Cubic structures applied to ideals of BCI-algebras
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A R T I C L E  I N F O

Article history:
Received 25 February 2011
Accepted 17 August 2011

Keywords:
(Cubic) subalgebra
(Cubic) ideal
(Cubic) q-ideal
(Cubic) a-ideal
(Cubic) p-ideal

A B S T R A C T

The notions of cubic a-ideals and cubic p-ideals are introduced, and several related properties are investigated. Characterizations of a cubic a-ideal are established. Relations between cubic p-ideals, cubic a-ideals and cubic q-ideals are discussed. The cubic extension property of a cubic a-ideal is discussed.

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doi:10.1016/j.camwa.2011.08.042

1. Introduction

The study of BCK/BCI-algebras was initiated by Iséki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Since then, a large volume of literature has been produced on the theory of BCK/BCI-algebras, in particular, emphasis has been given to the ideal theory of BCK/BCI-algebras. Fuzzy sets, which were introduced by Zadeh [1], deal with possibilistic uncertainty, connected with imprecision of states, perceptions and preferences. Based on the (interval-valued) fuzzy sets, Jun et al. [2] introduced the notion of (internal, external) cubic sets, and investigated several properties. Jun et al. applied the notion of cubic sets to BCK/BCI-algebras. They introduced the notions of cubic subalgebras/ideals, cubic a-subalgebras and closed cubic ideals in BCK/BCI-algebras, and then they investigated several properties (see [3–5]).

In this paper, we introduce the notion of cubic p-ideals and cubic a-ideals in BCI-algebras. We discuss the relationship between a cubic ideal, a cubic q-ideal, a cubic p-ideal and a cubic a-ideal. We consider characterizations of a cubic a-ideal. We provide conditions for a cubic ideal to be a cubic p-ideal. We establish a cubic extension property of a cubic a-ideal.

2. Preliminaries

In this section, we include some elementary aspects that are necessary for this paper.

An algebra (X; ∗, 0) of type (2, 0) is called a BCI-algebra if it satisfies the following axioms:

(I) (x ∗ y) ∗ (x ∗ z) ∗ (z ∗ y) = 0,
(II) (x ∗ (x ∗ y)) ∗ y = 0,
(III) x ∗ x = 0,
(IV) x ∗ y = 0, y ∗ x = 0 ⇒ x = y

where x, y and z are elements of X. If a BCI-algebra, X satisfies the following identity:
(V) 0 ∗ x = 0 for all x ∈ X,
then $X$ is called a BCK-algebra. Any BCI-algebra $X$ satisfies the following conditions:

(a1) $x \ast 0 = x$,

(a2) $x \ast y = 0 \Rightarrow (x \ast z) \ast (y \ast z) = 0$, $(z \ast y) \ast (z \ast x) = 0$,

(a3) $(x \ast y) \ast z = (x \ast z) \ast y$,

(a4) $(x \ast z) \ast (y \ast z) = (x \ast y) = 0$,

(a5) $x \ast (x \ast y) = x \ast y$,

(a6) $0 \ast (x \ast y) = (0 \ast x) \ast (0 \ast y)$,

where $x$, $y$, and $z$ are elements of $X$. We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x \ast y = 0$. A BCK-algebra $X$ is said to be with condition $(S)$ if for all $x, y \in X$, the set $[z \in X \mid z \ast x \leq y]$ has the greatest element, written $x \land y$. A BCI-algebra $X$ is said to be a $p$-semisimple if its BCP-part is equal to $\{0\}$. In a $p$-semisimple BCI-algebra, the following conditions are valid:

(a7) $0 \ast (x \ast y) = y \ast x$,

(a8) $x \ast (x \ast y) = y$,

where $x$ and $y$ are elements of $X$. A BCI-algebra $X$ is called a weakly BCK-algebra if $0 \ast x \leq x$ for all $x \in X$. A nonempty subset $S$ of a BCK/BCI-algebra $X$ is called a subalgebra of $X$ if $x \ast y \in S$ for all $x, y \in S$. A subset $I$ of a BCK/BCI-algebra $X$ is called an ideal of $X$ if it satisfies the following conditions:

(b1) $0 \in I$,

(b2) $(\forall x, y \in X) (x \ast y \in I \Rightarrow x \in I)$.

A subset $I$ of a BCI-algebra $X$ is called a q-ideal of $X$ (see [6]) if it satisfies (b1) and

(b3) $(\forall x, y, z \in X) (x \ast (y \ast z) \in I \Rightarrow x \ast z \in I)$.

A subset $I$ of a BCI-algebra $X$ is called an a-ideal of $X$ (see [6]) if it satisfies (b1) and

(b4) $(\forall x, y, z \in X) ((x \ast z) \ast (0 \ast y) \in I \Rightarrow z \in I \Rightarrow y \ast x \in I)$.

We refer the reader to the books [7,8] and the paper [9] for further information regarding BCK/BCI-algebras. Let $I$ be a closed unit interval, i.e., $I = [0, 1]$. By an interval number, we mean a closed subinterval $\bar{I} = [a^-, a^+]$ of $I$, where $0 \leq a^- \leq a^+ \leq 1$. Denote by $D[0, 1]$ the set of all interval numbers. Let us define what is known as refined minimum (briefly, rmin) of two elements in $D[0, 1]$. We also define the symbols $\geq n$, $\leq n$, $= n$ in case of two elements in $D[0, 1]$. Consider two interval numbers $\bar{a}_1 := [a^-_1, a^+_1]$ and $\bar{a}_2 := [a^-_2, a^+_2]$. Then

