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Cubic structures applied to ideals of BCI-algebras

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ABSTRACT

The notions of cubic *a*-ideals and cubic *p*-ideals are introduced, and several related properties are investigated. Characterizations of a cubic *a*-ideal are established. Relations between cubic *p*-ideals, cubic *a*-ideals and cubic *q*-ideals are discussed. The cubic extension property of a cubic *a*-ideal is discussed.

1. Introduction

The study of BCK/BCI-algebras was initiated by Iséki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Since then, a large volume of literature has been produced on the theory of BCK/BCI-algebras, in particular, emphasis has been given to the ideal theory of BCK/BCI-algebras. Fuzzy sets, which were introduced by Zadeh [1], deal with possibilistic uncertainty, connected with imprecision of states, perceptions and preferences. Based on the (interval-valued) fuzzy sets, Jun et al. [2] introduced the notion of (internal, external) cubic sets, and investigated several properties. Jun et al. applied the notion of cubic sets to BCK/BCI-algebras. They introduced the notions of cubic subalgebras/ideals, cubic $\circ$-subalgebras and closed cubic ideals in BCK/BCI-algebras, and then they investigated several properties (see [3–5]).

In this paper, we introduce the notion of cubic *p*-ideals and cubic *a*-ideals in BCI-algebras. We discuss the relationship between a cubic ideal, a cubic *q*-ideal, a cubic *p*-ideal and a cubic *a*-ideal. We consider characterizations of a cubic *a*-ideal. We provide conditions for a cubic ideal to be a cubic *p*-ideal. We establish a cubic extension property of a cubic *a*-ideal.

2. Preliminaries

In this section, we include some elementary aspects that are necessary for this paper.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a BCI-*algebra* if it satisfies the following axioms:

(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) $x * y = 0, y * x = 0 \Rightarrow x = y$

where x, y and z are elements of X. If a BCI-algebra, X satisfies the following identity:

(V) $0 * x = 0$ for all $x \in X$,

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then X is called a BCK-*algebra*. Any BCI-algebra X satisfies the following conditions:

(a1) $x * 0 = x$,
(a2) $x * y = 0 \Rightarrow (x * z) * (y * z) = 0,\ (z * y) * (z * x) = 0$,
(a3) $(x * y) * z = (x * z) * y$,
(a4) $((x * z) * (y * z)) * (x * y) = 0$,
(a5) $x * (x * (x * y)) = x * y$,
(a6) $0 * (x * y) = (0 * x) * (0 * y)$,

where x, y and z are elements of X. We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$. A BCK-algebra X is said to be with *condition* (*S*) if for all $x, y \in X$, the set $\{z \in X \mid z*x \leq y\}$ has the greatest element, written $x \circ y$. A BCI-algebra X is said to be *p-semisimple* if its BCK-part is equal to $\{0\}$. In a p-semisimple BCI-algebra, the following conditions are valid:

(a7) $0 * (x * y) = y * x$,
(a8) $x * (x * y) = y$,

where x and y are elements of X. A BCI-algebra X is called a *weakly* BCK-*algebra* if $0 * x \leq x$ for all $x \in X$. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies the following conditions:

(b1) $0 \in I$,
(b2) $(\forall x, y \in X)\ (x * y \in I,\ y \in I \Rightarrow x \in I)$.

A subset I of a BCI-algebra X is called a *q-ideal* of X (see [6]) if it satisfies (b1) and

(b3) $(\forall x, y, z \in X)\ (x * (y * z) \in I,\ y \in I \Rightarrow x * z \in I)$.

A subset I of a BCI-algebra X is called an *a-ideal* of X (see [6]) if it satisfies (b1) and

(b4) $(\forall x, y, z \in X)\ ((x * z) * (0 * y) \in I,\ z \in I \Rightarrow y * x \in I)$.

We refer the reader to the books [7,8] and the paper [9] for further information regarding BCK/BCI-algebras.

Let I be a closed unit interval, i.e., $I = [0, 1]$. By an *interval number*, we mean a closed subinterval $\bar{a} = [a^-, a^+]$ of I, where $0 \leq a^- \leq a^+ \leq 1$. Denote by $D[0, 1]$ the set of all interval numbers. Let us define what is known as *refined minimum* (briefly, rmin) of two elements in $D[0, 1]$. We also define the symbols "$\succeq$", "$\preceq$", "$=$" in case of two elements in $D[0, 1]$. Consider two interval numbers $\bar{a}_1 := [a_1^-, a_1^+]$ and $\bar{a}_2 := [a_2^-, a_2^+]$. Then

$$\text{rmin}\,\{\bar{a}_1, \bar{a}_2\} = [\min\{a_1^-, a_2^-\}, \min\{a_1^+, a_2^+\}], \quad \bar{a}_1 \succeq \bar{a}_2 \text{ if and only if } a_1^- \geq a_2^- \text{ and } a_1^+ \geq a_2^+,$$

