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Periodic bidirectional associative memory neural networks with distributed delays ☆

Anping Chen [a,*], Lihong Huang [b], Zhigang Liu [a], Jinde Cao [c]

[a] *Department of Mathematics, Xiangnan University, Chenzhou, 423000, China*
[b] *College of Mathematics and Econometrics, Hunan University, Changsha, 410082, China*
[c] *Department of Mathematics, Southeast University, Nanjing, 210096, China*

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Abstract

Some sufficient conditions are obtained for the existence and global exponential stability of a periodic solution to the general bidirectional associative memory (BAM) neural networks with distributed delays by using the continuation theorem of Mawhin's coincidence degree theory and the Lyapunov functional method and the Young's inequality technique. These results are helpful for designing a globally exponentially stable and periodic oscillatory BAM neural network, and the conditions can be easily verified and be applied in practice. An example is also given to illustrate our results.

Keywords: BAM neural networks; Stability; Periodic attractor; Distributed delay; Young's inequality

1. Introduction

It is well known that neural networks have been paid great attention in past decade due to the potential applications in many fields such as image and signal processing, pattern recognition,

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* Corresponding author.
E-mail address: anping_chen@hotmail.com (A. Chen).

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optimization and automatic control, etc. Various neural network models have been proposed and have been extensively studied in the literature (see, e.g., [1–7,9–11,13,14,16–30]). Among which, the dynamic property of delayed neural networks, including stable, unstable, oscillatory, and chaotic behavior, has recently become an important subject and has attracted many researchers attention (see, for example, [1–7,10,13,14,21–30]). For an introduction to the delayed neural networks, we refer to Wu's recent book [30].

Note that although the constant discrete delays in the models of delayed feedback systems serve good approximations in simple circuits consisting of a small number of neurons, neural networks usually have a spatial extent due to the presence of an amount of parallel pathways with a variety of axon sizes and lengths. Thus, there will be a distribution of transmission delays. In this case, the transmission of signals is no longer instantaneous and cannot be modelled with discrete delays. A more appropriate way is to incorporate distributed delays, since a neural network model with continuously distributed delays not only takes into account the local phenomena, but generalizes their representations. Indeed, with distributed time delays, the neural network model also takes into account the spatial effects through the multiple parallel pathways of the network. Moreover, a neural network model with distributed delay is more general than that with discrete delay and the distributed delay becomes a discrete delay when the delay kernel is a δ-function at a certain time (see Remark 1).

In the application of neural networks to some practical problems, the properties of equilibrium point play important roles. An equilibrium point can be looked as a special periodic solution of neural networks with arbitrary period. In this sense the analysis of periodic solutions of neural networks could be more general than that of equilibrium points. The experimental and theoretical studies [11,23] show also that a mammal's brain may be exploiting dynamic attractors for its encoding and subsequent associative recall rather than temporally static (equilibrium-type) attractors. In most studies of artificial neural networks, limit cycles, strange attractors and other dynamical phenomena have been used by many authors to represent encoded temporal patterns as associative memories [9,13,16,27,31]. Recently, some literature dealing with time-varying stimuli or network parameters in particular a periodic environment has been appeared [13,20,32–35]. The studies of existence of periodic solution together with global exponential stability of delayed neural networks dealing with time-varying stimuli or network parameters have been active market. Especially, in [13], the dynamical characteristics of neuron behavior in time-varying periodic environments has been studied, it shows a temporally varying can influence the dynamics of a single effective neuron of the Hopfield-type, thereat, some sufficient conditions for the heteroassociative stable and for the existence (or encoding) of a global attractive (heteroassociative recall) periodic solution (or a pattern) associated with a given periodic external stimulus has been obtained. In addition, Y. Zheng and T. Chen [35] have also obtained some excellent results for the global exponential stability of periodic solution to delayed cellular neural networks (DCNNS) in time-varying periodic environments. Ott et al. [22] have also shown theoretically that one can convert a chaotic attractor to one of possible attracting time periodic motions by making time dependent perturbations of system parameters. On the other hand, in population dynamics, one can assume that the system parameters are time dependent, periodic or almost periodic, analogically, If a neuron is operating under a periodic environment such as being excited or inhibited by periodic inputs, it is not unreasonable to assume that the dissipation and gain are also periodic. Hence, It is important to understand the dynamical periodic characteristics of neuron behavior in time-varying environments.

The existence of periodic solution of neural networks has applications in learning theory [27], which is motivated by the fact that most learning systems need repetition. One is also expected that it can be applied to association memory by storing targets in periodic solutions.

In this paper, we shall discuss the so-called bidirectional associative memory (BAM) networks with distributed delays, periodic external stimulus and the periodic connection weights. We obtain a sufficient condition for the existence (or encoding) of a global exponential attractive (heteroassociative recall) periodic solution (or a pattern). Note that the BAM neural network models were originally proposed and studied by Kosko [17–19]. Since then, for the delayed (with constant delays) BAM neural networks, Gopalsamy and He [14] established some delay-independent stability results by using the Lyapunov functional approach; Cao et al. [3–5] presented several new conditions for the exponential stability and periodic oscillatory solutions of BAM networks; recently, Chen et al. [7] studied the existence and the attractivity of the almost periodic solution of cellular neural networks with continuously distributed delays. Sree Hari Rao and Nagaraj [24] recently established two easily verifiable sufficient conditions for global exponential stability of a BAM neural network model's equilibrium pattern.

To the best of our knowledge, no paper in the literature has investigated periodic BAM neural networks with distributed delays and periodic connection weights. Hence, our goal in this paper is to study the periodic attractor of BAM neural networks with distributed delays and establish some sufficient conditions for the existence of the global exponential periodic attractor. More precisely, we will consider the following model:

$$x_i'(t) = -a_i h_i\big(x_i(t)\big) + \sum_{j=1}^{p} p_{ji}(t) \int_0^{\infty} K_{ji}(u) f_j\big(y_j(t-u)\big)\,du + I_i(t), \tag{1.1a}$$

$$y_j'(t) = -b_j c_j\big(y_j(t)\big) + \sum_{i=1}^{n} q_{ij}(t) \int_0^{\infty} S_{ij}(u) g_i\big(x_i(t-u)\big)\,du + J_j(t), \tag{1.1b}$$

where $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, p$. $a_i > 0$ and $b_j > 0$ represent the rate with which the ith neuron and jth neuron will reset their potential to the resting state in isolation when they are disconnected from the network and the external inputs. n and p correspond to the number of neurons in X-layer and Y-layer, respectively, $x_i(t)$ and $y_j(t)$ are the activations of the ith neuron and the jth neuron, respectively. $p_{ji}(t)$, $q_{ij}(t)$ are the connection weights at the time t, and $I_i(t)$ and $J_j(t)$ denote the external inputs at time t. K_{ji} and S_{ij} denote the refractoriness of the ith neuron and jth neuron after they have fired or responded. f_j ($j = 1, 2, \ldots, p$), g_i ($i = 1, 2, \ldots, n$) are signal transmission functions.