\[
\text{rmin} \{\bar{a}_1, \bar{a}_2\} = [\text{min}\{a^-_1, a^-_2\}, \text{min}\{a^+_1, a^+_2\}], \quad \bar{a}_1 \geq \bar{a}_2 \text{ if and only if } a^-_1 \geq a^-_2 \text{ and } a^+_1 \geq a^+_2,
\]

and similarly, we may have $\bar{a}_1 \leq \bar{a}_2$ and $\bar{a}_1 = \bar{a}_2$. To say $\bar{a}_1 > \bar{a}_2$ (resp. $\bar{a}_1 < \bar{a}_2$) we mean $\bar{a}_1 \geq \bar{a}_2$ and $\bar{a}_1 \neq \bar{a}_2$ (resp. $\bar{a}_1 \leq \bar{a}_2$ and $\bar{a}_1 \neq \bar{a}_2$). Let $\bar{a}_i \in D[0, 1], \ i \in \Lambda$, where $\Lambda$ is a set. We define

\[
r\inf \limits_{i \in \Lambda} \bar{a}_i = \left[\inf \limits_{i \in \Lambda} a^-_i, \inf \limits_{i \in \Lambda} a^+_i\right] \quad \text{and} \quad r\sup \limits_{i \in \Lambda} \bar{a}_i = \left[\sup \limits_{i \in \Lambda} a^-_i, \sup \limits_{i \in \Lambda} a^+_i\right].
\]

An interval-valued fuzzy set (briefly, IVF set) $\tilde{\mu}_A$ defined on a nonempty set $X$ is given by

\[
\tilde{\mu}_A := \{(x, [\mu^-_A(x), \mu^+_A(x)]) \mid x \in X\},
\]

which is briefly denoted by $\tilde{\mu}_A = ([\mu^-_A, \mu^+_A]$ where $\mu^-_A$ and $\mu^+_A$ are two fuzzy sets in $X$ such that $\mu^-_A(x) \leq \mu^+_A(x)$ for all $x \in X$. For any IVF set $\tilde{\mu}_A$ on $X$ and $x \in X$, $\tilde{\mu}_A(x) = [\mu^-_A(x), \mu^+_A(x)]$ is called the degree of membership of an element $x$ to $\tilde{\mu}_A$, in which $\mu^-_A(x)$ and $\mu^+_A(x)$ are referred to as the lower and upper degrees, respectively, of membership of $x$ to $\tilde{\mu}_A$.  

3. Cubic p-ideals

**Definition 3.1** ([3]). Let $X$ be a nonempty set. A cubic set $\mathcal{A}$ in $X$ is a structure

\[
\mathcal{A} = \{(x, \tilde{\mu}_A(x), \lambda(x)) : x \in X\}
\]

which is briefly denoted by $\mathcal{A} = \langle \tilde{\mu}_A, \lambda \rangle$ where $\tilde{\mu}_A = [\mu^-_A, \mu^+_A]$ is an IVF set in $X$ and $\lambda$ is a fuzzy set in $X$.

**Definition 3.2** ([3]). A cubic set $\mathcal{A} = \langle \tilde{\mu}_A, \lambda \rangle$ in $X$ is called a cubic subalgebra of a BCK/BCI-algebra $X$ if it satisfies: for all $x, y \in X$,

(a) $\tilde{\mu}_A(x \ast y) \geq \text{rmin}\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\}$.

(b) $\lambda(x \ast y) \leq \max\{\lambda(x), \lambda(y)\}$.

**Definition 3.3** ([3]). A cubic set $\mathcal{A} = \langle \tilde{\mu}_A, \lambda \rangle$ in a BCK/BCI-algebra $X$ is called a cubic ideal of $X$ if it satisfies: for all $x, y \in X$,

(a) $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$.

(b) $\lambda(0) \leq \lambda(x)$. 

\( \mu_A(x) \geq \min \{\mu_A(x * y), \mu_A(y)\} \).

(d) \( \lambda(x) \leq \max(\lambda(x * y), \lambda(y)) \).

In what follows, let \( X \) denote a BCI-algebra unless otherwise specified.

**Definition 3.4** ([5]). A cubic ideal \( \mathcal{A} = (\tilde{\mu}_A, \lambda) \) of \( X \) is said to be closed if \( \tilde{\mu}_A(0 * x) \geq \tilde{\mu}_A(x) \) and \( \lambda(0 * x) \leq \lambda(x) \) for all \( x \in X \).

**Definition 3.5.** A cubic set \( \mathcal{A} = (\tilde{\mu}_A, \lambda) \) in \( X \) is called a cubic \( p \)-ideal of \( X \) if it satisfies conditions (a) and (b) in **Definition 3.3** and for all \( x, y, z \in X \),

(a) \( \tilde{\mu}_A(x) \geq \min \{\tilde{\mu}_A((x * z) * (y * z)), \tilde{\mu}_A(y)\} \).

(b) \( \lambda(x) \leq \max\{\lambda((x * z) * (y * z)), \lambda(y)\} \).

**Example 3.6.** Consider a BCI-algebra \( X = \{0, a, b, c\} \) in which the \( * \)-operation is given by **Table 1**. We define \( \mu_A = [\mu_A^-, \mu_A^+] \) and \( \lambda \) by

\[
\begin{align*}
\tilde{\mu}_A &= \begin{cases}
0 & [0.5, 0.9] \\
0.3 & 0.4 & 0.8 & [0.3, 0.5] & 0.3 & 0.5
\end{cases} \\
\lambda &= \begin{cases}
0 & a & b & c \\
0.3 & 0.4 & 0.7 & 0.7
\end{cases}
\end{align*}
\]

Then \( \mathcal{A} = (\tilde{\mu}_A, \lambda) \) is a cubic \( p \)-ideal of \( X \).

If we put \( z = x \) and \( y = 0 \) in **Definition 3.5**, then we have the following proposition.

