\widehat{T_m f} = \widehat{g}$, a.e.

Finally we show that the operator T_m is bounded if and only if m is bounded. First, if m is bounded then there exists a $K > 0$ such that $|m(u)| < K$ for any $u \in \mathbb{R}$ and we get, for any $f \in \mathbf{L}_2(\mathbb{R})$,

$$\int_{\mathbb{R}} m(u)|\widehat{f}(u)|^2 \mathrm{d}u < K \int_{\mathbb{R}} |\widehat{f}(u)|^2 \mathrm{d}u,$$

so T_m is bounded since the Fourier transform is an isometry. Now if T_m is bounded, then there exists a $K \in \mathbb{R}$ such that for any $f \in \mathbf{L}_2(\mathbb{R})$

$$\int_{\mathbb{R}} m(u) \left|\widehat{f}(u)\right|^2 \mathrm{d}u \le K \int_{\mathbb{R}} \left|\widehat{f}(u)\right|^2 \mathrm{d}u.$$

Assume that m is unbounded. Then for any $N \in \mathbb{N}$ there exists a measurable set E_N such that $\lambda(E_N) > 0$ (λ denotes the Lebesgue measure) and $m(u) \ge N$ on E_N, where, without loss of generality, one may take E_N such that $\lambda(E_N) \le 1$. Define f_n such that $\widehat{f_n} = 1_{E_N}$ on E_N. We then have

$$N \int_{E_N} \left|\widehat{f_n}(u)\right|^2 \mathrm{d}u \le \int_{E_N} m(u) \left|\widehat{f_n}(u)\right|^2 \mathrm{d}u \le K \int_{E_N} \left|\widehat{f_n}(u)\right|^2 \mathrm{d}u,$$

and hence $N \le K$, but this is impossible, so m is bounded. □

For $m(u) = \frac{1}{2\pi}|u|^{1-2H}$, we have that

$$\text{supp } T_m I_t \subset \text{supp } I_t, \quad t \in \mathbb{R}. \tag{3.7}$$

In general this property will not hold, as we now illustrate with a counterexample. This example is of particular importance for marking the difference between our approach and the approach presented in [4].

Example 3.3. Let $m(u) = u^4 \mathrm{e}^{-2u^2}$. We have

$$\begin{aligned}(T_m 1_{[0,t]})(s) &= \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}su} u^2 \mathrm{e}^{-u^2} \cdot \frac{\mathrm{e}^{-\mathrm{i}tu} - 1}{-\mathrm{i}u} \mathrm{d}u \\ &= -\frac{1}{2\pi \mathrm{i}}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}su} u \mathrm{e}^{-u^2} (\mathrm{e}^{-\mathrm{i}tu} - 1)\mathrm{d}u \\ &= -\frac{1}{2\pi \mathrm{i}}\int_{\mathbb{R}} u \mathrm{e}^{\mathrm{i}(s-t)u} \mathrm{e}^{-u^2} \mathrm{d}u + \frac{1}{2\pi \mathrm{i}}\int_{\mathbb{R}} u \mathrm{e}^{\mathrm{i}su} \mathrm{e}^{-u^2} \mathrm{d}u \\ &= \Phi(s) - \Phi(s-t),\end{aligned}$$

where

$$\Phi(s) = \frac{1}{2\pi \mathrm{i}}\int_{\mathbb{R}} u \mathrm{e}^{\mathrm{i}su} \mathrm{e}^{-u^2} \mathrm{d}u.$$

We have

$$\Phi(s) = \frac{1}{2\pi \mathrm{i}}\int_{\mathbb{R}} u \mathrm{e}^{\mathrm{i}su} \mathrm{e}^{-u^2} \mathrm{d}u = \frac{1}{2\pi i} \mathrm{e}^{-\frac{s^2}{4}} \int_{\mathbb{R}} u \mathrm{e}^{-(u-\frac{is}{2})^2} \mathrm{d}u = \frac{s}{4\sqrt{\pi}} \mathrm{e}^{\frac{-s^2}{4}}.$$

Thus,

$$(T_m 1_{[0,t]})(s) = \frac{1}{4\sqrt{\pi}}\left\{(t-s)\mathrm{e}^{-\frac{(t-s)^2}{4}} + s\mathrm{e}^{-\frac{s^2}{4}}\right\}.$$

The support of the function $T_m(I_t)$ is not bounded, and in particular (3.7) is not in force.

For m bounded, we note that T_m is a translation invariant operator.

We now recall the definitions of the Hermite polynomials and of the Hermite functions. Then, in Proposition 3.6 we study the action of the operator T_m on Hermite functions.

Definition 3.4. The Hermite polynomials $\{h_k(u) k \in \mathbb{N}_0\}$ are defined by

$$h_k(u) \stackrel{\text{def.}}{=} (-1)^k \mathrm{e}^{\frac{u^2}{2}} \frac{\mathrm{d}^k}{\mathrm{d}u^k}\left(\mathrm{e}^{-\frac{u^2}{2}}\right), \quad k = 0, 1, 2 \ldots.$$

Definition 3.5. The Hermite functions are defined by

$$\widetilde{h}_k(u) \stackrel{\text{def.}}{=} \frac{h_{k-1}(\sqrt{2}u)\mathrm{e}^{-\frac{u^2}{2}}}{\pi^{\frac{1}{4}}\sqrt{(k-1)!}}, \quad k = 1, 2, \ldots.$$

The following proposition outlines the main properties of the Hermite functions which we will need; see [14, p. 349] and the references therein.

Proposition 3.6 (*[14, p. 349]*)**.** *The Hermite functions* $\{\widetilde{h}_k, k \in \mathbb{N}\}$ *form an orthonormal basis of* $\mathbf{L}_2(\mathbb{R})$. *Furthermore,*

$$|\widetilde{h}_k(u)| \leq \begin{cases} C & \textit{if } |u| \leq 2\sqrt{k}, \\ C\mathrm{e}^{-\gamma u^2} & \textit{if } |u| > 2\sqrt{k}, \end{cases} \tag{3.8}$$

where C and $\gamma > 0$ are constants independent of k. Finally, the Fourier transform of the Hermite function is given by

$$\widehat{\widetilde{h}}_k(u) = \sqrt{2\pi}(-1)^{k-1}\widetilde{h}_k(u). \tag{3.9}$$

Using the previous proposition, we now study the functions $T_m\widetilde{h}_k$.

Proposition 3.7. *Assume that the function m satisfies a bound of the type (1.7)*

$$m(u) \leq \begin{cases} K\,|u|^{-b} & \textit{if } |u| \leq 1, \\ K|u|^{2N} & \textit{if } |u| > 1, \end{cases}$$

where $b < 2$, $N \in \mathbb{N}_0$, and $0 < K < \infty$. Then,

$$\left|(T_m\widetilde{h}_k)(u)\right| \leq C_1 k^{\frac{N+1}{2}} + C_2, \tag{3.10}$$

where C_1 and C_2 are constants independent of k.

Before giving the proof of this proposition, we comment on (1.7): a positive (and say continuous) function m which satisfies (1.7) need not satisfy (3.1). What we need in the sequel are functions which satisfy both (1.7) and (3.1). The purpose of (1.7) is to give some bound on the behaviour of m at infinity and near the origin.