The initial conditions associated with (1.1) are of the form

$$x_i(s) = \phi_i(s), \quad s \in (-\infty, 0],\ i = 1, 2, \ldots, n, \tag{1.2a}$$

$$y_j(s) = \psi_j(s), \quad s \in (-\infty, 0],\ j = 1, 2, \ldots, p, \tag{1.2b}$$

where $\phi_i(s)$, $\psi_j(s)$ are continuous bounded functions.

The organization of this paper is as follows. In Section 2, we state a definition and some useful lemmas. In Section 3, we establish the existence of periodic solution of system (1.1), the result on the global exponential periodic attractor is given in Section 4. In Section 5, we shall give an example to illustrate our main results.

2. Preliminaries

In this section, we state some notations, definitions and lemmas.

Definition 1. A periodic solution

$$\big(x^{*T}(t), y^{*T}(t)\big)^T = \big(x_1^*(t), x_2^*(t), \dots, x_n^*(t), y_1^*(t), y_2^*(t), \dots, y_p^*(t)\big)^T$$

of system (1.1) is said to be a global exponential periodic attractor, if for any solution $(x^T(t), y^T(t))^T = (x_1(t), x_2(t), \dots, x_n(t), y_1(t), y_2(t), \dots, y_p(t))^T$ of system (1.1) with initial data (1.2), there exist constants $\alpha > 0$ and $M \geqslant 1$ such that

$$\sum_{i=1}^{n} \big|x_i(t) - x_i^*(t)\big| + \sum_{j=1}^{p} \big|y_j(t) - y_j^*(t)\big| \leqslant M \big\| \big(\phi^T, \psi^T\big)^T - \big(x^{*T}, y^{*T}\big)^T \big\| e^{-\alpha t},$$

for all $t > 0$, where $\|(\phi^T, \psi^T)^T - (x^{*T}, y^{*T})^T\|$ is defined as

$$\big\| \big(\phi^T, \psi^T\big)^T - \big(x^{*T}, y^{*T}\big)^T \big\| = \sum_{i=1}^{n} \sup_{t \leqslant 0} \big|\phi_i(t) - x_i^*(t)\big| + \sum_{j=1}^{p} \sup_{t \leqslant 0} \big|\psi_j(t) - y_j^*(t)\big|.$$

Throughout this paper, we assume that

(R_1) the signal transmission functions g_i, f_j ($i = 1, 2, \dots, n$; $j = 1, 2, \dots, p$) are Lipschitz continuous on R with Lipschitz constants α_i and β_j, that is,

$$0 \leqslant \frac{g_i(x) - g_i(y)}{x - y} \leqslant \alpha_i, \quad g_i(0) = 0, \ \forall x, y \in R,$$
$$0 \leqslant \frac{f_j(x) - f_j(y)}{x - y} \leqslant \beta_j, \quad f_j(0) = 0, \ \forall x, y \in R;$$

(R_2) the functions $h_i, c_j : R \to R$ are all continuous differentiable and

$$\gamma_i = \inf_{x \in R} h_i'(x) > 0, \quad h_i(0) = 0,$$
$$\mu_j = \inf_{x \in R} c_j'(x) > 0, \quad c_j(0) = 0;$$

(R_3) for any differentiable ω-periodic function $x(t)$, the functions h_i, c_j satisfy that

$$\int_0^\omega x'(t) h_i\big(x(t)\big)\, dt = 0, \qquad \int_0^\omega x'(t) c_j\big(x(t)\big)\, dt = 0;$$

(R_4) the delay kernels $K_{ji}, S_{ij} : [0, \infty) \to [0, \infty)$ are integrable and satisfy

$$\int_0^\infty K_{ji}(s)\, ds = k_{ji} > 0, \qquad \int_0^\infty S_{ij}(s)\, ds = s_{ij} > 0;$$

(R_5) there exists a positive constant μ^* such that

$$\int_0^\infty K_{ji}(s)e^{\mu^* s}\,ds < \infty, \qquad \int_0^\infty S_{ij}(s)e^{\mu^* s}\,ds < \infty,$$
$$\int_0^\infty sK_{ji}(s)e^{\mu^* s}\,ds < \infty, \qquad \int_0^\infty sS_{ij}(s)e^{\mu^* s}\,ds < \infty,$$

where $i = 1, 2, \dots, n$; $j = 1, 2, \dots, p$.

Definition 2. (See [8].) Matrix $A = (a_{ij})_{n\times n}$ is said to be a nonsingular M-matrix, if

(i) $a_{ii} > 0$, $i = 1, 2, \dots, n$;
(ii) $a_{ij} \leqslant 0$, for $i \neq j$, $i, j = 1, 2, \dots, n$;
(iii) $A^{-1} \geqslant 0$.

Lemma 1. (See [8].) *Assume that A is a nonsingular M-matrix and $Aw \leqslant d$, then $w \leqslant A^{-1}d$.*

Lemma 2. (Young's inequality [15].) *Assume that $a > 0$, $b > 0$, $\kappa > 1$, $\frac{1}{\kappa} + \frac{1}{q} = 1$, then the inequality*

$$ab \leqslant \frac{1}{\kappa}a^\kappa + \frac{1}{q}b^q \tag{2.1}$$

holds.

Suppose that $z^*(t) := (x_1^*(t), \dots, x_n^*(t), y_1^*(t), , \dots, y_p^*(t))^T$ is a periodic solution of (1.1), *$z(t) := (x_1(t), \dots, x_n(t), y_1(t), \dots, y_p(t))^T$ is a solution of* (1.1)–(1.2). *Set*

$$\begin{aligned}
&u_i(t) = x_i(t) - x_i^*(t), \qquad v_j(t) = y_j(t) - y_j^*(t),\\
&G_i\big(u_i(t)\big) = g_i\big(u_i(t) + x_i^*(t)\big) - g_i\big(x_i^*(t)\big),\\
&F_j\big(v_j(t)\big) = f_j\big(v_j(t) + y_j^*(t)\big) - f_j\big(y_j^*(t)\big),\\
&H_i\big(u_i(t)\big) = h_i\big(u_i(t) + x_i^*(t)\big) - h_i\big(x_i^*(t)\big),\\
&C_j\big(v_j(t)\big) = c_j\big(v_j(t) + y_j^*(t)\big) - c_j\big(y_j^*(t)\big),
\end{aligned}$$