**Proposition 3.7.** Every cubic \( p \)-ideal \( \mathcal{A} = (\tilde{\mu}_A, \lambda) \) of \( X \) satisfies the following inequalities:

\[
(\forall x \in X) \left( \tilde{\mu}_A(x) \geq \mu_A(0 * (0 * x)), \lambda(x) \leq \lambda((0 * (0 * x))) \right).
\] (3.1)

If we put \( z = 0 \) in **Definition 3.5** and use (a1), then we have the following theorem.

**Theorem 3.8.** Every cubic \( p \)-ideal is a cubic ideal.

The converse of **Theorem 3.8** may not be true as seen in the following example.

**Example 3.9.** Let \( X = \{0, a, 1, 2, 3\} \) be a set with the Cayley table given by **Table 2**. We define \( \mu_A = [\mu_A^-, \mu_A^+] \) and \( \lambda \) by

\[
\begin{align*}
\tilde{\mu}_A &= \begin{cases}
0 & [0.4, 0.8] \\
0.2 & 0.5 & 0.6 & 0.6 & 0.6 & 0.6
\end{cases} \\
\lambda &= \begin{cases}
0 & a & 1 & 2 & 3 \\
0.2 & 0.5 & 0.6 & 0.6 & 0.6
\end{cases}
\end{align*}
\]

Then \( \mathcal{A} = (\tilde{\mu}_A, \lambda) \) is a cubic ideal of \( X \). But it is not a cubic \( p \)-ideal of \( X \), since

\[
\tilde{\mu}_A(a) = [0.3, 0.6] \not\geq [0.4, 0.8] = \min \{\tilde{\mu}_A((a * 1) * (0 * 1)), \tilde{\mu}_A(0)\}
\]

and/or \( \lambda(a) = 0.5 \not\leq 0.2 = \max\{\lambda((a * 1) * (0 * 1)), \lambda(0)\} \).

**Proposition 3.10.** If \( \mathcal{A} = (\tilde{\mu}_A, \lambda) \) is a cubic \( p \)-ideal of \( X \), then

\[
\tilde{\mu}_A((x * z) * (y * z)) \leq \tilde{\mu}_A((x * z) * (y * z)) \quad \text{and} \quad \lambda((x * z) * (y * z)) \geq \lambda((x * z) * (y * z))
\]

for all \( x, y, z \in X \).

**Proof.** Let \( \mathcal{A} = (\tilde{\mu}_A, \lambda) \) be a cubic \( p \)-ideal of \( X \). Note that \((x * z) * (y * z) \leq x * y \), i.e., \((x * z) * (y * z)) * (x * y) = 0\), for all \( x, y, z \in X \). Since \( \mathcal{A} = (\tilde{\mu}_A, \lambda) \) is a cubic ideal of \( X \), by **Theorem 3.8**, it follows that

\[
\begin{align*}
\tilde{\mu}_A((x * z) * (y * z)) &\geq \min \{\tilde{\mu}_A(((x * z) * (y * z)) * (x * y)), \tilde{\mu}_A(x * y)\} \\
&= \min \{\mu_A(0), \tilde{\mu}_A(x * y)\} = \tilde{\mu}_A(x * y)
\end{align*}
\]
Theorem 3.11. Let \( \mathcal{A} = (\tilde{\mu}_A, \lambda) \) be a cubic ideal of \( X \) that satisfies:

\[
\tilde{\mu}_A(x \ast y) \geq \tilde{\mu}_A((x \ast z) \ast (y \ast z)) \quad \text{and} \quad \lambda(x \ast y) \leq \lambda((x \ast z) \ast (y \ast z))
\]

for all \( x, y, z \in X \). Then \( \mathcal{A} = (\tilde{\mu}_A, \lambda) \) is a cubic \( p \)-ideal of \( X \).

Proof. For any \( x, y, z \in X \), we have

\[
\tilde{\mu}_A(x) = \text{rmin} \{\tilde{\mu}_A(x \ast y), \tilde{\mu}_A(y)\}
\]

and \( \lambda(x) \leq \text{max} \{\lambda(x \ast y), \lambda(y)\} \leq \text{max} \{\lambda((x \ast z) \ast (y \ast z)), \lambda(y)\} \). This completes the proof. \( \square \)

Lemma 3.12. Every cubic ideal \( \mathcal{A} = (\tilde{\mu}_A, \lambda) \) satisfies the following inequalities:

\[
\tilde{\mu}_A(0 \ast (0 \ast x)) \geq \tilde{\mu}_A(x) \quad \text{and} \quad \lambda(0 \ast (0 \ast x)) \leq \lambda(x)
\]

for all \( x \in X \).

Proof. For any \( x \in X \), we have

\[
\tilde{\mu}_A(x) = \text{rmin} \{\tilde{\mu}_A(0), \tilde{\mu}_A(x)\}
\]

and \( \lambda(x) = \text{max} \{\lambda(0), \lambda(x)\} \geq \lambda(0 \ast (0 \ast x)) \). \( \square \)

Lemma 3.13 ([10]). Let \( X \) be a BCI-algebra. Then

(1) \( 0 \ast (0 \ast ((x \ast z) \ast (y \ast z))) = (0 \ast y) \ast (0 \ast x) \),

(2) \( 0 \ast (0 \ast (x \ast y)) = (0 \ast y) \ast (0 \ast x) \)

for all \( x, y, z \in X \).

Theorem 3.14. Let \( \mathcal{A} = (\tilde{\mu}_A, \lambda) \) be a cubic ideal of \( X \) that satisfies:

\[
\tilde{\mu}_A(0 \ast (0 \ast x)) \leq \tilde{\mu}_A(x) \quad \text{and} \quad \lambda(0 \ast (0 \ast x)) \geq \lambda(x)
\]

for all \( x \in X \). Then \( \mathcal{A} = (\tilde{\mu}_A, \lambda) \) is a cubic \( p \)-ideal of \( X \).