and similarly, we may have $\bar{a}_1 \preceq \bar{a}_2$ and $\bar{a}_1 = \bar{a}_2$. To say $\bar{a}_1 \succ \bar{a}_2$ (resp. $\bar{a}_1 \prec \bar{a}_2$) we mean $\bar{a}_1 \succeq \bar{a}_2$ and $\bar{a}_1 \neq \bar{a}_2$ (resp. $\bar{a}_1 \preceq \bar{a}_2$ and $\bar{a}_1 \neq \bar{a}_2$). Let $\bar{a}_i \in D[0, 1]$, where $i \in \Lambda$. We define

$$\underset{i\in\Lambda}{\text{rinf}}\ \bar{a}_i = \left[\inf_{i\in\Lambda} a_i^-, \inf_{i\in\Lambda} a_i^+\right] \quad \text{and} \quad \underset{i\in\Lambda}{\text{rsup}}\ \bar{a}_i = \left[\sup_{i\in\Lambda} a_i^-, \sup_{i\in\Lambda} a_i^+\right].$$

An *interval-valued fuzzy set* (briefly, *IVF set*) $\tilde{\mu}_A$ defined on a nonempty set X is given by

$$\tilde{\mu}_A := \{(x, [\mu_A^-(x), \mu_A^+(x)]) \mid x \in X\},$$

which is briefly denoted by $\tilde{\mu}_A = [\mu_A^-, \mu_A^+]$ where μ_A^- and μ_A^+ are two fuzzy sets in X such that $\mu_A^-(x) \leq \mu_A^+(x)$ for all $x \in X$. For any IVF set $\tilde{\mu}_A$ on X and $x \in X$, $\tilde{\mu}_A(x) = [\mu_A^-(x), \mu_A^+(x)]$ is called the degree of membership of an element x to $\tilde{\mu}_A$, in which $\mu_A^-(x)$ and $\mu_A^+(x)$ are referred to as the lower and upper degrees, respectively, of membership of x to $\tilde{\mu}_A$.

3. Cubic *p*-ideals

Definition 3.1 (*[3]*). Let X be a nonempty set. A *cubic set* $\mathscr{A}$ in X is a structure

$$\mathscr{A} = \{\langle x, \tilde{\mu}_A(x), \lambda(x)\rangle : x \in X\}$$

which is briefly denoted by $\mathscr{A} = \langle \tilde{\mu}_A, \lambda\rangle$ where $\tilde{\mu}_A = [\mu_A^-, \mu_A^+]$ is an IVF set in X and λ is a fuzzy set in X.

Definition 3.2 (*[3]*). A cubic set $\mathscr{A} = \langle \tilde{\mu}_A, \lambda\rangle$ in X is called a *cubic subalgebra* of a BCK/BCI-algebra X if it satisfies: for all $x, y \in X$,

(a) $\tilde{\mu}_A(x * y) \succeq \text{rmin}\,\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$.
(b) $\lambda(x * y) \leq \max\{\lambda(x), \lambda(y)\}$.

Definition 3.3 (*[3]*). A cubic set $\mathscr{A} = \langle \tilde{\mu}_A, \lambda\rangle$ in a BCK/BCI-algebra X is called a *cubic ideal* of X if it satisfies: for all $x, y \in X$,

(a) $\tilde{\mu}_A(0) \succeq \tilde{\mu}_A(x)$.
(b) $\lambda(0) \leq \lambda(x)$.

Table 1
$*$-operation.

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

(c) $\tilde{\mu}_A(x) \succeq \text{rmin}\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(y)\}$.
(d) $\lambda(x) \leq \max\{\lambda(x*y), \lambda(y)\}$.

In what follows, let X denote a BCI-algebra unless otherwise specified.

Definition 3.4 ([5]). A cubic ideal $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ of X is said to be *closed* if $\tilde{\mu}_A(0*x) \succeq \tilde{\mu}_A(x)$ and $\lambda(0*x) \leq \lambda(x)$ for all $x \in X$.

Definition 3.5. A cubic set $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ in X is called a *cubic p-ideal* of X if it satisfies conditions (a) and (b) in Definition 3.3 and for all $x, y, z \in X$,
(a) $\tilde{\mu}_A(x) \succeq \text{rmin}\{\tilde{\mu}_A((x*z)*(y*z)), \tilde{\mu}_A(y)\}$.
(b) $\lambda(x) \leq \max\{\lambda((x*z)*(y*z)), \lambda(y)\}$.

Example 3.6. Consider a BCI-algebra $X = \{0, a, b, c\}$ in which the $*$-operation is given by Table 1. We define $\tilde{\mu}_A = [\mu_A^-, \mu_A^+]$ and λ by

$$\tilde{\mu}_A = \begin{pmatrix} 0 & a & b & c \\ [0.5, 0.9] & [0.4, 0.8] & [0.3, 0.5] & [0.3, 0.5] \end{pmatrix}$$

and

$$\lambda = \begin{pmatrix} 0 & a & b & c \\ 0.3 & 0.4 & 0.7 & 0.7 \end{pmatrix}.$$

Then $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic p-ideal of X.