$i = 1, 2, \dots, n$ and $j = 1, 2, \dots, p$. It is easy to see that system (1.1) *can be reduced to the following system*:

$$u_i'(t) = -a_i H_i\big(u_i(t)\big) + \sum_{j=1}^{p} p_{ji}(t)\int_0^\infty K_{ji}(u)F_j\big(v_j(t-u)\big)\,du, \tag{2.2a}$$

$$v_j'(t) = -b_j C_j\big(v_j(t)\big) + \sum_{i=1}^{n} q_{ij}(t)\int_0^\infty S_{ij}(u)G_i\big(u_i(t-u)\big)\,du. \tag{2.2b}$$

By (R_1), *we have*

$$0 \leqslant \frac{G_i(u) - G_i(v)}{u - v} \leqslant \alpha_i, \quad G_i(0) = 0, \quad i = 1, 2, \ldots, n, \tag{2.3a}$$

$$0 \leqslant \frac{F_j(u) - F_j(v)}{u - v} \leqslant \beta_j, \quad F_j(0) = 0, \quad j = 1, 2, \ldots, p. \tag{2.3b}$$

$$\alpha_i \left|u_i(t)\right| \left|G_i\big(u_i(t)\big)\right|^{\kappa-1} \geqslant \left|G_i\big(u_i(t)\big)\right|^{\kappa}, \quad i = 1, 2, \ldots, n, \tag{2.4a}$$

$$\beta_j \left|v_j(t)\right| \left|F_j\big(v_j(t)\big)\right|^{\kappa-1} \geqslant \left|F_j\big(v_j(t)\big)\right|^{\kappa}, \quad j = 1, 2, \ldots, p, \tag{2.4b}$$

where $\kappa > 1$ is a constant.

Lemma 3. *Assume that* (R_1) *holds, then*

$$\int_0^u \operatorname{sgn}(s) \left|G_i(s)\right|^{\kappa-1} ds \leqslant |u| \left|G_i(u)\right|^{\kappa-1}, \quad i = 1, 2, \ldots, n, \tag{2.5a}$$

$$\int_0^u \operatorname{sgn}(s) \left|F_j(s)\right|^{\kappa-1} ds \leqslant |v| \left|F_j(u)\right|^{\kappa-1}, \quad j = 1, 2, \ldots, p, \tag{2.5b}$$

where $\kappa > 1$ is a constant.

Proof. We will divide the proof into two cases.

Case 1. $0 \leqslant s \leqslant u$. The condition (R_1) implies that (2.3a) holds, hence, for $\kappa > 1$, we have

$$0 \leqslant G_i(s) \leqslant G_i(u), \qquad 0 \leqslant \operatorname{sgn}(s) \left|G_i(s)\right|^{\kappa-1} \leqslant \operatorname{sgn}(u) \left|G_i(u)\right|^{\kappa-1}. \tag{2.6}$$

Integrating (2.6) from 0 to u, we obtain

$$\int_0^u \operatorname{sgn}(s) \left|G_i(s)\right|^{\kappa-1} ds \leqslant |u| \left|G_i(u)\right|^{\kappa-1}, \quad i = 1, 2, \ldots, n.$$

Case 2. $u \leqslant s \leqslant 0$. We have by (2.3a) that

$$0 \geqslant G_i(s) \geqslant G_i(u), \qquad 0 \geqslant \operatorname{sgn}(s) \left|G_i(s)\right|^{\kappa-1} \geqslant \operatorname{sgn}(u) \left|G_i(u)\right|^{\kappa-1}. \tag{2.7}$$

Integrating (2.7) from u to 0, we get

$$\int_u^0 \operatorname{sgn}(s) \left|G_i(s)\right|^{\kappa-1} ds \geqslant -u \operatorname{sgn}(u) \left|G_i(u)\right|^{\kappa-1}, \quad i = 1, 2, \ldots, n,$$

i.e.,

$$\int_0^u \operatorname{sgn}(s) \left|G_i(s)\right|^{\kappa-1} ds \leqslant |u| \left|G_i(u)\right|^{\kappa-1}, \quad i = 1, 2, \ldots, n.$$

Similarly, we can prove that

$$\int_0^u \operatorname{sgn}(s) \left|F_j(s)\right|^{\kappa-1} ds \leqslant |u| \left|F_j(u)\right|^{\kappa-1}, \quad j = 1, 2, \ldots, p. \qquad \Box$$

Lemma 4. *Assume that the assumption* (R_1) *holds, then for all* $u \in R$*, we have*

$$\int_0^u \operatorname{sgn}(s)\left|G_i(s)\right|^{\kappa-1} ds \geqslant \frac{1}{\kappa\alpha_i}\left|G_i(u)\right|^{\kappa}, \quad i = 1, 2, \ldots, n, \tag{2.8a}$$

$$\int_0^u \operatorname{sgn}(s)\left|F_j(s)\right|^{\kappa-1} ds \geqslant \frac{1}{\kappa\beta_j}\left|F_j(u)\right|^{\kappa}, \quad j = 1, 2, \ldots, p. \tag{2.8b}$$

Proof. We define the continuous functions

$$E_i(u) = \int_0^u \operatorname{sgn}(s)\left|G_i(s)\right|^{\kappa-1} ds - \frac{1}{\kappa\alpha_i}\left|G_i(u)\right|^{\kappa}, \quad i = 1, 2, \ldots, n,$$

then, we obtain that

$$\begin{aligned} D^+E_i(u) &= \operatorname{sgn}(u)\left|G_i(u)\right|^{\kappa-1} - \frac{1}{\alpha_i}\left|G_i(u)\right|^{\kappa-1} D^+\left|G_i(u)\right| \\ &= \left|G_i(u)\right|^{\kappa-1}\left[\operatorname{sgn}(u) - \frac{1}{\alpha_i} D^+\left|G_i(u)\right|\right], \quad i = 1, 2, \ldots, n. \end{aligned}$$

Since $0 \leqslant D^+|G_i(u)| \leqslant \alpha_i$ for all $u \in R$, we have

$$D^+E_i(u) \begin{cases} \geqslant 0, & \text{for } u > 0, \\ \leqslant 0, & \text{for } u < 0, \\ = 0, & \text{for } u = 0, \end{cases}$$

where $i = 1, 2, \ldots, n$. This shows that $u = 0$ is the minimum value point of the function $E_i(u)$. Thus

$$E_i(u) \geqslant E_i(0) = 0, \quad u \in R.$$

Hence

$$\int_0^u \operatorname{sgn}(s)\left|G_i(s)\right|^{\kappa-1} ds \geqslant \frac{1}{\kappa\alpha_i}\left|G_i(u)\right|^{\kappa}, \quad i = 1, 2, \ldots, n.$$

Similarly, we can obtain

$$\int_0^u \operatorname{sgn}(s)\left|F_j(s)\right|^{\kappa-1} ds \geqslant \frac{1}{\kappa\beta_j}\left|F_j(u)\right|^{\kappa}, \quad j = 1, 2, \ldots, p.$$

This completes the proof. □

To obtain the existence of the periodic solution of (1.1), we shall introduce some results from Gaines and Mawhin [12].