Proof. Let \( x, y, z \in X \). Using Lemmas 3.12 and 3.13, we have

\[
\tilde{\mu}_A((x \ast z) \ast (y \ast z)) \leq \tilde{\mu}_A(0 \ast ((x \ast z) \ast (y \ast z)))
\]

\[
= \tilde{\mu}_A((0 \ast y) \ast (0 \ast x))
\]

\[
= \tilde{\mu}_A(0 \ast (0 \ast (x \ast y)))
\]

\[
\leq \tilde{\mu}_A(x \ast y)
\]

and

\[
\lambda((x \ast z) \ast (y \ast z)) \geq \lambda(0 \ast ((x \ast z) \ast (y \ast z)))
\]

\[
= \lambda((0 \ast y) \ast (0 \ast x))
\]

\[
= \lambda(0 \ast (0 \ast (x \ast y)))
\]

\[
\geq \lambda(x \ast y).
\]

It follows from Theorem 3.11 that \( \mathcal{A} = (\tilde{\mu}_A, \lambda) \) is a cubic \( p \)-ideal of \( X \). \( \square \)
4. Cubic $a$-ideals

**Definition 4.1** ([4]). A cubic set $\mathcal{A} = \langle \mu_A, \lambda \rangle$ in $X$ is called a cubic $q$-ideal of $X$ if it satisfies conditions (a) and (b) in **Definition 3.3** and for all $x, y, z \in X$,

(a) $\mu_A(x \ast z) \geq \text{rmin} \{\mu_A(x \ast (y \ast z)), \mu_A(y)\}$.
(b) $\lambda(x \ast z) \leq \text{max}\{\lambda(x \ast (y \ast z)), \lambda(y)\}$.

**Definition 4.2.** A cubic set $\mathcal{A} = \langle \mu_A, \lambda \rangle$ in $X$ is called a cubic $a$-ideal of $X$ if it satisfies conditions (a) and (b) in **Definition 3.3** and for all $x, y, z \in X$,

(a) $\mu_A(y \ast x) \geq \text{rmin} \{\mu_A((x \ast z) \ast (0 \ast y)), \mu_A(z)\}$.
(b) $\lambda(y \ast x) \leq \text{max}\{\lambda((x \ast z) \ast (0 \ast y)), \lambda(z)\}$.

**Example 4.3.** Consider a BCI-algebra $X = \{0, a, b, c\}$ in which the $\ast$-operation is given by **Table 3**. We define $\mu_A = [\mu_A^-, \mu_A^+]$ and $\lambda$ by

$$
\mu_A = \begin{pmatrix}
0 & a & b & c \\
[0.4, 0.8] & [0.4, 0.8] & [0.2, 0.5] & [0.2, 0.5]
\end{pmatrix}
$$

and

$$
\lambda = \begin{pmatrix}
0 & a & b & c \\
0.3 & 0.3 & 0.6 & 0.6
\end{pmatrix}.
$$

Then $\mathcal{A} = \langle \mu_A, \lambda \rangle$ is a cubic $a$-ideal of $X$.

**Theorem 4.4.** Every cubic $a$-ideal is a closed cubic ideal.

**Proof.** Let $\mathcal{A} = \langle \mu_A, \lambda \rangle$ be a cubic $a$-ideal of $X$. Putting $z = y = 0$ in **Definition 4.2** and using (a1), **Definition 3.3**(a) and (b), we have

$$
\begin{align*}
\mu_A(0 \ast x) & \geq \text{rmin} \{\mu_A((x \ast 0) \ast (0 \ast 0)), \mu_A(0)\} = \mu_A(x) \\
\lambda(0 \ast x) & \leq \text{max}\{\lambda((x \ast 0) \ast (0 \ast 0)), \lambda(0)\} = \lambda(x)
\end{align*}
$$

for all $x \in X$. If we take $x = z = 0$ in **Definition 4.2** and use (a1), **Definition 3.3**(a) and (b), then

$$
\begin{align*}
\mu_A(y) & \geq \text{rmin} \{\mu_A(0 \ast (0 \ast y)), \mu_A(0)\} = \mu_A(0 \ast (0 \ast y)) \\
\lambda(y) & \leq \text{max}\{\lambda(0 \ast (0 \ast y)), \lambda(0)\} = \lambda(0 \ast (0 \ast y))
\end{align*}
$$

for all $y \in X$. It follows from (4.1) that

$$
\begin{align*}
\mu_A(x) & \geq \mu_A(0 \ast x) \quad \text{and} \quad \lambda(x) \leq \lambda(0 \ast x)
\end{align*}
$$

for all $x \in X$; so from **Definition 4.2**, that

$$
\begin{align*}
\mu_A(x) & \geq \mu_A(0 \ast x) \geq \text{rmin} \{\mu_A((x \ast z) \ast (0 \ast 0)), \mu_A(z)\} \\
& = \text{rmin} \{\mu_A(x \ast z), \mu_A(z)\} \\
\lambda(x) & \leq \lambda(0 \ast x) \leq \text{max}\{\lambda((x \ast z) \ast (0 \ast 0)), \lambda(z)\} \\
& = \text{max}\{\lambda(x \ast z), \lambda(z)\}
\end{align*}
$$

for all $x, z \in X$. Therefore $\mathcal{A} = \langle \mu_A, \lambda \rangle$ is a closed cubic ideal of $X$. \qed

The following example shows that the converse of **Theorem 4.4** may not be true.