If we put $z = x$ and $y = 0$ in Definition 3.5, then we have the following proposition.

Proposition 3.7. *Every cubic p-ideal $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ of X satisfies the following inequalities:*

$$(\forall x \in X)\ (\tilde{\mu}_A(x) \succeq \tilde{\mu}_A(0*(0*x)),\ \lambda(x) \leq \lambda(0*(0*x))). \tag{3.1}$$

If we put $z = 0$ in Definition 3.5 and use (a1), then we have the following theorem.

Theorem 3.8. *Every cubic p-ideal is a cubic ideal.*

The converse of Theorem 3.8 may not be true as seen in the following example.

Example 3.9. Let $X = \{0, a, 1, 2, 3\}$ be a set with the Cayley table given by Table 2. We define $\tilde{\mu}_A = [\mu_A^-, \mu_A^+]$ and λ by

$$\tilde{\mu}_A = \begin{pmatrix} 0 & a & 1 & 2 & 3 \\ [0.4, 0.8] & [0.3, 0.6] & [0.1, 0.4] & [0.1, 0.4] & [0.1, 0.4] \end{pmatrix}$$

and

$$\lambda = \begin{pmatrix} 0 & a & 1 & 2 & 3 \\ 0.2 & 0.5 & 0.6 & 0.6 & 0.6 \end{pmatrix}.$$

Then $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic ideal of X. But it is not a cubic p-ideal of X, since

$$\tilde{\mu}_A(a) = [0.3, 0.6] \not\succeq [0.4, 0.8] = \text{rmin}\{\tilde{\mu}_A((a*1)*(0*1)), \tilde{\mu}_A(0)\}$$

and/or $\lambda(a) = 0.5 \not\leq 0.2 = \max\{\lambda((a*1)*(0*1)), \lambda(0)\}$.

Proposition 3.10. *If $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic p-ideal of X, then*

$$\tilde{\mu}_A(x*y) \preceq \tilde{\mu}_A((x*z)*(y*z)) \quad \textit{and} \quad \lambda(x*y) \geq \lambda((x*z)*(y*z))$$

for all $x, y, z \in X$.

Proof. Let $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ be a cubic p-ideal of X. Note that $(x*z)*(y*z) \leq x*y$, i.e., $((x*z)*(y*z))*(x*y) = 0$, for all $x, y, z \in X$. Since $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic ideal of X, by Theorem 3.8, it follows that

$$\begin{aligned} \tilde{\mu}_A((x*z)*(y*z)) &\succeq \text{rmin}\{\tilde{\mu}_A(((x*z)*(y*z))*(x*y)), \tilde{\mu}_A(x*y)\} \\ &= \text{rmin}\{\tilde{\mu}_A(0), \tilde{\mu}_A(x*y)\} = \tilde{\mu}_A(x*y) \end{aligned}$$

Table 2
$*$-operation.

$*$	0	a	1	2	3
0	0	0	3	2	1
a	a	0	3	2	1
1	1	1	0	3	2
2	2	2	1	0	3
3	3	3	2	1	0

and

$$\begin{aligned}\lambda((x*z)*(y*z)) &\le \max\{\lambda(((x*z)*(y*z))*(x*y)), \lambda(x*y)\}\\ &= \max\{\lambda(0), \lambda(x*y)\} = \lambda(x*y)\end{aligned}$$

for all $x, y, z \in X$. This completes the proof. □

We provide conditions for a cubic ideal to be a cubic p-ideal.

Theorem 3.11. *Let $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ be a cubic ideal of X that satisfies:*

$$\tilde{\mu}_A(x*y) \succeq \tilde{\mu}_A((x*z)*(y*z)) \quad \text{and} \quad \lambda(x*y) \le \lambda((x*z)*(y*z))$$

for all $x, y, z \in X$. Then $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic p-ideal of X.

Proof. For any $x, y, z \in X$, we have

$$\tilde{\mu}_A(x) \succeq \text{rmin}\,\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(y)\} \succeq \text{rmin}\,\{\tilde{\mu}_A((x*z)*(y*z)), \tilde{\mu}_A(y)\}$$

and $\lambda(x) \le \max\{\lambda(x*y), \lambda(y)\} \le \max\{\lambda((x*z)*(y*z)), \lambda(y)\}$. This completes the proof. □

Lemma 3.12. *Every cubic ideal $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ satisfies the following inequalities:*

$$\tilde{\mu}_A(0*(0*x)) \succeq \tilde{\mu}_A(x) \quad \text{and} \quad \lambda(0*(0*x)) \le \lambda(x)$$

for all $x \in X$.