Proof of Proposition 3.7. Using (3.9) we have

$$\begin{aligned} \left|(T_m\widetilde{h}_k)(u)\right| &= \frac{1}{2\pi}\left|\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}uy}\widehat{\widetilde{h}}_k(y)\sqrt{m(y)}\mathrm{d}y\right| \\ &\leq \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left|\widetilde{h}_k(y)\right|\sqrt{m(y)}\mathrm{d}y. \end{aligned}$$

We now compute an upper bound for the integral

$$\int_{\mathbb{R}}\left|\widetilde{h}_k(y)\right|\sqrt{m(y)}\mathrm{d}y = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \int_{-\infty}^{-2\sqrt{k}}\left|\widetilde{h}_k(y)\right|\sqrt{m(y)}\mathrm{d}y, \\ I_2 &= \int_{-2\sqrt{k}}^{2\sqrt{k}}\left|\widetilde{h}_k(y)\right|\sqrt{m(y)}\mathrm{d}y, \\ I_3 &= \int_{2\sqrt{k}}^{\infty}\left|\widetilde{h}_k(y)\right|\sqrt{m(y)}\mathrm{d}y. \end{aligned}$$

By (3.8) we have

$$I_1 \leq C\int_{-\infty}^{-2\sqrt{k}}\mathrm{e}^{-\gamma y^2}\sqrt{m(y)}\mathrm{d}y,$$

$$I_2 \leq C \int_{-2\sqrt{k}}^{2\sqrt{k}} \sqrt{m(y)}\mathrm{d}y,$$
$$I_3 \leq C \int_{2\sqrt{k}}^{\infty} \mathrm{e}^{-\gamma y^2}\sqrt{m(y)}\mathrm{d}y.$$

We have

$$\begin{aligned}\int_{-2\sqrt{k}}^{2\sqrt{k}} \sqrt{m(y)}\mathrm{d}y &= \int_{-2\sqrt{k}}^{-1} \sqrt{m(y)}\mathrm{d}y + \int_{-1}^{1} \sqrt{m(y)}\mathrm{d}y + \int_{1}^{2\sqrt{k}} \sqrt{m(y)}\mathrm{d}y \\ &\leq \sqrt{K}\int_{-2\sqrt{k}}^{-1} |y|^N \,\mathrm{d}y + \sqrt{K}\int_{-1}^{1} |y|^{-\frac{b}{2}}\,\mathrm{d}y + \sqrt{K}\int_{1}^{2\sqrt{k}} |y|^N\,\mathrm{d}y \\ &= 2\sqrt{K}\left(\frac{2^{N+1}k^{\frac{N+1}{2}-1}}{N+1}\right) + 4\sqrt{K}\frac{1}{2-b},\end{aligned}$$

so we get

$$I_2 \leq \tilde{K}_1 k^{\frac{N+1}{2}} + \tilde{K}_2$$

for appropriate constants $\tilde{K}_1$ and $\tilde{K}_2$ (which depend on N). Furthermore,

$$\begin{aligned}\int_{2\sqrt{k}}^{\infty} \mathrm{e}^{-\gamma y^2}\sqrt{m(y)}\mathrm{d}y &\leq \sqrt{K}\int_{2\sqrt{k}}^{\infty} |y|^N\,\mathrm{e}^{-\gamma y^2}\mathrm{d}y \\ &\leq \sqrt{K}\int_0^{\infty} y^N \mathrm{e}^{-\gamma y^2}\mathrm{d}y \\ &= \sqrt{K}\frac{\Gamma(\frac{N+1}{2})}{2\gamma^{\frac{N+1}{2}}}.\end{aligned}$$

Finally we get

$$\left|(T_m\widetilde{h}_k)(y)\right| \leq C_1 k^{\frac{N+1}{2}} + C_2$$

for appropriate strictly positive constants C_1 and C_2 (which depend on N). □

Lemma 3.8. *Assume that the spectral density m satisfies (1.7). The function $T_m\widetilde{h}_k$ is uniformly continuous for every $k \in \mathbb{N}$. More precisely, it holds that*

$$\left|(T_m\widetilde{h}_k)(t) - (T_m\widetilde{h}_k)(s)\right| \leq |t-s|\left\{C_1 k^{\frac{N+2}{2}} + C_2\right\}, \tag{3.11}$$

where C_1 and C_2 are constants independent of k.

Proof. Let $t, s \in \mathbb{R}$. We have

$$(T_m\widetilde{h}_k)(t) - (T_m\widetilde{h}_k)(s) = \frac{1}{\sqrt{2\pi}}(-1)^{k-1}\int_{\mathbb{R}} \left(\mathrm{e}^{-\mathrm{i}ut} - \mathrm{e}^{-\mathrm{i}us}\right)\sqrt{m(u)}\widetilde{h}_k(u)\mathrm{d}u.$$

Taking into account that

$$\left|\mathrm{e}^{-\mathrm{i}ut} - \mathrm{e}^{-\mathrm{i}us}\right| \leq |u(t-s)|,$$

we get

$$\left|(T_m\widetilde{h}_k)(t) - (T_m\widetilde{h}_k)(s)\right| \leq \frac{|t-s|}{\sqrt{2\pi}}\int_{\mathbb{R}} |u|\sqrt{m(u)}\left|\widetilde{h}_k(u)\right|\mathrm{d}u.$$

To conclude the proof, it suffices to show that $\int_{\mathbb{R}} |u| \sqrt{m(u)} \left|\widetilde{h}_k(u)\right| \mathrm{d}u < \infty$. By (1.7) and (3.8) we have

$$\begin{aligned}\int_{\mathbb{R}} |u| \sqrt{m(u)} \left|\widetilde{h}_k(u)\right| \mathrm{d}u &\leq A \int_{|u|\leq 1} |u|^{1-\frac{b}{2}} \mathrm{d}u + B \int_{1<|u|\leq 2\sqrt{k}} |u|^{N+1} \mathrm{d}u \\ &\quad + C \int_{|u|>2\sqrt{k}} |u|^{N+1} \mathrm{e}^{-\gamma u^2} \mathrm{d}u \\ &\leq Dk^{\frac{N+2}{2}} + E,\end{aligned}$$

where all the constants are independent of k. □

We conclude this section with a remark.

Remark 3.9. When

$$\int_{\mathbb{R}} \frac{\ln m(u)}{u^2+1} \mathrm{d}u > -\infty,$$

the function m admits a factorization $m(u) = |h(u)|^2$, where h is an outer function. One can then define an operator $\widetilde{T}_m$ through $\widetilde{T}_m f \overset{\text{def.}}{=} \widehat{h} * f$ rather than the operator T_m. We will not pursue this direction here.