**Example 4.5.** Consider a BCI-algebra $X = \{0, a, b\}$ in which the $\ast$-operation is given by **Table 4**. We define $\mu_A = [\mu_A^-, \mu_A^+]$ and $\lambda$ by

$$
\mu_A = \begin{pmatrix}
0 & a & b \\
[0.4, 0.8] & [0.2, 0.5] & [0.2, 0.5]
\end{pmatrix}
$$

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Assume that
\[ b \leq a \quad \text{and} \quad c \leq a \]
and we can say that it is a cubic a-ideal of \( X \), since

\[
\mu_A(b \cdot a) = \mu_A(a) < \min \{ \mu_A((a \cdot 0) \cdot (0 \cdot b)), \mu_A(0) \}
\]
and/or \( \lambda(a \cdot b) > \max \{ \lambda((0 \cdot 0) \cdot (0 \cdot a)), \lambda(0) \} \).

We provide characterizations of a cubic a-ideal.

**Lemma 4.6** ([3]). Let \( \mathcal{A} = \langle \mu_A, \lambda \rangle \) be a cubic ideal of \( X \). If the inequality \( x \cdot y \leq z \) holds in \( X \), then

\[
\mu_A(x) \geq \min \{ \mu_A(y), \mu_A(z) \} \quad \text{and} \quad \lambda(x) \leq \max \{ \lambda(y), \lambda(z) \}.
\]

**Theorem 4.7.** If \( \mathcal{A} = \langle \mu_A, \lambda \rangle \) is a cubic ideal of \( X \), then the following are equivalent:

1. \( \mathcal{A} = \langle \mu_A, \lambda \rangle \) is a cubic a-ideal of \( X \).
2. \( \mathcal{A} = \langle \mu_A, \lambda \rangle \) satisfies the following inequalities:

\[
\mu_A(y \cdot (x \cdot z)) \geq \mu_A((x \cdot z) \cdot (0 \cdot y)) \quad \text{and} \quad \lambda(y \cdot (x \cdot z)) \leq \lambda((x \cdot z) \cdot (0 \cdot y))
\]

for all \( x, y, z \in X \).
3. \( \mathcal{A} = \langle \mu_A, \lambda \rangle \) satisfies the following inequalities:

\[
\mu_A(y \cdot x) \geq \mu_A(x \cdot (0 \cdot y)), \quad \lambda(y \cdot x) \leq \lambda(x \cdot (0 \cdot y))
\]

for all \( x, y \in X \).

**Proof.** Assume that \( \mathcal{A} = \langle \mu_A, \lambda \rangle \) is a cubic a-ideal of \( X \). Then

\[
\mu_A(y \cdot (x \cdot z)) \geq \min \{ \mu_A((x \cdot z) \cdot 0) \cdot (0 \cdot y)), \mu_A(0) \} = \mu_A(((x \cdot z) \cdot 0) \cdot (0 \cdot y)) = \mu_A((x \cdot z) \cdot (0 \cdot y))
\]

and

\[
\lambda(y \cdot (x \cdot z)) \leq \max \{ \lambda((x \cdot z) \cdot 0) \cdot (0 \cdot y)), \lambda(0) \} = \lambda((x \cdot z) \cdot 0) \cdot (0 \cdot y)) = \lambda((x \cdot z) \cdot (0 \cdot y))
\]

for all \( x, y, z \in X \), and so (4.3) is valid. (4.4) is induced by taking \( z = 0 \) in (4.3) and using (a1). Suppose that \( \mathcal{A} = \langle \mu_A, \lambda \rangle \) satisfies (4.4). Note that

\[
(x \cdot (0 \cdot y)) \circ ((x \cdot z) \circ (0 \cdot y)) \leq x \cdot (x \cdot z) \leq x
\]

for all \( x, y, z \in X \). It follows from (4.4) and Lemma 4.6 that

\[
\mu_A(y \cdot x) \geq \mu_A(x \cdot (0 \cdot y)) \geq \min \{ \mu_A((x \cdot z) \cdot (0 \cdot y)), \mu_A(0) \}
\]

and

\[
\lambda(y \cdot x) \leq \lambda(x \cdot (0 \cdot y)) \leq \max \{ \lambda((x \cdot z) \cdot (0 \cdot y)), \lambda(x) \} \quad \text{for all} \ x, y, z \in X.
\]

Therefore, \( \mathcal{A} = \langle \mu_A, \lambda \rangle \) is a cubic a-ideal of \( X \).

Let \( \mathcal{A} = \langle \mu_A, \lambda \rangle \) be a cubic set in \( X \). For any \( r \in [0, 1] \) and \( s, t \in D[0, 1] \), we define \( U(\mathcal{A}; [s, t], r) \) as follows:

\[
U(\mathcal{A}; [s, t], r) = \{ x \in X \mid \mu_A(x) \geq [s, t], \lambda(x) \leq r \},
\]

and we can say that it is a cubic level set of \( \mathcal{A} = \langle \mu_A, \lambda \rangle \).
Lemma 4.8 ([5]). For a cubic set $\mathcal{A} = (\tilde{\mu}_A, \lambda)$ in $X$, the following are equivalent:

1. $\mathcal{A} = (\tilde{\mu}_A, \lambda)$ is a cubic ideal of $X$.
2. Every nonempty cubic level set of $\mathcal{A} = (\tilde{\mu}_A, \lambda)$ is an ideal of $X$.

Lemma 4.9 ([6]). A subset $I$ of $X$ is an a-ideal of $X$ if and only if it is an ideal of $X$ which satisfies the following implication:

$$\forall x, y \in X \ (x \ast (0 \ast y) \in I \Rightarrow y \ast x \in I).$$

Theorem 4.10. For a cubic set $\mathcal{A} = (\tilde{\mu}_A, \lambda)$ in $X$, the following are equivalent:

1. $\mathcal{A} = (\tilde{\mu}_A, \lambda)$ is a cubic a-ideal of $X$.
2. Every nonempty cubic level set of $\mathcal{A} = (\tilde{\mu}_A, \lambda)$ is an ideal of $X$.