Proof. For any $x \in X$, we have

$$\begin{aligned}\tilde{\mu}_A(x) &= \text{rmin}\,\{\tilde{\mu}_A(0), \tilde{\mu}_A(x)\}\\ &= \text{rmin}\,\{\tilde{\mu}_A(0*(0*x)), \tilde{\mu}_A(x)\}\\ &\preceq \tilde{\mu}_A(0*(0*x))\end{aligned}$$

and $\lambda(x) = \max\{\lambda(0), \lambda(x)\} = \max\{\lambda(0*(0*x)), \lambda(x)\} \ge \lambda(0*(0*x))$. □

Lemma 3.13 ([10]). *Let X be a BCI-algebra. Then*
(1) $0*(0*((x*z)*(y*z))) = (0*y)*(0*x)$,
(2) $0*(0*(x*y)) = (0*y)*(0*x)$
for all $x, y, z \in X$.

Theorem 3.14. *Let $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ be a cubic ideal of X that satisfies:*

$$\tilde{\mu}_A(0*(0*x)) \preceq \tilde{\mu}_A(x) \quad \text{and} \quad \lambda(0*(0*x)) \ge \lambda(x)$$

for all $x \in X$. Then $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic p-ideal of X.

Proof. Let $x, y, z \in X$. Using Lemmas 3.12 and 3.13, we have

$$\begin{aligned}\tilde{\mu}_A((x*z)*(y*z)) &\preceq \tilde{\mu}_A(0*(0*((x*z)*(y*z))))\\ &= \tilde{\mu}_A((0*y)*(0*x))\\ &= \tilde{\mu}_A(0*(0*(x*y)))\\ &\preceq \tilde{\mu}_A(x*y)\end{aligned}$$

and

$$\begin{aligned}\lambda((x*z)*(y*z)) &\ge \lambda(0*(0*((x*z)*(y*z))))\\ &= \lambda((0*y)*(0*x))\\ &= \lambda(0*(0*(x*y)))\\ &\ge \lambda(x*y).\end{aligned}$$

It follows from Theorem 3.11 that $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic p-ideal of X. □

Table 3
$*$-operation.

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

4. Cubic *a*-ideals

Definition 4.1 ([4]). A cubic set $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ in X is called a *cubic q-ideal* of X if it satisfies conditions (a) and (b) in Definition 3.3 and for all $x, y, z \in X$,

(a) $\tilde{\mu}_A(x * z) \succeq \text{rmin}\,\{\tilde{\mu}_A(x * (y * z)), \tilde{\mu}_A(y)\}$.
(b) $\lambda(x * z) \leq \max\{\lambda(x * (y * z)), \lambda(y)\}$.

Definition 4.2. A cubic set $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ in X is called a *cubic a-ideal* of X if it satisfies conditions (a) and (b) in Definition 3.3 and for all $x, y, z \in X$,

(a) $\tilde{\mu}_A(y * x) \succeq \text{rmin}\,\{\tilde{\mu}_A((x * z) * (0 * y)), \tilde{\mu}_A(z)\}$.
(b) $\lambda(y * x) \leq \max\{\lambda((x * z) * (0 * y)), \lambda(z)\}$.

Example 4.3. Consider a BCI-algebra $X = \{0, a, b, c\}$ in which the $*$-operation is given by Table 3. We define $\tilde{\mu}_A = [\mu_A^-, \mu_A^+]$ and λ by

$$\tilde{\mu}_A = \begin{pmatrix} 0 & a & b & c \\ [0.4, 0.8] & [0.4, 0.8] & [0.2, 0.5] & [0.2, 0.5] \end{pmatrix}$$

and

$$\lambda = \begin{pmatrix} 0 & a & b & c \\ 0.3 & 0.3 & 0.6 & 0.6 \end{pmatrix}.$$

Then $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic a-ideal of X.

Theorem 4.4. *Every cubic a-ideal is a closed cubic ideal.*

Proof. Let $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ be a cubic a-ideal of X. Putting $z = y = 0$ in Definition 4.2 and using (a1), Definition 3.3(a) and (b), we have

$$\begin{aligned} &\tilde{\mu}_A(0 * x) \succeq \text{rmin}\,\{\tilde{\mu}_A((x * 0) * (0 * 0)), \tilde{\mu}_A(0)\} = \tilde{\mu}_A(x) \\ &\lambda(0 * x) \leq \max\{\lambda((x * 0) * (0 * 0)), \lambda(0)\} = \lambda(x) \end{aligned} \tag{4.1}$$

for all $x \in X$. If we take $x = z = 0$ in Definition 4.2 and use (a1), Definition 3.3(a) and (b), then

$$\begin{aligned} &\tilde{\mu}_A(y) \succeq \text{rmin}\,\{\tilde{\mu}_A(0 * (0 * y)), \tilde{\mu}_A(0)\} = \tilde{\mu}_A(0 * (0 * y)) \\ &\lambda(y) \leq \max\{\lambda(0 * (0 * y)), \lambda(0)\} = \lambda(0 * (0 * y)) \end{aligned} \tag{4.2}$$

for all $y \in X$. It follows from (4.1) that

$$\tilde{\mu}_A(x) \succeq \tilde{\mu}_A(0 * x) \quad \text{and} \quad \lambda(x) \leq \lambda(0 * x)$$