4. The white noise space and the Brownian motion

In this section we review the construction of the white noise space and recall some results related to the Brownian motion. We refer the reader to [31,33,42,34] for additional information and references. To build the white noise space one considers the subspace $\mathcal{S}_{\mathbb{R}}(\mathbb{R})$ of the Schwartz space which consists of *real-valued* functions. Denote by $\mathcal{S}_{\mathbb{R}}(\mathbb{R})'$ its dual. Let $\mathcal{F}$ be the σ-algebra of Borel sets in the space $\mathcal{S}_{\mathbb{R}}(\mathbb{R})'$. The function

$$K(s_1 - s_2) = \exp(-\|s_1 - s_2\|^2_{\mathbf{L}_2(\mathbb{R})}/2)$$

is positive in $\mathcal{S}_{\mathbb{R}}(\mathbb{R})$ in the sense of reproducing kernels since

$$\begin{aligned}\exp(-\|s_1 - s_2\|^2_{\mathbf{L}_2(\mathbb{R})}/2) &= \exp(-\|s_1\|^2_{\mathbf{L}_2(\mathbb{R})}/2) \times \exp\langle s_1, s_2\rangle_{\mathbf{L}_2(\mathbb{R})} \\ &\quad \times \exp(-\|s_2\|^2_{\mathbf{L}_2(\mathbb{R})}/2).\end{aligned}$$

The space $\mathcal{S}_{\mathbb{R}}(\mathbb{R})$ is nuclear, and therefore the Bochner–Minlos theorem (see for instance [33, Appendix A, p. 193]) implies that there exists a probability measure P on $(\mathcal{S}_{\mathbb{R}}(\mathbb{R})', \mathcal{F})$ such that, for all $s \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$,

$$E(\mathrm{e}^{\mathrm{i}Q_s(s')}) \overset{\text{def.}}{=} \int_{\mathcal{S}_{\mathbb{R}}(\mathbb{R})'} \mathrm{e}^{\mathrm{i}Q_s(s')} \mathrm{d}P(s') = \mathrm{e}^{-\frac{\|s\|^2_{\mathbf{L}_2(\mathbb{R})}}{2}}, \tag{4.1}$$

where Q_s denotes the linear functional $Q_s(s') = \langle s', s\rangle_{\mathcal{S}_{\mathbb{R}}(\mathbb{R})', \mathcal{S}_{\mathbb{R}}(\mathbb{R})}$; see [14, (2.1), p. 348], [27, (2.3), p. 303]. Note that Q_s is the canonical isomorphism of the Schwartz space $\mathcal{S}_{\mathbb{R}}(\mathbb{R})$ onto its bidual; see [35, p. 7]. Definition (4.1) implies in particular that

$$E(Q_s) = 0 \quad \text{and} \quad E(Q_s^2) = \|s\|^2_{\mathbf{L}_2(\mathbb{R})}. \tag{4.2}$$

In view of (4.2), the map

$$s \to Q_s \tag{4.3}$$

is an isometry from the real Hilbert space $\mathcal{S}_{\mathbb{R}}(\mathbb{R}) \subset \mathbf{L}_2(\mathbb{R})$ into the real Hilbert space $\mathbf{L}_2(\mathcal{S}_{\mathbb{R}}(\mathbb{R})', \mathcal{F}, \mathrm{d}P)$. It extends to an isometry from $\mathbf{L}_2(\mathbb{R})$ into $\mathbf{L}_2(\mathcal{S}_{\mathbb{R}}(\mathbb{R})', \mathcal{F}, \mathrm{d}P)$, and we define for $f \in \mathbf{L}_2(\mathbb{R})$

$$Q_f(s') \stackrel{\text{def.}}{=} \lim_{n\to\infty} Q_{f_n}(s'), \tag{4.4}$$

where the limit is in $\mathbf{L}_2(\mathcal{S}_{\mathbb{R}}(\mathbb{R})', \mathcal{F}, \mathrm{d}P)$ and where $f_n \to f$ in $\mathbf{L}_2(\mathbb{R})$. The limit is easily shown not to depend on (f_n).

In the sequel we consider complex-valued functions. The map (4.3) extends to an isometry between the complexified spaces of $\mathbf{L}_2(\mathbb{R})$ and $\mathbf{L}_2(\mathscr{S}'(\mathbb{R}), \mathcal{F}, \mathrm{d}P)$. See for instance [15, pp. V4–V5] for the complexification of Hilbert spaces.

The triplet $\mathbf{L}_2(\mathcal{S}_{\mathbb{R}}(\mathbb{R})', \mathcal{F}, P)$ is called the white noise space. In accordance with the notation standard in probability theory, we set

$$\Omega = \mathcal{S}_{\mathbb{R}}(\mathbb{R})',$$

and denote by

$$\mathcal{W} = \mathbf{L}_2(\Omega, \mathcal{F}, P)$$

the complexified space of $\mathbf{L}_2(\mathcal{S}_{\mathbb{R}}(\mathbb{R})', \mathcal{F}, P)$.

The Brownian motion is a family $\{B(t,\omega)\}$ of random variables in the white noise space with the following properties:

(1) $B(0,\omega) = 0$ almost surely with respect to P.
(2) $\{B(t,\omega)\}$ is a Gaussian stochastic process with mean zero and $B(t,\omega)$ and $B(s,\omega)$ have the covariance $\min(t,s)$.
(3) $s \to B(s,\omega)$ is continuous for almost all ω with respect to P.

Define the stochastic process

$$B(t,\omega) = Q_{I_t}(\omega), \quad t \in \mathbb{R}.$$

Then, for $t, s \geq 0$,

$$E(B(t,\omega)B(s,\omega)^*) = \langle I_t, I_s\rangle_{\mathbf{L}_2(\mathbb{R})} = \int_{\mathbb{R}} I_t(u)(I_s(u))^* \mathrm{d}u = \min(t,s).$$

By Kolmogorov's continuity theorem the process $\{B(t,\omega)\}$ has a continuous version, which is a Brownian motion. For $F \in \mathcal{W}$ we now recall Wiener–Ito chaos expansion. In the stochastic process literature this expansion is for real-valued functions. We write it for the complexification of the underlying Hilbert spaces. This creates no technical problem.

The white noise probability space $\mathcal{W}$ admits a special orthonormal basis $\{H_\alpha\}$, indexed by the set ℓ (defined by (1.11)). The definition of this basis is recalled in the following proposition. We refer the reader to [33, Definition 2.2.1, p. 19] for more information.

Proposition 4.1 (*Wiener–Ito Chaos Expansion [33, Theorem 2.2.4, p. 23]*)**.** *Every $F \in \mathcal{W}$ can be written as*

$$F = \sum_{\alpha\in\ell} c_\alpha H_\alpha,$$

with $\alpha \in \ell$, $c_\alpha \in \mathbb{C}$, and

$$\|F\|_{\mathcal{W}}^2 = \sum_{\alpha \in \ell} \alpha! \, |c_\alpha|^2 < \infty,$$

where $\alpha! = \alpha_1!\alpha_2!\alpha_3! \cdots$ *and*

$$H_\alpha(\omega) \stackrel{\text{def.}}{=} \prod_{k=1}^{\infty} h_{\alpha_k}\Big(Q_{\widetilde{h}_k}(\omega)\Big), \quad \omega \in \Omega. \tag{4.5}$$

5. The Kondratiev space and the Wick product

The Wick product is defined through

Definition 5.1. Let $\alpha, \beta \in \ell$; then

$$H_\alpha \diamond H_\beta = H_{\alpha+\beta}.$$

Definition 5.2. Let F, G be two elements in $\mathcal{W}$:

$$F = \sum_{\alpha \in \ell} a_{\alpha \in \ell} H_\alpha, \quad \text{and} \quad G = \sum_{\alpha \in \ell} b_\alpha H_\alpha,$$

where $\alpha \in \ell$, $a_\alpha, b_\alpha \in \mathbb{C}$ and $a_\alpha, b_\alpha \neq 0$ for only a finite number of indexes α. The Wick product of F and G is defined by

$$(F \diamond G)(\omega) = \sum_{\alpha,\beta \in \ell} a_\alpha b_\beta H_{\alpha+\beta}(\omega) = \sum_\gamma \left(\sum_{\gamma=\alpha+\beta} a_\alpha b_\beta \right) H_\gamma(\omega).$$

The basis $\{H_\alpha\}_{\alpha \in \ell}$ is computed from the Hermite functions. The choice of another orthonormal basis for $\mathbf{L}_2(\mathbb{R})$ will lead to another basis for the white noise space, but to the same Wick product. See [33, Appendix D, p. 213]. The Wick product is not defined for all pairs of elements in the white noise space. See [33].