Consider an abstract equation in a Banach space X,

$$Lz = \lambda Nz, \quad \lambda \in (0, 1), \tag{2.9}$$

where $L : \operatorname{Dom} L \cap X \to X$ is a linear operator and λ is a parameter. Let P and Q denote two projectors

$$P : X \cap \operatorname{Dom} L \to \operatorname{Ker} L \quad \text{and} \quad Q : X \to \mathrm{X}/\operatorname{Im} L.$$

Lemma 5. (See [12].) *Let X be a Banach space and L be a Fredholm mapping of index zero. Assume that $N:\overline{\Omega}\to X$ is a L-compact on $\overline{\Omega}$ with Ω bounded in X. Furthermore, assume that*

(a) *for each $\lambda\in(0,1)$, $z\in\partial\Omega\cap\operatorname{Dom}L$,*

$$Lz\neq\lambda Nz;$$

(b) *for each $z\in\partial\Omega\cap\operatorname{Ker}L$,*

$$QNz\neq 0\quad\text{and}\quad\deg\{QNz,\Omega\cap\operatorname{Ker}L,0\}\neq 0.$$

Then $Lz=Nz$ has at least one solution in $\overline{\Omega}$.

Throughout this paper, we always assume that $p_{ji}(t), q_{ij}(t), I_i(t), J_j(t)$ are continuous ω-periodic functions, a_i, b_j are positive constants,

$$p_{ji}^{+}=\max_{0\leqslant t\leqslant\omega}\bigl|p_{ji}(t)\bigr|,\qquad q_{ij}^{+}=\max_{0\leqslant t\leqslant\omega}\bigl|q_{ij}(t)\bigr|,\qquad \tilde{p}_{ji}=\frac{1}{\omega}\int_0^{\omega}p_{ji}(t)\,dt,$$

$$I_i^{+}=\max_{0\leqslant t\leqslant\omega}\bigl|I_i(t)\bigr|,\qquad J_j^{+}=\max_{0\leqslant t\leqslant\omega}\bigl|J_j(t)\bigr|,\qquad \tilde{q}_{ij}=\frac{1}{\omega}\int_0^{\omega}q_{ij}(t)\,dt,$$

$$\tilde{I}_i=\frac{1}{\omega}\int_0^{\omega}I_i(t)\,dt,\qquad \tilde{J}_j=\frac{1}{\omega}\int_0^{\omega}J_j(t)\,dt.$$

3. Existence of periodic solutions of (1.1)

In this section, we study the existence of periodic solution of system (1.1) by using the continuation theorem of Mawhin's coincidence degree theory (Lemma 5).

Theorem 1. *Assume that* (R_1)–(R_4) *hold and*

(i) *A is a nonsingular M-matrix,*
(ii) $\min(\min_{1\leqslant i\leqslant n}(a_i\gamma_i-\alpha_i\sum_{j=1}^{p}|\tilde{q}_{ij}|s_{ij}),\min_{1\leqslant j\leqslant p}(b_j\mu_j-\beta_j\sum_{i=1}^{n}|\widetilde{p}_{ji}|k_{ji}))>0$,

then system (1.1) *has at least one ω-periodic solution, where*

$$A=\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix},\qquad A_{11}=(a_i\gamma_i\delta_{il})_{n\times n},\ A_{22}=(b_j\mu_j\delta_{jr})_{p\times p},$$
$$A_{12}=\bigl(-p_{ji}^{+}k_{ji}\beta_j(1+\omega a_i\gamma_i)\bigr)_{n\times p},\ A_{21}=\bigl(-q_{ij}^{+}s_{ij}\alpha_i(1+\omega b_j\mu_j)\bigr)_{p\times n}.$$

Proof. Let $x(t)=(x_1(t),x_2(t),\dots,x_n(t))^T$, $y(t)=(y_1(t),y_2(t),\dots,y_p(t))^T$. In order to apply Lemma 5 to systems (1.1), we take $X=\{(x^T(t),y^T(t))^T\in C(R,R^{n+p})\colon\ x(t+\omega)=x(t),\ y(t+\omega)=y(t),$ for some $\omega>0\}$ equipped with the norm

$$\bigl\|\bigl(x^T,y^T\bigr)^T\bigr\|=\sum_{i=1}^{n}\max_{t\in[0,\omega]}\bigl|x_i(t)\bigr|+\sum_{j=1}^{p}\max_{t\in[0,\omega]}\bigl|y_j(t)\bigr|,$$

then X is a Banach space. For any

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \in X,$$

define

$$N\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \left(-a_i h_i(x_i(t)) + \sum_{j=1}^{p} p_{ji}(t)\int_0^\infty K_{ji}(u) f_j(y_j(t-u))\,du + I_i(t)\right)_{n\times 1} \\ \left(-b_j c_j(y_j(t)) + \sum_{i=1}^{n} q_{ij}(t)\int_0^\infty S_{ij}(u) g_i(x_i(t-u))\,du + J_j(t)\right)_{p\times 1} \end{pmatrix},$$

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}, \qquad P\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = Q\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \left(\frac{1}{\omega}\int_0^\omega x_i(t)\,dt\right)_{n\times 1} \\ \left(\frac{1}{\omega}\int_0^\omega y_j(t)\,dt\right)_{p\times 1} \end{pmatrix}.$$

It is easy to prove that L is a Fredholm mapping of index zero, that $P : X \cap \operatorname{Dom} L \to \operatorname{Ker} L$ and $Q : X \to X/\operatorname{Im} L$ are two projectors, and N is L-compact on $\overline{\Omega}$ for any given open bounded set.