for all $x \in X$; so from Definition 4.2, that

$$\begin{aligned} \tilde{\mu}_A(x) &\succeq \tilde{\mu}_A(0 * x) \succeq \text{rmin}\,\{\tilde{\mu}_A((x * z) * (0 * 0)), \tilde{\mu}_A(z)\} \\ &= \text{rmin}\,\{\tilde{\mu}_A(x * z), \tilde{\mu}_A(z)\} \end{aligned}$$

$$\begin{aligned} \lambda(x) &\leq \lambda(0 * x) \leq \max\{\lambda((x * z) * (0 * 0)), \lambda(z)\} \\ &= \max\{\lambda(x * z), \lambda(z)\} \end{aligned}$$

for all $x, z \in X$. Therefore $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a closed cubic ideal of X. □

The following example shows that the converse of Theorem 4.4 may not be true.

Example 4.5. Consider a BCI-algebra $X = \{0, a, b\}$ in which the $*$-operation is given by Table 4. We define $\tilde{\mu}_A = [\mu_A^-, \mu_A^+]$ and λ by

$$\tilde{\mu}_A = \begin{pmatrix} 0 & a & b \\ [0.4, 0.8] & [0.2, 0.5] & [0.2, 0.5] \end{pmatrix}$$

Table 4
$*$-operation.

$*$	0	a	b
0	0	b	a
a	a	0	b
b	b	a	0

and

$$\lambda = \begin{pmatrix} 0 & a & b \\ 0.3 & 0.6 & 0.6 \end{pmatrix}.$$

Then $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a closed cubic ideal of X. But it is not a cubic a-ideal of X, since

$$\tilde{\mu}_A(b * a) = \tilde{\mu}_A(a) \prec \text{rmin}\,\{\tilde{\mu}_A((a * 0) * (0 * b)), \tilde{\mu}_A(0)\}$$

and/or $\lambda(a * b) > \max\{\lambda((b * 0) * (0 * a)), \lambda(0)\}$.

We provide characterizations of a cubic a-ideal.

Lemma 4.6 ([3]). *Let $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ be a cubic ideal of X. If the inequality $x * y \le z$ holds in X, then*

$$\tilde{\mu}_A(x) \succeq \text{rmin}\,\{\tilde{\mu}_A(y), \tilde{\mu}_A(z)\} \quad \textit{and} \quad \lambda(x) \le \max\{\lambda(y), \lambda(z)\}.$$

Theorem 4.7. *If $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic ideal of X, then the following are equivalent:*

(1) *$\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic a-ideal of X.*
(2) *$\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ satisfies the following inequalities:*

$$\begin{aligned} &\tilde{\mu}_A(y * (x * z)) \succeq \tilde{\mu}_A((x * z) * (0 * y)) \\ &\lambda(y * (x * z)) \le \lambda((x * z) * (0 * y)) \end{aligned} \tag{4.3}$$

for all $x, y, z \in X$.
(3) *$\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ satisfies the following inequalities:*

$$\tilde{\mu}_A(y * x) \succeq \tilde{\mu}_A(x * (0 * y)), \qquad \lambda(y * x) \le \lambda(x * (0 * y)) \tag{4.4}$$

for all $x, y \in X$.

Proof. Assume that $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic a-ideal of X. Then

$$\begin{aligned} \tilde{\mu}_A(y * (x * z)) &\succeq \text{rmin}\,\{\tilde{\mu}_A(((x * z) * 0) * (0 * y)), \tilde{\mu}_A(0)\} \\ &= \tilde{\mu}_A(((x * z) * 0) * (0 * y)) \\ &= \tilde{\mu}_A((x * z) * (0 * y)) \end{aligned}$$

and

$$\begin{aligned} \lambda(y * (x * z)) &\le \max\{\lambda(((x * z) * 0) * (0 * y)), \lambda(0)\} \\ &= \lambda(((x * z) * 0) * (0 * y)) \\ &= \lambda((x * z) * (0 * y)) \end{aligned}$$

for all $x, y, z \in X$, and so (4.3) is valid. (4.4) is induced by taking $z = 0$ in (4.3) and using (a1). Suppose that $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ satisfies (4.4). Note that

$$(x * (0 * y)) * ((x * z) * (0 * y)) \le x * (x * z) \le x$$

for all $x, y, z \in X$. It follows from (4.4) and Lemma 4.6 that

$$\tilde{\mu}_A(y * x) \succeq \tilde{\mu}_A(x * (0 * y)) \succeq \text{rmin}\,\{\tilde{\mu}_A((x * z) * (0 * y)), \tilde{\mu}_A(x)\}$$

and $\lambda(y * x) \le \lambda(x * (0 * y)) \le \max\{\lambda((x * z) * (0 * y)), \lambda(x)\}$ for all $x, y, z \in X$. Therefore, $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic a-ideal of X. □

Let $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ be a cubic set in X. For any $r \in [0, 1]$ and $[s, t] \in D[0, 1]$, we define $U(\mathscr{A}; [s, t], r)$ as follows:

$$U(\mathscr{A}; [s, t], r) = \{x \in X \mid \tilde{\mu}_A(x) \succeq [s, t],\ \lambda(x) \le r\},$$

and we can say that it is a *cubic level set* of $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$.