The Kondratiev space S_{-1} seems to be the most convenient space within which the Wick product is well defined. It is a space of distributions. We first recall on which space of test functions its elements operate.

Definition 5.3. The Kondratiev space S_1 of stochastic test functions consists of the elements in the form $f = \sum_{\alpha \in \ell} a_\alpha H_\alpha \in \mathcal{W}$ such that

$$\sum_{\alpha \in \ell} |a_\alpha|^2 \, (\alpha!)^2 (2\mathbb{N})^{k\alpha} < \infty, \quad k = 1, 2, \ldots,$$

where

$$(2\mathbb{N})^\alpha \stackrel{\text{def.}}{=} 2^{\alpha_1}(2 \cdot 2)^{\alpha_2}(2 \cdot 3)^{\alpha_3} \cdots, \quad \alpha \in \ell.$$

Definition 5.4. The Kondratiev space S_{-1} of stochastic distributions consists of the elements in the form $F = \sum_{\alpha \in \ell} b_\alpha H_\alpha$ with the property that

$$\sum_{\alpha \in \ell} |b_\alpha|^2 \, (2\mathbb{N})^{-q\alpha} < \infty,$$

for some $q \in \mathbb{N}$.

S_{-1} can be identified with the dual of S_1 and the action of $F \in S_{-1}$ on $f = \sum_{\alpha\in\ell} a_\alpha H_\alpha \in S_1$ is given by

$$\langle F, f\rangle_{S_{-1},S_1} \stackrel{\text{def.}}{=} \sum_{\alpha\in\ell} \alpha! a_\alpha b_\alpha. \tag{5.1}$$

We also note the following: let $\alpha \in \ell$. By (4.5), using a Wick product calculation, we have

$$H_\alpha(\omega) = \prod_{k=1}^{\infty} \left(Q_{\widetilde{h}_k}(\omega)\right)^{\diamond\alpha_k}$$

for $\alpha = \epsilon^{(k)} = (0, 0, \ldots, 0, 1, 0, \ldots)$, $\alpha_i = 0$ for $i \neq k$ and $\alpha_k = 1$; we get

$$H_{\epsilon^{(k)}} = Q_{\widetilde{h}_k} = \int_{\mathbb{R}} \widetilde{h}_k(t)\mathrm{d}B(t).$$

We now review the main results associated with the Wick product and the Hermite transform.

A key property of the basis $\{H_\alpha,\ \alpha \in \ell\}$ is the following: define a map $\mathbf{I}$ such that

$$\mathbf{I}(H_\alpha) = z^\alpha,$$

where $\alpha \in \ell$, $z = (z_1, z_2, \ldots) \in \mathbb{C}^{\mathbb{N}}$ (the set of all sequences of complex numbers) and

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots.$$

Then

$$\mathbf{I}(H_\alpha \diamond H_\beta) = \mathbf{I}(H_\alpha)\mathbf{I}(H_\beta).$$

The map $\mathbf{I}$ is called the Hermite transform.

We note that the spaces S_{-1} and S_1 are closed under the Wick product; see [33, Lemma 2.4.4, p 42].

Definition 5.5. Let $F = \sum_{\alpha\in\ell} a_\alpha H_\alpha \in S_{-1}$. Then the Hermite transform of F, denoted by $\mathbf{I}(F)$ or $\widetilde{F}$, is defined by

$$\mathbf{I}(F)(z) = \widetilde{F}(z) = \sum_{\alpha\in\ell} a_\alpha z^\alpha.$$

Proposition 5.6 (*[33, Proposition 2.6.6, p. 59]*). *Let $F, G \in S_{-1}$. Then*

$$\mathbf{I}(F \diamond G)(z) = (\mathbf{I}(F)(z)) \cdot (\mathbf{I}(G)(z)).$$

6. The stochastic process associated with T_m

Let m be a spectral density subject to (3.1). Using Lemma 3.1, we have $T_m I_t \in \mathbf{L}_2(\mathbb{R})$ and by the expansion in $\mathbf{L}_2(\mathbb{R})$ in terms of the Hermite functions $\widetilde{h}_k$ we obtain

$$T_m I_t = \sum_{k=1}^{\infty} \langle T_m I_t, \widetilde{h}_k\rangle \widetilde{h}_k,$$

where

$$\langle T_m I_t, \widetilde{h}_k\rangle = \langle I_t, T_m\widetilde{h}_k\rangle = \int_{\mathbb{R}} I_t(y)(T_m\widetilde{h}_k)(y)\mathrm{d}y.$$

We define the stochastic process $\{X_m(t,\omega),\ t\in\mathbb{R}\}$ defined through

$$X_m(t,\omega) \stackrel{\text{def.}}{=} Q_{T_m I_t}(\omega),$$

where $t\in\mathbb{R}$ and $\omega\in\Omega$. As already noted in Section 3, when the function m is given by (3.5) (or, equivalently, the function r is given by (1.8)), the operator T_m is equal to the operator M_H defined in [27,14]. Then, X_m reduces to the fractional Brownian motion with Hurst parameter $H\in(0,1)$.

Lemma 6.1. *Let m be a spectral density subject to (3.1). The stochastic process X_m has the following properties:*

(1) $E(X_m(t,\omega)X_m(s,\omega)^*) = K_r(s,t)$.
(2) $E\left(|X_m(t,\omega)-X_m(s,\omega)|^2\right) = 2\,\mathrm{Re}\, r(t-s)$.
(3) *Assume moreover that m satisfies (1.7), with $N=0$. Then*

$$\mathrm{Re}\, r(t) \le C_1 t^2 + C_2 t.$$

for some positive constants C_1 and C_2.