Proof. Assume that $\mathcal{A} = (\tilde{\mu}_A, \lambda)$ is a cubic a-ideal of $X$. Then $\mathcal{A} = (\tilde{\mu}_A, \lambda)$ is a cubic ideal of $X$ by Theorem 4.4. Hence every nonempty cubic level set of $\mathcal{A} = (\tilde{\mu}_A, \lambda)$ is an ideal of $X$ by Lemma 4.8. Let $[s, t] \in D[0, 1]$ and $r \in [0, 1]$ be such that $U(\mathcal{A}; [s, t], r) \neq \emptyset$. Let $x, y \in X$ be such that $x \ast (0 \ast y) \in U(\mathcal{A}; [s, t], r)$. Then $\tilde{\mu}_A(x \ast (0 \ast y)) \geq [s, t]$ and $\lambda(x \ast (0 \ast y)) \leq r$. It follows from (4.4) that

$$\tilde{\mu}_A(y \ast x) \geq \tilde{\mu}_A(x \ast (0 \ast y)) \geq [s, t]$$

and $\lambda(y \ast x) \leq \lambda(x \ast (0 \ast y)) \leq r$ so that $y \ast x \in U(\mathcal{A}; [s, t], r)$. Using Lemma 4.9, we conclude that $U(\mathcal{A}; [s, t], r)$ is an a-ideal of $X$.

Conversely, suppose that (2) is valid, that is, $U(\mathcal{A}; [s, t], r)$ is nonempty and is an a-ideal of $X$ for all $r \in [0, 1]$. Since any a-ideal is an ideal (see [6]), it follows from Lemma 4.8 that $\mathcal{A} = (\tilde{\mu}_A, \lambda)$ is a cubic ideal of $X$. Assume that two inequalities in (4.4) are false. Then there exist $a, b \in X$ such that $\tilde{\mu}_A(b \ast a) < \tilde{\mu}_A(a \ast (0 \ast b))$ and $\lambda(b \ast a) > \lambda(a \ast (0 \ast b))$. Thus $\tilde{\mu}_A(b \ast a) < [s_0, t_0]$, $\tilde{\mu}_A(a \ast (0 \ast b))$ and $\lambda(b \ast a) > r_0$, $\lambda(a \ast (0 \ast b))$ for some $[s_0, t_0] \in D[0, 1]$ and $r_0 \in [0, 1]$. It follows that $a \ast (0 \ast b) \in U(\mathcal{A}; [s_0, t_0], r_0)$ but $b \ast a \not\in U(\mathcal{A}; [s_0, t_0], r_0)$. This is a contradiction. Suppose that

$$\tilde{\mu}_A(y \ast x) \geq \tilde{\mu}_A(x \ast (0 \ast y))$$

for all $x, y \in X$, and there exist $a, b \in X$ such that $\lambda(b \ast a) < \lambda(a \ast (0 \ast b))$, $\lambda(b \ast a) > r_0 \geq \lambda(a \ast (0 \ast b))$ for some $r_0 \in [0, 1]$, and so $a \ast (0 \ast b) \in U(\mathcal{A}; [s_0, t_0], r_0)$ but $b \ast a \not\in U(\mathcal{A}; [s_0, t_0], r_0)$. This is also a contradiction. For the case that $\lambda(y \ast x) \leq \lambda(x \ast (0 \ast y))$ for all $x, y \in X$ and $\tilde{\mu}_A(b \ast a) < \tilde{\mu}_A(a \ast (0 \ast b))$ for some $a, b \in X$, we can induce a contradiction. Therefore, (4.4) is valid, which implies from Theorem 4.7 that $\mathcal{A} = (\tilde{\mu}_A, \lambda)$ is a cubic a-ideal of $X$.

Theorem 4.10 combines with (a) and (b) of Definition 3.3 to induce the following corollary.

Corollary 4.11. If $\mathcal{A} = (\tilde{\mu}_A, \lambda)$ is a cubic a-ideal of $X$, then the set

$$I := \{x \in X \mid \tilde{\mu}_A(x) = \tilde{\mu}_A(0), \lambda(x) = \lambda(0)\}$$

is an a-ideal of $X$.

Theorem 4.12. Every cubic a-ideal is a cubic p-ideal.

Proof. Let $\mathcal{A} = (\tilde{\mu}_A, \lambda)$ be a cubic a-ideal of $X$. Then $\mathcal{A} = (\tilde{\mu}_A, \lambda)$ is a cubic ideal of $X$ (see Theorem 4.4). If we take $x = z = 0$ in (4.3), then $\tilde{\mu}_A(0 \ast (0 \ast y)) \leq \tilde{\mu}_A(y)$ and $\lambda(0 \ast (0 \ast y)) \geq \lambda(y)$ for all $y \in X$. Hence, by Theorem 3.14, we conclude that $\mathcal{A} = (\tilde{\mu}_A, \lambda)$ is a cubic p-ideal of $X$.

Note that the cubic set $\mathcal{A} = (\tilde{\mu}_A, \lambda)$ in Example 4.5 is a cubic p-ideal which is not a cubic $a$-ideal. Hence the converse of Theorem 4.12 is not true in general.

Lemma 4.13 ([4]). For a cubic ideal $\mathcal{A} = (\tilde{\mu}_A, \lambda)$ of $X$, the following are equivalent:

1. $\mathcal{A} = (\tilde{\mu}_A, \lambda)$ is a cubic q-ideal of $X$.
2. $\tilde{\mu}_A(x \ast y) \geq \tilde{\mu}_A(x \ast (0 \ast y))$ and $\lambda(x \ast y) \leq \lambda(x \ast (0 \ast y))$ for all $x, y \in X$.