Lemma 4.8 ([5]). *For a cubic set $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ in X, the following are equivalent:*

(1) $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ *is a cubic ideal of* X.
(2) *Every nonempty cubic level set of* $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ *is an ideal of* X.

Lemma 4.9 ([6]). *A subset I of X is an a-ideal of X if and only if it is an ideal of X which satisfies the following implication:*

$$(\forall x, y \in X)\ (x * (0 * y) \in I \ \Rightarrow\ y * x \in I).$$

Theorem 4.10. *For a cubic set $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ in X, the following are equivalent:*

(1) $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ *is a cubic a-ideal of* X.
(2) *Every nonempty cubic level set of* $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ *is an a-ideal of* X.

Proof. Assume that $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic a-ideal of X. Then $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic ideal of X by Theorem 4.4. Hence every nonempty cubic level set of $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is an ideal of X by Lemma 4.8. Let $[s, t] \in D[0, 1]$ and $r \in [0, 1]$ be such that $U(\mathscr{A}; [s, t], r) \neq \emptyset$. Let $x, y \in X$ be such that $x * (0 * y) \in U(\mathscr{A}; [s, t], r)$. Then $\tilde{\mu}_A(x * (0 * y)) \succeq [s, t]$ and $\lambda(x * (0 * y)) \leq r$. It follows from (4.4) that

$$\tilde{\mu}_A(y * x) \succeq \tilde{\mu}_A(x * (0 * y)) \succeq [s, t]$$

and $\lambda(y * x) \leq \lambda(x * (0 * y)) \leq r$ so that $y * x \in U(\mathscr{A}; [s, t], r)$. Using Lemma 4.9, we conclude that $U(\mathscr{A}; [s, t], r)$ is an a-ideal of X.

Conversely, suppose that (2) is valid, that is, $U(\mathscr{A}; [s, t], r)$ is nonempty and is an a-ideal of X for all $r \in [0, 1]$ and $[s, t] \in D[0, 1]$. Since any a-ideal is an ideal (see [6]), it follows from Lemma 4.8 that $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic ideal of X. Assume that two inequalities in (4.4) are false. Then there exist $a, b \in X$ such that $\tilde{\mu}_A(b * a) \prec \tilde{\mu}_A(a * (0 * b))$ and $\lambda(b * a) > \lambda(a * (0 * b))$. Thus $\tilde{\mu}_A(b * a) \prec [s_0, t_0] \preceq \tilde{\mu}_A(a * (0 * b))$ and $\lambda(b * a) > r_0 \geq \lambda(a * (0 * b))$ for some $[s_0, t_0] \in D[0, 1]$ and $r_0 \in [0, 1]$. It follows that $a * (0 * b) \in U(\mathscr{A}; [s_0, t_0], r_0)$ but $b * a \notin U(\mathscr{A}; [s_0, t_0], r_0)$. This is a contradiction. Suppose that

$$\tilde{\mu}_A(y * x) \succeq \tilde{\mu}_A(x * (0 * y))$$

for all $x, y \in X$, and there exist $a, b \in X$ such that $\lambda(b * a) > \lambda(a * (0 * b))$. Then $\lambda(b * a) > r_0 \geq \lambda(a * (0 * b))$ for some $r_0 \in [0, 1]$, and so $a * (0 * b) \in U(\mathscr{A}; \tilde{\mu}_A(a * (0 * b)), r_0)$ but $b * a \notin U(\mathscr{A}; \tilde{\mu}_A(a * (0 * b)), r_0)$. This is also a contradiction. For the case that $\lambda(y * x) \leq \lambda(x * (0 * y))$ for all $x, y \in X$ and $\tilde{\mu}_A(b * a) \prec \tilde{\mu}_A(a * (0 * b))$ for some $a, b \in X$, we can induce a contradiction. Therefore, (4.4) is valid, which implies from Theorem 4.7 that $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic a-ideal of X. □

Theorem 4.10 combines with (a) and (b) of Definition 3.3 to induce the following corollary.

Corollary 4.11. *If $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic a-ideal of X, then the set*

$$I := \{x \in X \mid \tilde{\mu}_A(x) = \tilde{\mu}_A(0),\ \lambda(x) = \lambda(0)\}$$

is an a-ideal of X.