Proof. To prove item (1) we note that

$$\begin{aligned}
E(X_m(t,\omega)X_m(s,\omega)^*) &= \langle T_m I_t, T_m I_s\rangle_{\mathbf{L}_2(\mathbb{R})}\\
&= \int_{\mathbb{R}} (T_m I_t)(u)((T_m I_t)(u))^*\mathrm{d}u\\
&= \int_{\mathbb{R}} \widehat{(T_m I_t)}(u)(\widehat{(T_m I_s)}(u))^*\mathrm{d}u\\
&= \int_{\mathbb{R}} m(u)\widehat{(1_{[0,t]})}(u)\left(\widehat{(1_{[0,s]})}(u)\right)^*\mathrm{d}u\\
&= \int_{\mathbb{R}} m(u)\left\{\int_0^t \mathrm{e}^{-\mathrm{i}ux}\mathrm{d}x\right\}\left\{\int_0^s \mathrm{e}^{\mathrm{i}uy}\mathrm{d}y\right\}\mathrm{d}u\\
&= \int_{\mathbb{R}} \frac{\mathrm{e}^{-\mathrm{i}tu}-1}{u}\frac{\mathrm{e}^{\mathrm{i}su}-1}{u}m(u)\mathrm{d}u\\
&= K_r(s,t).
\end{aligned}$$

The proof of the second statement is carried out by direct computations:

$$\begin{aligned}
E\left(|X_m(t,\omega)-X_m(s,\omega)|^2\right) &= E((X_m(t,\omega)-X_m(s,\omega))(X_m(t,\omega)-X_m(s,\omega))^*)\\
&= E\left\{X_m(t,\omega)X_m(t,\omega)^* - X_m(t,\omega)X_m(s,\omega)^*\right.\\
&\quad \left. -\ X_m(s,\omega)X_m(t,\omega)^* + X_m(s,\omega)X_m(s,\omega)^*\right\}.
\end{aligned}$$

Using the first statement we get

$$\begin{aligned}
&= E(X_m(t,\omega)X_m(t,\omega)^*) - E(X_m(t,\omega)X_m(s,\omega)^*)\\
&\quad - E(X_m(s,\omega)X_m(t,\omega)^*) + E(X_m(s,\omega)X_m(s,\omega)^*)\\
&= \{K_r(t,t) - K_r(s,t) - K_r(t,s) + K_r(s,s)\}\\
&= \big(r(t) + r(t)^* - (r(s) + r(t)^* - r(s-t))\\
&\quad - (r(t) + r(s)^* - r(t-s)) + r(s) + r(s)^*\big)
\end{aligned}$$

$$
\begin{aligned}
&= (r(t-s) + r(t-s)^*)\\
&= 2\,\mathrm{Re}\, r(t-s).
\end{aligned}
$$

Finally, recall that we have

$$\mathrm{Re}\, r(t) = \int_{\mathbb{R}} \Big\{1 - \cos(tu)\Big\} \frac{m(u)}{u^2} \mathrm{d}u.$$

Then, when $m(u)$ satisfies (1.7) with $N = 0$, we get

$$
\begin{aligned}
\mathrm{Re}\, r(t) &\le 2 \left\{ K \int_0^1 \frac{|1-\cos(tu)|}{u^{2+b}} \mathrm{d}u + K' \int_1^\infty \frac{|1-\cos(tu)|}{u^2} \mathrm{d}u \right\}\\
&= 2 \left\{ K \int_0^1 \frac{2\sin^2(\frac{tu}{2})}{u^{2+b}} \mathrm{d}u + K' \int_1^\infty \frac{|1-\cos(tu)|}{u^2} \mathrm{d}u \right\}.
\end{aligned}
$$

But

$$\int_0^1 \frac{2\sin^2(\frac{tu}{2})}{u^{2+b}} \mathrm{d}u \le t^2 \int_0^1 \frac{u^2}{2u^{2+b}} \mathrm{d}u = \frac{t^2}{2} \int_0^1 \frac{1}{u^b} \mathrm{d}u = \frac{t^2}{2(1-b)}$$

since for $t \in [0, 1]$ we get $tu \in [0, 1]$ and $\sin^2(\frac{tu}{2}) \le \frac{(tu)^2}{4}$. Furthermore,

$$
\begin{aligned}
\int_1^\infty \frac{|1-\cos(tu)|}{u^2} \mathrm{d}u &\le \int_0^\infty \frac{|1-\cos(tu)|}{u^2} \mathrm{d}u\\
&= t \int_0^\infty \frac{|1-\cos v|}{v^2} \mathrm{d}v\\
&= t \int_0^1 \frac{|1-\cos v|}{v^2} \mathrm{d}v + t \int_1^\infty \frac{|1-\cos v|}{v^2} \mathrm{d}v\\
&\le t \int_0^1 \frac{2\sin^2\left(\frac{v}{2}\right)}{v^2} \mathrm{d}v + 2t \int_1^\infty \frac{1}{v^2} \mathrm{d}v\\
&\le \frac{t}{2} \int_0^1 \frac{v^2}{v^2} \mathrm{d}v + 2t \int_1^\infty \frac{1}{v^2} \mathrm{d}v\\
&= \frac{t}{2} + 2t = \frac{5t}{2}.
\end{aligned}
\tag{6.1}
$$

Thus,

$$\mathrm{Re}\, r(t) \le |\mathrm{Re}\, r(t)| \le 2 \left\{ K \frac{t^2}{2(1-b)} + K' \frac{5t}{2} \right\} = C_1 t^2 + C_2 t. \quad \square$$

We note that the bounds in (6.1) make use of the assumption $N = 0$, and do not seem to extend to the case $N \in \mathbb{N}$. Still under the assumption $N = 0$ we now show that $\{X_m(t,\omega),\ t \in \mathbb{R}\}$ meets the criterion of Kolmogorov's theorem concerning the existence of a continuous version of a given stochastic process. Using the fact (see for instance [35, p. 5] with $p = 2n$) that

$$E\Big(|X_m(t,\omega)|^{2n}\Big) = \kappa(2n)^{2n} \Big(E\left(|X_m(t,\omega)|^2\right)\Big)^{\frac{2n}{2}} \tag{6.2}$$

where

$$\kappa(2n) = \sqrt{2}\left(\frac{\Gamma(\frac{2n+1}{2})}{\sqrt{\pi}}\right)^{\frac{1}{2n}} = \sqrt{2}\left(\frac{2n!}{4^n n!}\right)^{\frac{1}{2n}}$$

we have

$$E\left(|X_m(t,\omega)X_m(s,\omega)|^4\right) = \kappa(4)^4 E\Big(|X_m(t,\omega) - X_m(s,\omega)|^2\Big)^2.$$

By (2), (3) we get

$$\begin{aligned}\kappa(4)^4 E\Big(|X_m(t,\omega) - X_m(s,\omega)|^2\Big)^2 &= \kappa(4)^4\,(\mathrm{Re}\, r(t-s))^2\\ &\le \kappa(4)^4\left(C_1(t-s)^2 + C_2(t-s)\right)^2 = (t-s)^2(A + B(t-s))^2,\end{aligned}$$

for $t - s \in [0, 1]$. By Kolmogorov's continuity theorem the process $\{X_m(t,\omega)\}$ has a continuous version where $t \in [0, 1]$. One can show in a similar way that a continuous version exists on every finite interval.

In the following, computations of moments of X_m are presented. These are essential in the construction of stochastic integrals within the white noise space framework; see e.g. [9].

Proposition 6.2. *Let m be a spectral density, subject to (3.1). Then, $X_m(t)$ is a Gaussian random variable with*

$$E(X_m^n(t,\omega)) = \begin{cases} 0, & \text{if } n = 2k-1\\ \dfrac{(2k)!}{2^k k!}\,\|T_m I_t\|^{2k}, & \text{if } n = 2k\end{cases}$$

for $k = 1, 2, \ldots$.