Corresponding to Eq. (2.9), we have

$$x_i'(t) = -\lambda a_i h_i\big(x_i(t)\big) + \lambda \sum_{j=1}^{p} p_{ji}(t) \int_0^\infty K_{ji}(u) f_j\big(y_j(t-u)\big)\,du + \lambda I_i(t), \tag{3.1a}$$

$$y_j'(t) = -\lambda b_j c_j\big(y_j(t)\big) + \lambda \sum_{i=1}^{n} q_{ij}(t) \int_0^\infty S_{ij}(u) g_i\big(x_i(t-u)\big)\,du + \lambda J_j(t), \tag{3.1b}$$

where $i = 1, \ldots, n$, $j = 1, \ldots, p$. Suppose that $z(t) \in X$ is a solution of system (3.1) for a certain $\lambda \in (0, 1)$. For the sake of convenience, define $\|x\|_2$ by

$$\|x\|_2 = \left(\int_0^\omega \big|x(t)\big|^2\,dt\right)^{\frac{1}{2}}, \quad \text{for } x \in C(R, R).$$

From (R_1), we have

$$\big|g_i(x)\big| \leqslant \alpha_i |x|, \quad i = 1, 2, \ldots, n,$$
$$\big|f_j(x)\big| \leqslant \beta_j |x|, \quad j = 1, 2, \ldots, p.$$

Multiplying both sides of Eq. (3.1a) by x_i', integrating over $[0, \omega]$ and in view of the assumption (R_3), we obtain

$$\begin{aligned} \|x_i'\|_2^2 &= \lambda \sum_{j=1}^{p} \int_0^\omega p_{ji}(t) x_i'(t) \int_0^\infty K_{ji}(u) f_j\big(y_j(t-u)\big)\,du\,dt + \lambda \int_0^\omega x_i'(t) I_i(t)\,dt \\ &\leqslant \sum_{j=1}^{p} p_{ji}^{+} \beta_j \int_0^\omega \big|x_i'(t)\big| \int_0^\infty K_{ji}(u)\big|y_j(t-u)\big|\,du\,dt + I_i^{+} \int_0^\omega \big|x_i'(t)\big|\,dt \\ &\leqslant \sum_{j=1}^{p} p_{ji}^{+} \beta_j \int_0^\omega \int_0^\infty \big|x_i'(t)\big|\big|y_j(t-u)\big| K_{ji}(u)\,du\,dt + \sqrt{\omega} I_i^{+} \left(\int_0^\omega \big|x_i'\big|^2\,dt\right)^{\frac{1}{2}} \end{aligned}$$

$$= \sum_{j=1}^{p} p_{ji}^{+}\beta_j \int_0^\infty K_{ji}(u) \int_0^\omega \left|x_i'(t)\right|\left|y_j(t-u)\right| dt\, du + \sqrt{\omega} I_i^{+} \left\|x_i'\right\|_2$$

$$\leqslant \sum_{j=1}^{p} p_{ji}^{+}\beta_j \int_0^\infty K_{ji}(u) \left(\int_0^\omega \left|x_i'(t)\right|^2 dt\right)^{\frac{1}{2}} \left(\int_0^\omega \left|y_j(t-u)\right|^2 dt\right)^{\frac{1}{2}} du + \sqrt{\omega} I_i^{+} \left\|x_i'\right\|_2$$

$$\leqslant \sum_{j=1}^{p} p_{ji}^{+} k_{ji}\beta_j \left\|x_i'\right\|_2 \|y_j\|_2 + \sqrt{\omega} I_i^{+} \left\|x_i'\right\|_2.$$

This implies that

$$\left\|x_i'\right\|_2 \leqslant \sum_{j=1}^{p} p_{ji}^{+} k_{ji}\beta_j \|y_j\|_2 + \sqrt{\omega} I_i^{+}, \tag{3.2}$$

where

$$k_{ji} = \int_0^\infty K_{ji}(u)\, du.$$

A similar argument shows we have

$$\left\|y_j'\right\|_2 \leqslant \sum_{i=1}^{n} q_{ij}^{+} s_{ij}\alpha_i \|x_i\|_2 + \sqrt{\omega} J_j^{+}, \tag{3.3}$$

where

$$s_{ij} = \int_0^\infty S_{ij}(u)\, du.$$

Integrating Eq. (3.1a) over $[0, \omega]$, we get

$$a_i \int_0^\omega h_i\big(x_i(t)\big)\, dt = \sum_{j=1}^{p} \int_0^\omega p_{ji}(t) \int_0^\infty K_{ji}(u) f_j\big(y_j(t-u)\big)\, du\, dt + \int_0^\omega I_i(t)\, dt.$$

Thus there exists a point $\xi_i \in (0, \omega)$ such that

$$\omega a_i h_i\big(x_i(\xi_i)\big) = \sum_{j=1}^{p} \int_0^\omega p_{ji}(t) \int_0^\infty K_{ji}(u) f_j\big(y_j(t-u)\big)\, du\, dt + \int_0^\omega I_i(t)\, dt.$$

So, we have

$$\omega a_i \left|h_i\big(x_i(\xi_i)\big)\right| \leqslant \sum_{j=1}^{p} p_{ji}^{+}\beta_j \int_0^\omega \int_0^\infty K_{ji}(u)\left|y_j(t-u)\right| du\, dt + \omega I_i^{+}$$

$$= \sum_{j=1}^{p} p_{ji}^{+}\beta_j \int_0^\infty K_{ji}(u) \int_0^\omega \left|y_j(t-u)\right| dt\, du + \omega I_i^{+}$$

$$= \sum_{j=1}^{p} p_{ji}^{+} \beta_j \int_0^{\infty} K_{ji}(u) \int_0^{\omega} |y_j(t)|\, dt\, du + \omega I_i^{+}$$
$$= \sum_{j=1}^{p} p_{ji}^{+} k_{ji} \beta_j \int_0^{\omega} |y_j(t)|\, dt + \omega I_i^{+}$$
$$\leqslant \sqrt{\omega} \sum_{j=1}^{p} p_{ji}^{+} k_{ji} \beta_j \|y_j\|_2 + \omega I_i^{+}.$$

That is

$$\sqrt{\omega} a_i \big| h_i\big(x_i(\xi_i)\big)\big| \leqslant \sum_{j=1}^{p} p_{ji}^{+} k_{ji} \beta_j \|y_j\|_2 + \sqrt{\omega} I_i^{+}.$$

By the assumption (R_2), we have

$$\gamma_i \big|x_i(\xi_i)\big| \leqslant \big|h_i\big(x_i(\xi_i)\big)\big|.$$

Therefore,

$$\sqrt{\omega} a_i \gamma_i \big|x_i(\xi_i)\big| \leqslant \sum_{j=1}^{p} p_{ji}^{+} k_{ji} \beta_j \|y_j\|_2 + \sqrt{\omega} I_i^{+}. \tag{3.4}$$

Again, since

$$\big|x_i(t)\big| \leqslant \big|x_i(\xi_i)\big| + \int_0^{\omega} \big|x_i'(t)\big|\, dt \leqslant \big|x_i(\xi_i)\big| + \sqrt{\omega} \big\|x_i'\big\|_2, \quad \text{for } t \in [0, \omega], \tag{3.5}$$

and so

$$a_i \gamma_i \|x_i\|_2 = a_i \gamma_i \left(\int_0^{\omega} \big|x_i(t)\big|^2\, dt \right)^{\frac{1}{2}} \leqslant \sqrt{\omega} a_i \gamma_i \max_{0 \leqslant t \leqslant \omega} \big|x_i(t)\big|$$
$$\leqslant \sqrt{\omega} a_i \gamma_i \big|x_i(\xi_i)\big| + \omega a_i \gamma_i \big\|x_i'\big\|_2$$
$$\leqslant \sum_{j=1}^{p} p_{ji}^{+} k_{ji} \beta_j \|y_j\|_2 + \sqrt{\omega} I_i^{+} + \omega a_i \gamma_i \big\|x_i'\big\|_2. \tag{3.6}$$