Theorem 4.12. *Every cubic a-ideal is a cubic p-ideal.*

Proof. Let $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ be a cubic a-ideal of X. Then $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic ideal of X (see Theorem 4.4). If we take $x = z = 0$ in (4.3), then $\tilde{\mu}_A(0 * (0 * y)) \preceq \tilde{\mu}_A(y)$ and $\lambda(0 * (0 * y)) \geq \lambda(y)$ for all $y \in X$. Hence, by Theorem 3.14, we conclude that $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic p-ideal of X. □

Note that the cubic set $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ in Example 4.5 is a cubic p-ideal which is not a cubic a-ideal. Hence the converse of Theorem 4.12 is not true in general.

Lemma 4.13 ([4]). *For a cubic ideal $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ of X, the following are equivalent:*

(1) $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ *is a cubic q-ideal of* X.
(2) $\tilde{\mu}_A(x * y) \succeq \tilde{\mu}_A(x * (0 * y))$ *and* $\lambda(x * y) \leq \lambda(x * (0 * y))$ *for all* $x, y \in X$.

Theorem 4.14. *Every cubic a-ideal is a cubic q-ideal.*

Proof. Let $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ be a cubic a-ideal of X. Then $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic ideal of X by Theorem 4.4. Note that

$$\begin{aligned}(0 * (0 * (y * (0 * x)))) * (x * (0 * y)) &= ((0 * (0 * y)) * (0 * (0 * (0 * x)))) * (x * (0 * y)) \\ &= ((0 * (0 * y)) * (0 * x)) * (x * (0 * y)) \\ &\leq (x * (0 * y)) * (x * (0 * y)) = 0\end{aligned}$$

Table 5
$*$-operation.

$*$	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

for all $x, y \in X$. Since $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic p-ideal of X (see Theorem 4.12), it follows from (4.4), Proposition 3.10 and Lemma 4.6 that

$$\begin{aligned} \tilde{\mu}_A(x*y) &\succeq \tilde{\mu}_A(y*(0*x)) \succeq \tilde{\mu}_A(0*(0*(y*(0*x)))) \\ &\succeq \mathrm{rmin}\,\{\tilde{\mu}_A(x*(0*y)), \tilde{\mu}_A(0)\} \\ &= \tilde{\mu}_A(x*(0*y)) \end{aligned}$$

and

$$\begin{aligned} \lambda(x*y) &\geq \lambda(y*(0*x)) \geq \lambda(0*(0*(y*(0*x)))) \\ &\geq \max\{\lambda(x*(0*y)), \lambda(0)\} \\ &= \lambda(x*(0*y)) \end{aligned}$$

for all $x, y \in X$. Using Lemma 4.13, we conclude that $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ be a cubic q-ideal of X. $\square$

The following example shows that the converse of Theorem 4.14 may not be true.

Example 4.15. Consider a BCI-algebra $X = \{0, a, b\}$ with the $*$-operation which is given in Table 5. We define $\tilde{\mu}_A = [\mu_A^-, \mu_A^+]$ and λ by

$$\tilde{\mu}_A = \begin{pmatrix} 0 & a & b \\ [0.5, 0.8] & [0.3, 0.6] & [0.3, 0.6] \end{pmatrix}$$

and

$$\lambda = \begin{pmatrix} 0 & a & b \\ 0.3 & 0.6 & 0.6 \end{pmatrix}.$$

Then $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic q-ideal of X. But it is not a cubic a-ideal of X since $\tilde{\mu}_A(a*0) \not\succeq \mathrm{rmin}\,\{\tilde{\mu}_A((0*0)*(0*a)), \tilde{\mu}_A(0)\}$.

Lemma 4.16 ([3]). *Let $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ be a cubic ideal of X. If the inequality $x \leq y$ holds in X, then $\tilde{\mu}_A(x) \succeq \tilde{\mu}_A(y)$ and $\lambda(x) \leq \lambda(y)$.*

Theorem 4.17. *For a cubic set $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ in X, the following are equivalent.*

(1) *$\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic a-ideal of X.*
(2) *$\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is both a cubic p-ideal and a cubic q-ideal of X.*

Proof. By means of Theorems 4.12 and 4.14, every cubic a-ideal is both a cubic p-ideal and a cubic q-ideal.

Conversely, let $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ be both a cubic p-ideal and a cubic q-ideal. Note that $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic ideal of X (see [4]). Taking $z = y$ at (a) and (b) in Definition 4.1, we have $\tilde{\mu}_A(x*y) \succeq \mathrm{rmin}\,\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$ and $\lambda(x*y) \leq \max\{\lambda(x), \lambda(y)\}$. Hence $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic subalgebra of X, and so $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a closed cubic ideal of X. Using Lemma 4.13, we get

$$\begin{aligned} &\tilde{\mu}_A(x*y) \succeq \tilde{\mu}_A(x*(0*y)) \\ &\lambda(x*y) \leq \lambda(x*(0*y)) \end{aligned} \tag{4.5}$$

for all $x, y \in X$. Since $0*(y*x) \leq x*y$ for all $x, y \in X$, it follows from Lemma 4.16, (4.5) that