Proof. By (4.1) with $X_m(t,\omega) = Q_{T_m I_t}(\omega)$, we have with $\epsilon \in \mathbb{R}$,

$$E(\exp(\mathrm{i}\epsilon X_m(t,\omega))) = \mathrm{e}^{-\epsilon^2\frac{\|T_m I_t\|^2}{2}},$$

and therefore $X_m(t,\omega)$ is a centered Gaussian random variable. Thus $E(X_m^{2n-1}(t,\omega)) = 0$ for $n = 1, 2, \ldots$, and we get

$$\sum_{n=0}^{\infty}\frac{(i\epsilon)^n}{n!}E(X_m^n(t,\omega)) = \sum_{n=0}^{\infty}\frac{(-1)^n\epsilon^{2n}}{2n!}E(X_m^{2n}(t,\omega)).$$

We have to verify that

$$E\Big(\sum_{n=0}^{\infty}\frac{(-1)^n\epsilon^{2n}}{2n!}X_m^{2n}(t,\omega)\Big) = \sum_{n=0}^{\infty}\frac{(-1)^n\epsilon^{2n}}{2n!}E(X_m^{2n}(t,\omega)).$$

Using (6.2) we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{|\epsilon|^{2n}\,E(|X_m(t,\omega)|^{2n})}{2n!} &= \sum_{n=0}^{\infty}\frac{|\epsilon|^{2n}\,\kappa(2n)^{2n}\left(E\,|X_m(t,\omega)|^2\right)^n}{2n!}\\ &= \sum_{n=0}^{\infty}\frac{|\epsilon|^{2n}\left(E\,|X_m(t,\omega)|^2\right)^n}{2^n n!} < \infty.\end{aligned}$$

We can thus use the dominated convergence theorem to obtain that

$$E\Big(\sum_{n=0}^{\infty}\epsilon^n\frac{\mathrm{i}^n}{n!}X_m^n(t,\omega)\Big)=\sum_{n=0}^{\infty}\epsilon^n\frac{\mathrm{i}^n}{n!}E(X_m^n(t,\omega))$$
$$=\sum_{\ell=0}^{\infty}(-1)^{\ell}\epsilon^{2\ell}\frac{\|T_mI_t\|^{2\ell}}{2^{\ell}\ell!}.$$

The proof is completed by comparing the powers of ϵ on both sides. □

Remark 6.3. In view of (1.6) we have (when $N=0$ in (1.7))

$$\|T_mI_t\|^2=K_r(t,t).$$

Remark 6.4. Since for any $t\in\mathbb{R}$, $X_m(t)$ is written as a weighted sum of the $\{H_\alpha,\ \alpha\in\ell\}$ (for an explicit expression, see (7.3)), in turn being jointly Gaussian random variables, it follows that $\{X_m(t),\ t\in\mathbb{R}\}$ is a Gaussian process.

The following proposition will be used in a subsequent paper, where, as already noted, we develop the stochastic analysis associated with the processes X_m.

Proposition 6.5. *Let m be a spectral density subset of (3.1). Let $f\in$ dom (T_m) and $n\in\mathbb{N}$. It holds that*

$$Q_{T_mf}^{\diamond n}(\omega)=n!\sum_{k=\lceil\frac{n}{2}\rceil}^{n}\Big(-\frac{1}{2}\Big)^{n-k}\frac{Q_{T_mf}^{2k-n}(\omega)}{(2k-n)!}\frac{(\|T_mf\|^2)^{n-k}}{(n-k)!}.$$

In particular, for $f=I_t$, it holds that

$$X_m^{\diamond n}(t)=n!\sum_{k=\lceil\frac{n}{2}\rceil}^{n}\Big(-\frac{1}{2}\Big)^{n-k}\frac{X_m^{2k-n}(t)}{(2k-n)!}\frac{(\|T_mI_t\|^2)^{n-k}}{(n-k)!}.$$

Proof. Let $\epsilon\in\mathbb{R}$; then

$$\exp^{\diamond}(Q_{\epsilon T_mf}(\omega))=\sum_{n=0}^{\infty}\frac{(Q_{\epsilon T_mf}(\omega))^{\diamond n}}{n!}=\sum_{n=0}^{\infty}\frac{\epsilon^n(Q_{T_mf}(\omega))^{\diamond n}}{n!}.$$

By [35, Theorem 3.33, p. 32], we have

$$\exp^{\diamond}(Q_{\epsilon T_mf}(\omega))=\exp\left(Q_{\epsilon T_mf}(\omega)-\frac{1}{2}\|\epsilon T_mf\|^2\right)$$
$$=\sum_{k=0}^{\infty}\frac{(\epsilon Q_{T_mf}(\omega)-\frac{1}{2}\epsilon^2\|T_mf\|^2)^k}{k!}$$
$$=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\epsilon^{2k-j}\frac{(Q_{T_mf}(\omega))^j}{j!}\frac{(-\frac{1}{2}\|T_mf\|^2)^{k-j}}{(k-j)!}.$$

Hence,

$$\sum_{n=0}^{\infty}\frac{\epsilon^n(Q_{T_mf}(\omega))^{\diamond n}}{n!}=\sum_{k=0}^{\infty}\sum_{j=0}^{k}\epsilon^{2k-j}\frac{(Q_{T_mf}(\omega))^j}{j!}\frac{(-\frac{1}{2}\|T_mf\|^2)^{k-j}}{(k-j)!},$$

and comparing the powers of ϵ leads to

$$\frac{(Q_{T_m f}(\omega))^{\diamond n}}{n!} = \sum_{k=\lceil \frac{n}{2} \rceil}^{n} \left(-\frac{1}{2}\right)^{n-k} \frac{(Q_{T_m f}(\omega))^{2k-n}}{(2k-n)!} \frac{(\|T_m f\|^2)^{n-k}}{(n-k)!}. \qquad \square$$

7. The derivative of X_m

One important aspect of the white noise space theory is that the Brownian motion admits a derivative, which belongs to the Hida space $(S)^*$ (the definition of which we do not recall here), and in particular the Kondratiev space S_{-1}. See [33, p. 53]. In this section we prove that this result still holds for the stochastic process X_m. For the next definition, see also [33, Definition 2.5.5, p. 49], where the integral is defined to be an element in the Hida space $(S)^*$.

Definition 7.1. Suppose that $Z : \mathbb{R} \to S_{-1}$ is a given function with the property that

$$\langle Z(t), f \rangle \in \mathbf{L}_1(\mathbb{R}, \mathrm{d}t)$$

for all $f \in S_1$. Then $\int_{\mathbb{R}} Z(t)\mathrm{d}t$ is defined to be the unique element of S_{-1} such that

$$\left\langle \int_{\mathbb{R}} Z(t)\mathrm{d}t, f \right\rangle = \int_{\mathbb{R}} \langle Z(t), f \rangle \, \mathrm{d}t$$

for all $f \in S_1$.

In view of Lemma 3.8 the coefficients of the expansion (7.1) are continuous functions, and not merely elements of $\mathbf{L}_2(\mathbb{R})$.