Theorem 4.14. Every cubic a-ideal is a cubic q-ideal.

Proof. Let $\mathcal{A} = (\tilde{\mu}_A, \lambda)$ be a cubic a-ideal of $X$. Then $\mathcal{A} = (\tilde{\mu}_A, \lambda)$ is a cubic ideal of $X$ by Theorem 4.4. Note that

\[
\begin{align*}
(0 \ast (0 \ast (y \ast (0 \ast x)))) \ast (x \ast (0 \ast y)) &= ((0 \ast (0 \ast y)) \ast (0 \ast (0 \ast (0 \ast x)))) \ast (x \ast (0 \ast y)) \\
&= ((0 \ast (0 \ast y)) \ast (0 \ast x)) \ast (x \ast (0 \ast y)) \\
&\leq (x \ast (0 \ast y)) \ast (x \ast (0 \ast y)) = 0
\end{align*}
\]
Table 5

<table>
<thead>
<tr>
<th>*</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>b</td>
<td></td>
</tr>
<tr>
<td>a</td>
<td>b</td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>0</td>
</tr>
</tbody>
</table>

for all \( x, y \in X \). Since \( \mathcal{A} = \langle \tilde{\mu}_A, \lambda \rangle \) is a cubic \( p \)-ideal of \( X \) (see Theorem 4.12), it follows from (4.4), Proposition 3.10 and Lemma 4.6 that

\[
\tilde{\mu}_A(x \ast y) \geq \tilde{\mu}_A(y \ast (0 \ast x)) \geq \tilde{\mu}_A(0 \ast (0 \ast (y \ast (0 \ast x))))
\]

\[
\geq \min\{\tilde{\mu}_A(x \ast (0 \ast y)), \tilde{\mu}_A(0)\}
\]

\[
= \tilde{\mu}_A(x \ast (0 \ast y))
\]

and

\[
\lambda(x \ast y) \geq \lambda(y \ast (0 \ast x)) \geq \lambda(0 \ast (0 \ast (y \ast (0 \ast x))))
\]

\[
\geq \max\{\lambda(x \ast (0 \ast y)), \lambda(0)\}
\]

\[
= \lambda(x \ast (0 \ast y))
\]

for all \( x, y \in X \). Using Lemma 4.13, we conclude that \( \mathcal{A} = \langle \tilde{\mu}_A, \lambda \rangle \) be a cubic q-ideal of \( X \).

The following example shows that the converse of Theorem 4.14 may not be true.

Example 4.15. Consider a BCI-algebra \( X = \{0, a, b\} \) with the \( \ast \)-operation which is given in Table 5. We define \( \tilde{\mu}_A = [\mu_A^+, \mu_A^-] \) and \( \lambda \) by

\[
\tilde{\mu}_A = \begin{pmatrix}
0 & a & b \\
[0.5, 0.8] & [0.3, 0.6] & [0.3, 0.6]
\end{pmatrix}
\]

and

\[
\lambda = \begin{pmatrix}
0 & a & b \\
0.3 & 0.6 & 0.6
\end{pmatrix}
\]

Then \( \mathcal{A} = \langle \tilde{\mu}_A, \lambda \rangle \) is a cubic q-ideal of \( X \). But it is not a cubic a-ideal of \( X \) since \( \tilde{\mu}_A(a \ast 0) \not\geq \min\{\tilde{\mu}_A((0 \ast 0) \ast (0 \ast a)), \tilde{\mu}_A(0)\} \).

Lemma 4.16 ([3]). Let \( \mathcal{A} = \langle \tilde{\mu}_A, \lambda \rangle \) be a cubic ideal of \( X \). If the inequality \( x \leq y \) holds in \( X \), then \( \tilde{\mu}_A(x) \geq \tilde{\mu}_A(y) \) and \( \lambda(x) \leq \lambda(y) \).

Theorem 4.17. For a cubic set \( \mathcal{A} = \langle \tilde{\mu}_A, \lambda \rangle \) in \( X \), the following are equivalent.

(1) \( \mathcal{A} = \langle \tilde{\mu}_A, \lambda \rangle \) is a cubic a-ideal of \( X \).

(2) \( \mathcal{A} = \langle \tilde{\mu}_A, \lambda \rangle \) is both a cubic \( p \)-ideal and a cubic q-ideal of \( X \).

Proof. By means of Theorems 4.12 and 4.14, every cubic a-ideal is both a cubic \( p \)-ideal and a cubic q-ideal.

Conversely, let \( \mathcal{A} = \langle \tilde{\mu}_A, \lambda \rangle \) be both a cubic \( p \)-ideal and a cubic q-ideal. Note that \( \mathcal{A} = \langle \tilde{\mu}_A, \lambda \rangle \) is a cubic ideal of \( X \) (see [4]). Taking \( z = \gamma \) at (a) and (b) in Definition 4.1, we have \( \tilde{\mu}_A(x \ast y) \geq \min\{\tilde{\mu}_A(x), \tilde{\mu}_A(y)\} \) and \( \lambda(x \ast y) \leq \max\{\lambda(x), \lambda(y)\} \).