$$\begin{aligned} &\tilde{\mu}_A(0*(y*x)) \succeq \tilde{\mu}_A(x*y) \succeq \tilde{\mu}_A(x*(0*y)) \\ &\lambda(0*(y*x)) \leq \lambda(x*y) \leq \lambda(x*(0*y)) \end{aligned} \tag{4.6}$$

for all $x, y \in X$. Using Proposition 3.7, Definition 3.4 and (4.6), we have

$$\tilde{\mu}_A(y*x) \succeq \tilde{\mu}_A(0*(0*(y*x))) \succeq \tilde{\mu}_A(0*(y*x)) \succeq \tilde{\mu}_A(x*(0*y))$$

and $\lambda(y*x) \leq \lambda(0*(0*(y*x))) \leq \lambda(0*(y*x)) \leq \lambda(x*(0*y))$ for all $x, y \in X$. It follows from Theorem 4.7 that $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic a-ideal of X. $\square$

Theorem 4.18 (*Cubic extension property for a cubic a-ideal*). *Let $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ and $\mathscr{B} = \langle \tilde{\mu}_B, \kappa \rangle$ be cubic ideals of X such that $\mathscr{A} \precsim \mathscr{B}$ and $\tilde{\mu}_A(0) = \tilde{\mu}_B(0)$ and $\lambda(0) = \kappa(0)$. If $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic a-ideal of X, then so is $\mathscr{B} = \langle \tilde{\mu}_B, \kappa \rangle$.*

Proof. Suppose that $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic a-ideal of X. Then $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is both a cubic p-ideal and a cubic q-ideal of X by Theorems 4.12 and 4.14. Using Lemma 4.13, (a3) and (III), we have

$$\begin{aligned} \tilde{\mu}_B((x*y)*(x*(0*y))) &= \tilde{\mu}_B((x*(x*(0*y)))*y) \\ &\succeq \tilde{\mu}_A((x*(x*(0*y)))*y) \succeq \tilde{\mu}_A((x*(x*(0*y)))*(0*y)) \\ &= \tilde{\mu}_A((x*(0*y))*(x*(0*y))) = \tilde{\mu}_A(0) = \tilde{\mu}_B(0) \\ &\succeq \tilde{\mu}_B(x*(0*y)) \end{aligned}$$

and

$$\begin{aligned} \kappa((x*y)*(x*(0*y))) &= \kappa((x*(x*(0*y)))*y) \\ &\leq \lambda((x*(x*(0*y)))*y) \leq \lambda((x*(x*(0*y)))*(0*y)) \\ &= \lambda((x*(0*y))*(x*(0*y))) = \lambda(0) = \kappa(0) \\ &\leq \kappa(x*(0*y)). \end{aligned}$$

Since $\mathscr{B} = \langle \tilde{\mu}_B, \kappa \rangle$ is a cubic ideal of X, we get

$$\begin{aligned} \tilde{\mu}_B(x*y) &\succeq \operatorname{rmin}\{\tilde{\mu}_B((x*y)*(x*(0*y))), \tilde{\mu}_B(x*(0*y))\} \\ &= \tilde{\mu}_B(x*(0*y)) \end{aligned}$$

and

$$\kappa(x*y) \leq \max\{\kappa((x*y)*(x*(0*y))), \kappa(x*(0*y))\} = \kappa(x*(0*y)).$$

Therefore $\mathscr{B} = \langle \tilde{\mu}_B, \kappa \rangle$ is a cubic q-ideal of X by Lemma 4.13. Since $\mathscr{A} = \langle \tilde{\mu}_A, \lambda \rangle$ is a cubic p-ideal of X, it follows from Proposition 3.7 that

$$\begin{aligned} \tilde{\mu}_B(x*(0*(0*x))) &\succeq \tilde{\mu}_A(x*(0*(0*x))) \succeq \tilde{\mu}_A(0*(0*(x*(0*(0*x))))) \\ &= \tilde{\mu}_A(0) = \tilde{\mu}_B(0) \succeq \tilde{\mu}_B(0*(0*x)) \end{aligned}$$

and

$$\begin{aligned} \kappa(x*(0*(0*x))) &\leq \lambda(x*(0*(0*x))) \\ &\leq \lambda(0*(0*(x*(0*(0*x))))) \\ &= \lambda(0) = \kappa(0) \leq \kappa(0*(0*x)). \end{aligned}$$

Hence

$$\tilde{\mu}_B(x) \succeq \operatorname{rmin}\{\tilde{\mu}_B(x*(0*(0*x))), \tilde{\mu}_B(0*(0*x))\} = \tilde{\mu}_B(0*(0*x))$$

and $\kappa(x) \leq \max\{\kappa(x*(0*(0*x))), \kappa(0*(0*x))\} = \kappa(0*(0*x))$. Using Theorem 3.14, we conclude that $\mathscr{B} = \langle \tilde{\mu}_B, \kappa \rangle$ is a cubic p-ideal of X. Therefore $\mathscr{B} = \langle \tilde{\mu}_B, \kappa \rangle$ is a cubic a-ideal of X by Theorem 4.17. □

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