Theorem 7.2. *Assume that the spectral density satisfies (3.1) and (1.7). Then, for every real t we have that*

$$W_m(t) = \sum_{k=1}^{\infty} \left(T_m \widetilde{h}_k\right)(t) H_{\epsilon^{(k)}} \in S_{-1}, \tag{7.1}$$

and it holds that

$$X_m(t) = \int_0^t W_m(s)\mathrm{d}s, \quad t \in \mathbb{R}. \tag{7.2}$$

Proof. Let $q \geq N + 3$ where $N \in \mathbb{N}$. Then, using (3.10), we have

$$\sum_{k=1}^{\infty} \left|\left(T_m \widetilde{h}_k\right)(t)\right|^2 (2k)^{-q} \leq \sum_{k=1}^{\infty} \left(C_1 k^{\frac{N+1}{2}} + C_2\right)^2 (2k)^{-q} < \infty,$$

and so $W_m(t) \in S_{-1}$. We now prove (7.2). By construction, $X_m(t) \in \mathcal{W}$ for every $t \in \mathbb{R}$, and we can write

$$X_m(t) = \sum_{k=1}^{\infty} b_k(t) H_{\epsilon^{(k)}}, \tag{7.3}$$

where

$$b_k(t) = \int_0^t \left(T_m \widetilde{h}_k\right)(s)\mathrm{d}s,$$

with the convergence in the topology of $\mathcal{W}$. We want to show that, for every $f \in S_1$, we have

$$\langle X_m(t), f\rangle_{S_{-1},S_1} = \int_0^t \langle W_m(u), f\rangle_{S_{-1},S_1} \mathrm{d}u,$$

where $\langle\cdot,\cdot\rangle_{S_{-1},S_1}$ denotes the duality between S_1 and S_{-1} (see (5.1)). With that purpose, let $q \geq N+3$ where $N \in \mathbb{N}$. By using the estimate (3.10), then, with $f = \sum_{\alpha\in\ell} f_\alpha H_\alpha$, we have for $u \in [0,t]$ (and in fact for every $u \geq 0$),

$$\begin{aligned}\sum_{k=1}^{\infty} |(T_m\widetilde{h}_k)(u) f_k| &= \sum_{k=1}^{\infty} |(T_m\widetilde{h}_k)(u)|(2k)^{-\frac{q}{2}}(2k)^{\frac{q}{2}}|f_k| \\ &\leq \left(\sum_{k=1}^{\infty}\left(C_1 k^{\frac{N+1}{2}} + C_2\right)^2 (2k)^{-q}\right)^{\frac12} \cdot \left(\sum_{k=1}^{\infty}(2k)^q |f_k|^2\right)^{\frac12} \\ &< \infty,\end{aligned}$$

since $f \in S_1$. Therefore the series

$$\sum_{k=1}^{\infty} |(T_m\widetilde{h}_k)(u) f_k|$$

converges absolutely. Using the dominated convergence theorem we can write

$$\begin{aligned}\int_0^t \langle W_m(u), f\rangle_{S_{-1},S_1}\mathrm{d}u &= \int_0^t \left(\sum_{k=1}^{\infty}(T_m\widetilde{h}_k)(u) f_k\right)\mathrm{d}u \\ &= \sum_{k=1}^{\infty}\left(\int_0^t (T_m\widetilde{h}_k)(u)\mathrm{d}u\right) f_k \\ &= \langle X_m(t), f\rangle_{S_{-1},S_1}. \quad \square\end{aligned}$$

We now show that, conversely,

$$X_m(t)' = W_m(t),$$

in the sense of S_{-1}-processes; see [33, p. 77] and further below in the current section. In the following statements, the set $K_q(\delta)$ is defined by

$$K_q(\delta) = \left\{ z \in \mathbb{C}^{\mathbb{N}} : \sum_{\alpha\in\ell} \left|z^\alpha\right|^2 (2\mathbb{N})^{q\alpha} < \delta^2 \right\}.$$

See [33, Definition 2.6.4, p. 59].

Proposition 7.3. *Assume that the spectral density m satisfies (3.1) and (1.7). Then, the function $\mathbf{I}(W_m(t))(z)$ is bounded for $(t,z) \in \mathbb{R} \times K_{N+3}(\delta)$ where $N \in \mathbb{N}$.*

Proof. Write

$$W_m(t) = \sum_{k=1}^{\infty}(T_m\widetilde{h}_k)(t) Q_{\widetilde{h}_k}.$$

Taking the Hermite transform we have

$$\mathbf{I}(W_m(t))(z) = \sum_{k=1}^{\infty}(T_m\widetilde{h}_k)(t) z_k.$$

Thus, for every $q \geq N+3$ where $N \in \mathbb{N}$ and using (3.10) we have

$$\begin{aligned}
|\mathbf{I}(W_m(t))| &= \left|\sum_{k=1}^{\infty}(T_m\widetilde{h}_k)(t)z_k\right| \\
&= \left|\sum_{k=1}^{\infty}(T_m\widetilde{h}_k)(t)(2k)^{\frac{q}{2}}(2k)^{-\frac{q}{2}}z_k\right| \\
&= \left(\sum_{k=1}^{\infty}\left|(T_m\widetilde{h}_k)(t)\right|^2(2k)^{-q}\right)^{\frac{1}{2}}\left(\sum_{k=1}^{\infty}(2k)^q\left|z^{\epsilon_k}\right|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{k=1}^{\infty}\left\{C_1k^{\frac{N+1}{2}}+C_2\right\}^2(2k)^{-q}\right)^{\frac{1}{2}}\left(\sum_{\alpha\in\ell}(2\mathbb{N})^{q\alpha}\left|z^{\alpha}\right|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

The first sum converges when $q \geq N+3$ and the second converges since $z \in K_{N+3}(\delta)$. We conclude that the function $\mathbf{I}(W_m(t))(z)$ is bounded for any pair $(t,z) \in \mathbb{R}\times K_{N+3}(\delta)$. $\square$

Theorem 7.4. *Assume that the spectral density satisfies* (3.1) *and* (1.7) *. Then, the function* $\mathbf{I}(W_m(t))(z)$ *is uniformly continuous in t for* $z \in K_{N+4}(\delta)$.

Proof. Using the Cauchy–Schwartz inequality, we have

$$\begin{aligned}
|\mathbf{I}(W_m(t))(z)-\mathbf{I}(W_m(s))(z)| &= \left|\sum_{k=1}^{\infty}\left\{(T_m\widetilde{h}_k)(t)-(T_m\widetilde{h}_k)(s)\right\}z_k\right| \\
&= \left|\sum_{k=1}^{\infty}\left\{(T_m\widetilde{h}_k)(t)-(T_m\widetilde{h}_k)(s)\right\}(2k)^{-\frac{q}{2}}(2k)^{\frac{q}{2}}z_k\right| \\
&\leq \left(\sum_{k=1}^{\infty}\left|\left\{(T_m\widetilde{h}_k)(t)-(T_m\widetilde{h}_k)(s)\right\}\right|^2(2k)^{-q}\right)^{\frac{1}{2}} \\
&\quad\times\left(\sum_{k=1}^{\infty}\left|z^{\epsilon_k}\right|^2(2k)^q\right)^{\frac{1}{2}},
\end{aligned}$$

and thus

$$\begin{aligned}
|\mathbf{I}(W_m(t))(z)-\mathbf{I}(W_m(s))(z)| &\leq \left(\sum_{k=1}^{\infty}\left|\left\{(T_m\widetilde{h}_k)(t)-(T_m\widetilde{h}_k)(s)\right\}\right|^2(2k)^{-q}\right)^{\frac{1}{2}} \\
&\quad\times\left(\sum_{\alpha\in\ell}\left|z^{\alpha}\right|^2(2\mathbb{N})^{\alpha q}\right)^{\frac{1}{2}} \\
&\leq |t-s|\left(\sum_{k=1}^{\infty}\left\{C_1k^{\frac{N+2}{2}}+C_2\right\}^2(2k)^{-q}\right)^{\frac{1}{2}} \\
&\quad\times\left(\sum_{\alpha\in\ell}|z|^{\alpha}(2\mathbb{N})^{\alpha q}\right)^{\frac{1}{2}},
\end{aligned}$$

where we have used (3.11) to go from the first inequality to the second.