Hence \( \mathcal{A} = \langle \tilde{\mu}_A, \lambda \rangle \) is a cubic subalgebra of \( X \), and so \( \mathcal{A} = \langle \tilde{\mu}_A, \lambda \rangle \) is a closed cubic ideal of \( X \). Using Lemma 4.13, we get

\[
\tilde{\mu}_A(x \ast y) \geq \tilde{\mu}_A(x \ast (0 \ast y))
\]

\[
\lambda(x \ast y) \leq \lambda(x \ast (0 \ast y))
\]

for all \( x, y \in X \). Since \( 0 \ast (y \ast x) \leq x \ast y \) for all \( x, y \in X \), it follows from Lemma 4.16, (4.5) that

\[
\tilde{\mu}_A(0 \ast (y \ast x)) \geq \tilde{\mu}_A(x \ast (0 \ast y))
\]

\[
\lambda(0 \ast (y \ast x)) \leq \lambda(x \ast (0 \ast y))
\]

for all \( x, y \in X \). Using Proposition 3.7, Definition 3.4 and (4.6), we have

\[
\tilde{\mu}_A(y \ast x) \geq \tilde{\mu}_A(0 \ast (y \ast x)) \geq \tilde{\mu}_A(x \ast (0 \ast y))
\]

and \( \lambda(0 \ast (y \ast x)) \leq \lambda(0 \ast (0 \ast (y \ast x))) \leq \lambda(0 \ast (0 \ast y)) \) for all \( x, y \in X \). It follows from Theorem 4.7 that \( \mathcal{A} = \langle \tilde{\mu}_A, \lambda \rangle \) is a cubic a-ideal of \( X \).

Theorem 4.18 (Cubic extension property for a cubic a-ideal). Let \( \mathcal{A} = \langle \tilde{\mu}_A, \lambda \rangle \) and \( \mathcal{B} = \langle \tilde{\mu}_B, \kappa \rangle \) be cubic ideals of \( X \) such that \( \mathcal{A} \subseteq \mathcal{B} \) and \( \tilde{\mu}_A(0) = \tilde{\mu}_B(0) \) and \( \lambda(0) = \kappa(0) \). If \( \mathcal{A} = \langle \tilde{\mu}_A, \lambda \rangle \) is a cubic a-ideal of \( X \), then so is \( \mathcal{B} = \langle \tilde{\mu}_B, \kappa \rangle \).
Proof. Suppose that \( \mathcal{I} = \langle \tilde{\mu}_A, \lambda \rangle \) is a cubic \( a \)-ideal of \( X \). Then \( \mathcal{I} = \langle \tilde{\mu}_A, \lambda \rangle \) is both a cubic \( p \)-ideal and a cubic \( q \)-ideal of \( X \) by Theorems 4.12 and 4.14. Using Lemma 4.13, (a3) and (I), we have

\[
\tilde{\mu}_B((x \ast y) \ast (x \ast (0 \ast y))) = \tilde{\mu}_B((x \ast (0 \ast y)) \ast (x \ast (0 \ast y))) \\
\geq \tilde{\mu}_A((x \ast (0 \ast y)) \ast (x \ast (0 \ast y))) = \tilde{\mu}_A((x \ast (0 \ast y)) \ast (x \ast (0 \ast y))) = \tilde{\mu}_A(0) = \tilde{\mu}_B(0)
\]

and

\[
\kappa((x \ast y) \ast (x \ast (0 \ast y))) = \kappa((x \ast (0 \ast y)) \ast (x \ast (0 \ast y))) \\
\leq \lambda((x \ast (0 \ast y)) \ast (x \ast (0 \ast y))) = \lambda((x \ast (0 \ast y)) \ast (x \ast (0 \ast y))) = \lambda(0) = \kappa(0)
\]

Since \( \mathcal{R} = \langle \tilde{\mu}_B, \kappa \rangle \) is a cubic ideal of \( X \), we get

\[
\tilde{\mu}_B(x \ast y) \geq \min\{\tilde{\mu}_B((x \ast y) \ast (x \ast (0 \ast y))), \tilde{\mu}_B(x \ast (0 \ast y))\} \\
= \tilde{\mu}_B(x \ast (0 \ast y))
\]

and

\[
\kappa(x \ast y) \leq \max\{\kappa((x \ast y) \ast (x \ast (0 \ast y))), \kappa(x \ast (0 \ast y))\} = \kappa(x \ast (0 \ast y)).
\]

Therefore \( \mathcal{R} = \langle \tilde{\mu}_B, \kappa \rangle \) is a cubic \( q \)-ideal of \( X \) by Lemma 4.13. Since \( \mathcal{I} = \langle \tilde{\mu}_A, \lambda \rangle \) is a cubic \( p \)-ideal of \( X \), it follows from Proposition 3.7 that

\[
\tilde{\mu}_B((x \ast (0 \ast (0 \ast x)))) \geq \tilde{\mu}_A((x \ast (0 \ast (0 \ast x)))) \geq \tilde{\mu}_A(0) = \mu_B(0) \geq \mu_B(0 \ast (0 \ast x))
\]

and

\[
\kappa(x \ast (0 \ast (0 \ast x))) \leq \lambda(0 \ast (0 \ast (x \ast (0 \ast x)))) \\
\leq \lambda(0 \ast (0 \ast (x \ast (0 \ast x)))) = \lambda(0) = \kappa(0) \leq \kappa(0 \ast (0 \ast x)).
\]

Hence

\[
\tilde{\mu}_B(x) \geq \min\{\tilde{\mu}_B(x \ast (0 \ast (0 \ast x))), \tilde{\mu}_B(0 \ast (0 \ast x))\} = \mu_B(0 \ast (0 \ast x))
\]

and \( \kappa(x) \leq \max\{\kappa(x \ast (0 \ast (0 \ast x))), \kappa(0 \ast (0 \ast x))\} = \kappa(0 \ast (0 \ast x)) \). Using Theorem 3.14, we conclude that \( \mathcal{R} = \langle \tilde{\mu}_B, \kappa \rangle \) is a cubic \( p \)-ideal of \( X \). Therefore \( \mathcal{R} = \langle \tilde{\mu}_B, \kappa \rangle \) is a cubic \( a \)-ideal of \( X \) by Theorem 4.17. 

References