Substituting (3.2) into (3.6), we get

$$a_i \gamma_i \|x_i\|_2 \leqslant \sum_{j=1}^{p} p_{ji}^{+} k_{ji} \beta_j \|y_j\|_2 + \sqrt{\omega} I_i^{+} + \omega a_i \gamma_i \left[\sum_{j=1}^{p} p_{ji}^{+} k_{ji} \beta_j \|y_j\|_2 + \sqrt{\omega} I_i^{+} \right]$$
$$= \sum_{j=1}^{p} p_{ji}^{+} k_{ji} \beta_j (1 + \omega a_i \gamma_i) \|y_j\|_2 + \sqrt{\omega} (1 + \omega a_i \gamma_i) I_i^{+},$$

i.e.,

$$a_i \gamma_i \|x_i\|_2 - \sum_{j=1}^{p} p_{ji}^{+} k_{ji} \beta_j (1 + \omega a_i \gamma_i) \|y_j\|_2 \leqslant \sqrt{\omega} (1 + \omega a_i \gamma_i) I_i^{+} \triangleq d_i. \tag{3.7}$$

Integrating Eq. (3.1b) from 0 to ω, and using a similar argument to (3.6), we get

$$b_j\mu_j\|y_j\|_2 \leqslant \sum_{i=1}^{n} q_{ij}^{+} s_{ij}\alpha_i\|x_i\|_2 + \sqrt{\omega}J_j^{+} + \omega b_j\mu_j\left\|y_j'\right\|_2. \tag{3.8}$$

Substituting (3.3) into (3.8), we get

$$\begin{aligned} b_j\mu_j\|y_j\|_2 &\leqslant \sum_{i=1}^{n} q_{ij}^{+} s_{ij}\alpha_i\|x_i\|_2 + \omega b_j\mu_j\left[\sum_{i=1}^{n} q_{ij}^{+} s_{ij}\alpha_i\|x_i\|_2 + \sqrt{\omega}J_j^{+}\right] + \sqrt{\omega}J_j^{+} \\ &= \sum_{i=1}^{n} q_{ij}^{+} s_{ij}\alpha_i(1+\omega b_j\mu_j)\|x_i\|_2 + (1+\omega b_j\mu_j)\sqrt{\omega}J_j^{+}, \end{aligned}$$

that is

$$-\sum_{i=1}^{n} q_{ij}^{+} s_{ij}\alpha_i(1+\omega b_j\mu_j)\|x_i\|_2 + b_j\mu_j\|y_j\|_2 \leqslant (1+\omega b_j\mu_j)\sqrt{\omega}J_j^{+} \triangleq d_{n+j}. \tag{3.9}$$

The formulas (3.7) and (3.9) may be rewritten in the form

$$Aw \leqslant d, \tag{3.10}$$

where

$$\begin{aligned} &A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad A_{11} = (a_i\gamma_i\delta_{il})_{n\times n},\ A_{22} = (b_j\mu_j\delta_{jr})_{p\times p}, \\ &A_{12} = \left(-p_{ji}^{+}k_{ji}\beta_j(1+\omega a_i\gamma_i)\right)_{n\times p},\ A_{21} = \left(-q_{ij}^{+}s_{ij}\alpha_i(1+\omega b_j\mu_j)\right)_{p\times n}, \\ &w = \left(\|x_1\|_2, \|x_2\|_2, \ldots, \|x_n\|_2, \|y_1\|_2, \|y_2\|_2, \ldots, \|y_p\|_2\right)^T, \\ &d = (d_1, d_2, \ldots, d_{n+p})^T. \end{aligned}$$

From condition (i) of Theorem 1 and Lemma 2 it follows that

$$w \leqslant A^{-1}d \triangleq \left(R_1^*, R_2^*, \ldots, R_{n+p}^*\right)^T,$$

that is

$$\|x_i\|_2 \leqslant R_i^*, \qquad \|y_j\|_2 \leqslant R_{n+j}^*, \quad i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, p. \tag{3.11}$$

From (3.2)–(3.5) and (3.11), there exist $n + p$ positive constants R_l ($l = 1, 2, \ldots, n + p$) such that

$$\left|x_i(t)\right| \leqslant R_i, \qquad \left|y_j(t)\right| \leqslant R_{n+j}, \quad i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, p,\ \text{for } \forall t \in [0, \omega].$$

Clearly, R_l ($l = 1, 2, \ldots, n + p$) are independent of λ. Denote

$$M^* = \sum_{l=1}^{n+p} R_l + C,$$

where $C > 0$ is taken sufficiently large such that

$$\begin{aligned} &\min\left[\min_{1\leqslant i\leqslant n}\left(a_i\gamma_i - \alpha_i\sum_{j=1}^{p}|\widetilde{q}_{ij}|s_{ij}\right), \min_{1\leqslant j\leqslant p}\left(b_j\mu_j - \beta_j\sum_{i=1}^{n}|\tilde{p}_{ji}|k_{ji}\right)\right]M^* \\ &\quad > \sum_{i=1}^{n}|\tilde{I}_i| + \sum_{j=1}^{p}|\tilde{J}_j|. \end{aligned} \tag{3.12}$$

Now we take $\Omega = \{(x^T(t), y^T(t))^T \in X\colon\ \|(x^T, y^T)^T\| < M^*\}$, clearly condition (a) of Lemma 5 is satisfied.

When $(x^T, y^T)^T \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^{n+p}$, $(x^T, y^T)^T$ is a constant vector in R^{n+p} with $\sum_{i=1}^n |x_i| + \sum_{j=1}^p |y_j| = M^*$, then

$$QN\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \left(-a_i h_i(x_i) + \sum_{j=1}^p \tilde{p}_{ji} k_{ji} f_j(y_j) + \tilde{I}_i\right)_{n\times 1} \\ \left(-b_j c_j(y_j) + \sum_{i=1}^n \tilde{q}_{ij} s_{ij} g_i(x_i) + \tilde{J}_j\right)_{p\times 1} \end{pmatrix}.$$