The first sum converges for $q \geq N+4$ and the second converges since $z \in K_{N+4}(\delta)$, so we conclude that $\mathbf{I}(W_m(t))(z)$ is continuous in t for every $z \in K_{N+4}(\delta)$. □

We now recall the following result of [33], called *the differentiation of* S_{-1} *processes*.

Proposition 7.5 (*[33, Lemma 2.8.4, p. 77]*). *Suppose* $\{X(t,\omega)\}$ *and* $\{F(t,\omega)\}$ *are* S_{-1}*-valued processes such that*

$$\frac{\mathrm{d}(\mathbf{I}(X)(t))(z)}{\mathrm{d}t} = (\mathbf{I}(F)(t))(z)$$

for each $t \in (a,b),\ z \in K_q(\delta)$ *and that* $(\mathbf{I}(F)(t))(z)$ *is bounded for* $(t,z) \in (a,b) \times K_q(\delta)$, *and is a continuous function of* t *for every* $z \in K_q(\delta)$. *Then* $X(t,\omega)$ *is a differentiable process and*

$$\frac{\mathrm{d}X(t,\omega)}{\mathrm{d}t} = F(t,\omega)$$

for all $t \in (a,b)$.

In view of this proposition, the first step toward showing that W_m is the derivative of X_m is to show that this fact holds for the Hermite transforms. This is done in the following lemma.

Lemma 7.6. *Assume that the spectral density* m *satisfies* (3.1) *and* (1.7), *let* $t \in \mathbb{R}$ *and* $z \in K_{N+4}(\delta)$. *Then*

$$\frac{\mathrm{d}\mathbf{I}(X_m(t))(z)}{\mathrm{d}t} = \mathbf{I}(W_m(t))(z).$$

Proof. Let $h \in \mathbb{R}$. Then

$$\begin{aligned}
&\left| \frac{\mathbf{I}(X_m(t+h))(z) - \mathbf{I}(X_m(t))(z)}{h} - \mathbf{I}(W_m(t))(z) \right| \\
&= \frac{1}{|h|} \left| \sum_{k=1}^{\infty} \int_t^{t+h} \left((T_m \widetilde{h}_k)(s) - (T_m \widetilde{h}_k)(t) \right) \mathrm{d}s\, z_k \right| \\
&= \frac{1}{|h|} \left| \sum_{k=1}^{\infty} \int_t^{t+h} \left((T_m \widetilde{h}_k)(s) - (T_m \widetilde{h}_k)(t) \right) \mathrm{d}s\, (2k)^{-\frac{q}{2}} (2k)^{\frac{q}{2}} z_k \right| \\
&\leq \frac{1}{|h|} \left(\sum_{k=1}^{\infty} \left| \int_t^{t+h} \left((T_m \widetilde{h}_k)(s) - (T_m \widetilde{h}_k)(t) \right) \mathrm{d}s \right|^2 (2k)^{-q} \right)^{\frac{1}{2}} \cdot \left(\sum_{k=1}^{\infty} (2k)^q \left| z^{\epsilon_k} \right|^2 \right)^{\frac{1}{2}} \\
&\leq \frac{1}{|h|} \left(\sum_{k=1}^{\infty} \int_t^{t+h} \left| (T_m \widetilde{h}_k)(s) - (T_m \widetilde{h}_k)(t) \right|^2 \mathrm{d}s\, (2k)^{-q} \right)^{\frac{1}{2}} \cdot \left(\sum_{\alpha \in \ell} (2\mathbb{N})^{q\alpha} \left| z^{\alpha} \right|^2 \right)^{\frac{1}{2}},
\end{aligned}$$

and therefore

$$\begin{aligned}
&\left| \frac{\mathbf{I}(X_m(t+h))(z) - \mathbf{I}(X_m(t))(z)}{h} - \mathbf{I}(W_m(t))(z) \right| \\
&\leq \frac{1}{|h|} \left(\sum_{k=1}^{\infty} \int_t^{t+h} |t-s|^2 \,\mathrm{d}s \left\{ C_1 k^{\frac{N+2}{2}} + C_2 \right\}^2 (2k)^{-q} \right)^{\frac{1}{2}} \times \left(\sum_{\alpha \in \ell} (2\mathbb{N})^{q\alpha} \left| z^{\alpha} \right|^2 \right)^{\frac{1}{2}}
\end{aligned}$$

$$\leq \frac{|h|^{\frac{3}{2}}}{\sqrt{3}\,|h|}\left(\sum_{k=1}^{\infty}\left\{C_1 k^{\frac{N+2}{2}}+C_2\right\}^2(2k)^{-q}\right)^{\frac{1}{2}}$$
$$\times\left(\sum_{\alpha\in\ell}(2\mathbb{N})^{q\alpha}\left|z^{\alpha}\right|^2\right)^{\frac{1}{2}}\longrightarrow 0,\quad \text{as } |h|\to 0.\quad\square$$

We are now ready for the main result of this section:

Theorem 7.7. *Assume that the spectral density m satisfies (3.1) and (1.7). Then, it holds that*

$$\frac{\mathrm{d}X_m(t)}{\mathrm{d}t}=W_m(t)$$

in the sense that

$$\frac{\mathrm{d}(\mathbf{I}(X_m(t))(z))}{\mathrm{d}t}=\mathbf{I}(W_m(t))(z)$$

for all $t\in\mathbb{R}$, pointwise boundedly.

Proof. Taking in Proposition 7.5, $X(t,\omega)=X_m(t)$, $F(t,\omega)=W_m(t)$ we have, due to Lemma 7.6,

$$\frac{\mathrm{d}(\mathbf{I}(X_m(t))(z))}{\mathrm{d}t}=\mathbf{I}(W_m(t)(z))$$

for all $(t,z)\in\mathbb{R}\times K_{N+4}(\delta)$, and by Proposition 7.3, $\mathbf{I}(W_m(t))(z)$ is a bounded function for all $(t,z)\in\mathbb{R}\times K_{N+3}(\delta)$, then for all $(t,z)\in\mathbb{R}\times K_{N+4}(\delta)$ the pair $(\mathbf{I}(X_m(t))(z),\mathbf{I}(W_m(t)))(z)$ satisfies the condition of Proposition 7.5. We therefore may conclude that $X_m(t)$ is a differentiable S_{-1} process, which completes the proof. $\square$

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