Therefore,

$$\begin{aligned}
&\left\| QN\left(x^T, y^T\right)^T \right\| \\
&\quad = \sum_{i=1}^n \left| -a_i h_i(x_i) + \sum_{j=1}^p \tilde{p}_{ji} k_{ji} f_j(y_j) + \widetilde{I}_i \right| + \sum_{j=1}^p \left| -b_j c_j(y_j) + \sum_{i=1}^n \tilde{q}_{ij} s_{ij} g_i(x_i) + \widetilde{J}_j \right| \\
&\quad \geqslant \sum_{i=1}^n a_i \left| h_i(x_i) \right| - \sum_{i=1}^n \sum_{j=1}^p |k_{ji} \tilde{p}_{ji}| \left| f_j(y_j) \right| - \sum_{i=1}^n |\widetilde{I}_i| \\
&\qquad + \sum_{j=1}^p b_j \left| c_j(y_j) \right| - \sum_{j=1}^p \sum_{i=1}^n |\tilde{q}_{ij}| s_{ij} \left| g_i(x_i) \right| - \sum_{j=1}^p |\widetilde{J}_j| \\
&\quad \geqslant \sum_{i=1}^n a_i \gamma_i |x_i| - \sum_{i=1}^n \sum_{j=1}^p |\widetilde{p}_{ji}| k_{ji} \beta_j |y_j| - \sum_{i=1}^n |\tilde{I}_i| \\
&\qquad + \sum_{j=1}^p b_j \mu_j |y_j| - \sum_{j=1}^p \sum_{i=1}^n |\widetilde{q}_{ij}| s_{ij} \alpha_i |x_i| - \sum_{j=1}^p |\tilde{J}_j| \\
&\quad = \sum_{i=1}^n \left(a_i \gamma_i - \alpha_i \sum_{j=1}^p |\tilde{q}_{ij}| s_{ij} \right) |x_i| + \sum_{j=1}^p \left(b_j \mu_j - \beta_j \sum_{i=1}^n |\tilde{p}_{ji}| k_{ji} \right) |y_j| \\
&\qquad - \left(\sum_{i=1}^n |\tilde{I}_i| + \sum_{j=1}^p |\tilde{J}_j| \right) \\
&\quad \geqslant \min\left[\min_{1\leqslant i\leqslant n} \left(a_i \gamma_i - \alpha_i \sum_{j=1}^p s_{ij} |\tilde{q}_{ij}| \right), \min_{1\leqslant j\leqslant p} \left(b_j \mu_j - \beta_j \sum_{i=1}^n |\tilde{p}_{ji}| k_{ji} \right) \right] M^* \\
&\qquad - \left(\sum_{i=1}^n |\tilde{I}_i| + \sum_{j=1}^p |\tilde{J}_j| \right) \\
&\quad > 0.
\end{aligned}$$

Consequently,

$$QN\left(x^T, y^T\right)^T \neq (0, 0)^T, \quad \text{for } \left(x^T, y^T\right)^T \in \partial\Omega \cap \operatorname{Ker} L.$$

This shows that condition (b) of Lemma 5 holds.

Define $\Phi : \mathrm{Dom}\, L \times [0,1] \to X$ by

$$\Phi(x, y, \mu) = \mu \begin{pmatrix} (-a_i h_i(x_i))_{n\times 1} \\ (-b_j c_j(y_j))_{p\times 1} \end{pmatrix} + (1-\mu) \begin{pmatrix} \left(-a_i h_i(x_i) + \sum_{j=1}^{p} \tilde{p}_{ji} k_{ji} f_j(y_j) + \tilde{I}_i\right)_{n\times 1} \\ \left(-b_j c_j(y_j) + \sum_{i=1}^{n} \tilde{q}_{ij} s_{ij} g_i(x_i) + \tilde{J}_j\right)_{p\times 1} \end{pmatrix},$$

where $(x^T, y^T)^T \in R^{n+p}$, $\mu \in [0,1]$.

When $(x^T, y^T)^T \in \partial\Omega \cap \mathrm{Ker}\, L$ and $\mu \in [0,1]$, $(x^T, y^T)^T$ is a constant vector in R^{n+p} with $\sum_{i=1}^{n} |x_i| + \sum_{j=1}^{p} |y_j| = M^*$. Thus

$$\begin{aligned}
&\left\| \Phi\left(x^T, y^T, \mu\right) \right\| \\
&\quad = \sum_{i=1}^{n} \left| -\mu a_i h_i(x_i) + (1-\mu)\left(-a_i h_i(x_i) + \sum_{j=1}^{p} \tilde{p}_{ji} k_{ji} f_j(y_j) + \widetilde{I}_i \right) \right| \\
&\qquad + \sum_{j=1}^{p} \left| -\mu b_j c_j(y_j) + (1-\mu)\left(-b_j c_j(y_j) + \sum_{i=1}^{n} \tilde{q}_{ij} s_{ij} g_i(x_i) + \tilde{J}_j \right) \right| \\
&\quad = \sum_{i=1}^{n} \left| a_i h_i(x_i) - (1-\mu)\left[\sum_{j=1}^{p} \tilde{p}_{ji} k_{ji} f_j(y_j) + \tilde{I}_i \right] \right| \\
&\qquad + \sum_{j=1}^{p} \left| b_j c_j(y_j) - (1-\mu)\left[\sum_{i=1}^{n} \tilde{q}_{ij} s_{ij} g_i(x_i) + \tilde{J}_j \right] \right| \\
&\quad \geqslant \min\left[\min_{1\leqslant i\leqslant n} \left(a_i \gamma_i - \alpha_i \sum_{j=1}^{p} |\widetilde{q}_{ij}| s_{ij} \right), \min_{1\leqslant j\leqslant p} \left(b_j \mu_j - \beta_j \sum_{i=1}^{n} |\tilde{p}_{ji}| k_{ji} \right) \right] M^* \\
&\qquad - \left(\sum_{i=1}^{n} |\tilde{I}_i| + \sum_{j=1}^{p} |\tilde{J}_j| \right) > 0.
\end{aligned}$$

Therefore,

$$\Phi\left(x^T, y^T, \mu\right) \neq (0,0)^T, \quad \text{for } \left(x^T, y^T\right)^T \in \partial\Omega \cap \mathrm{Ker}\, L.$$

As a result, we have

$$\begin{aligned}
&\deg\left(QN\left(x^T, y^T\right)^T, \Omega \cap \mathrm{Ker}\, L, (0,0)^T \right) \\
&\quad = \deg\left(\left(\left(-a_i h_i(x_i)\right)_{n\times 1}, \left(-b_j c_j(y_j)\right)_{p\times 1} \right)^T, \Omega \cap \mathrm{Ker}\, L\, (0,0) \right)^T \\
&\quad \neq 0.
\end{aligned}$$

By now, we know that all conditions of Lemma 5 are satisfied and hence system (1.1) admits at least one ω-periodic solution and the proof is complete. □

4. Global exponential periodic attractor

In this section, we shall discuss the global exponential periodic attractor of BAM with continuously distributed delays, which associate with the given periodic external stimulus and the connection weights.

Theorem 2. *Assume that all conditions of Theorem* 1 *and* (R_5) *hold. If there exist constants* λ_i, λ_{n+j} *(*$i = 1, 2, \dots, n$, $j = 1, 2, \dots, p$*) and* $\kappa > 1$ *satisfying the following conditions denoted by* (R_6):

$$\lambda_i\left[-\kappa a_i\gamma_i + (\kappa-1)\alpha_i\sum_{j=1}^{p} p_{ji}^{+}k_{ji}\right] + \sum_{j=1}^{p}\lambda_{n+j}\beta_j q_{ij}^{+}s_{ij} < 0, \tag{4.1a}$$

$$\lambda_{n+j}\left[-\kappa b_j\mu_j + (\kappa-1)\beta_j\sum_{i=1}^{n} q_{ij}^{+}s_{ij}\right] + \sum_{i=1}^{n}\lambda_i\alpha_i p_{ji}^{+}k_{ji} < 0, \tag{4.1b}$$

then ω*-periodic solution of the system* (1.1) *is global exponential periodic attractor.*

Proof. In view of (R_5), (R_6), we can choose a suitable constant

$$\alpha \in \left(0, \min\left\{\mu^*, \min_{1\leqslant i\leqslant n,\, 1\leqslant j\leqslant p}\{a_i\gamma_i, b_j\mu_j\}\right\}\right) \tag{4.2}$$

such that

$$\lambda_i\left[\kappa(\alpha - a_i\gamma_i) + (\kappa-1)\alpha_i\sum_{j=1}^{p} p_{ji}^{+}k_{ji}\right] + \sum_{j=1}^{p}\lambda_{n+j}\beta_j q_{ij}^{+}\int_0^\infty S_{ij}(u)e^{\alpha u}\,du < 0, \tag{4.3a}$$

$$\lambda_{n+j}\left[\kappa(\alpha - b_j\mu_j) + (\kappa-1)\beta_j\sum_{i=1}^{n} q_{ij}^{+}s_{ij}\right] + \sum_{i=1}^{n}\lambda_i\alpha_i p_{ji}^{+}\int_0^\infty K_{ji}(u)e^{\alpha u}\,du < 0. \tag{4.3b}$$

Consider a Lyapunov functional $V(t)$ defined by

$$V(t) = V_1(t) + V_2(t),$$

where

$$\begin{aligned} V_1(t) = {} & \sum_{i=1}^{n}\kappa\lambda_i\alpha_i e^{\alpha t}\int_0^{u_i(t)}\operatorname{sgn}(s)\bigl|G_i(s)\bigr|^{\kappa-1}\,ds \\ & + \sum_{i=1}^{n}\lambda_i\sum_{j=1}^{p} p_{ji}^{+}\alpha_i\int_0^\infty\int_{t-u}^{t} K_{ji}(u)\bigl|F_j\bigl(v_j(s)\bigr)\bigr|^{\kappa}e^{\alpha(s+u)}\,ds\,du, \end{aligned} \tag{4.4a}$$

$$\begin{aligned} V_2(t) = {} & \sum_{j=1}^{p}\kappa\lambda_{n+j}\beta_j e^{\alpha t}\int_0^{v_j(t)}\operatorname{sgn}(s)\bigl|F_j(s)\bigr|^{\kappa-1}\,ds \\ & + \sum_{j=1}^{p}\lambda_{n+j}\sum_{i=1}^{n} q_{ij}^{+}\beta_j\int_0^\infty\int_{t-u}^{t} S_{ij}(u)\bigl|G_i\bigl(u_i(s)\bigr)\bigr|^{\kappa}e^{\alpha(s+u)}\,ds\,du. \end{aligned} \tag{4.4b}$$

Calculating the upper-right derivative of the $V_i(t)$ $(i = 1, 2)$ along solutions of (2.2), we have

$$\begin{aligned}
&\left.\frac{dV_1}{dt}\right|_{(2.2)}\\
&= e^{\alpha t}\sum_{i=1}^{n}\lambda_i\left\{\kappa\alpha\alpha_i\int_0^{u_i(t)}\operatorname{sgn}(s)\left|G_i(s)\right|^{\kappa-1}ds+\kappa\alpha_i\operatorname{sgn}\bigl(u_i(t)\bigr)\left|G_i\bigl(u_i(t)\bigr)\right|^{\kappa-1}u_i'(t)\right.\\
&\quad\left.+\sum_{j=1}^{p}p_{ji}^{+}\alpha_i\int_0^{\infty}K_{ji}(u)\left[\left|F_j\bigl(v_j(t)\bigr)\right|^{\kappa}e^{\alpha u}-\left|F_j\bigl(v_j(t-u)\bigr)\right|^{\kappa}\right]du\right\}\\
&= e^{\alpha t}\sum_{i=1}^{n}\lambda_i\left\{\kappa\alpha\alpha_i\int_0^{u_i(t)}\operatorname{sgn}(s)\left|G_i(s)\right|^{\kappa-1}ds\right.\\
&\quad-\kappa\alpha_i a_i\operatorname{sgn}\bigl(u_i(t)\bigr)H_i\bigl(u_i(t)\bigr)\left|G_i\bigl(u_i(t)\bigr)\right|^{\kappa-1}\\
&\quad+\sum_{j=1}^{p}\kappa\alpha_i p_{ji}(t)\operatorname{sgn}\bigl(u_i(t)\bigr)\left|G_i\bigl(u_i(t)\bigr)\right|^{\kappa-1}\int_0^{\infty}K_{ji}(u)F_j\bigl(v_j(t-u)\bigr)\,du\\
&\quad+\sum_{j=1}^{p}p_{ji}^{+}\alpha_i\int_0^{\infty}K_{ji}(u)e^{\alpha u}\,du\left|F_j\bigl(v_j(t)\bigr)\right|^{\kappa}\\
&\quad\left.-\sum_{j=1}^{p}p_{ji}^{+}\alpha_i\int_0^{\infty}K_{ji}(u)\left|F_j\bigl(v_j(t-u)\bigr)\right|^{\kappa}du\right\}.
\end{aligned}$$

By assumption (R_2), we have

$$\operatorname{sgn}\bigl(u_i(t)\bigr)H_i\bigl(u_i(t)\bigr)\geqslant\gamma_i\left|u_i(t)\right|.$$

From the above formula, (2.4), Lemmas 3 and 4, we have

$$\begin{aligned}
\left.\frac{dV_1}{dt}\right|_{(2.2)} &\leqslant e^{\alpha t}\sum_{i=1}^{n}\lambda_i\left\{\kappa\alpha\alpha_i\left|u_i(t)\right|\left|G_i\bigl(u_i(t)\bigr)\right|^{\kappa-1}-\kappa\alpha_i a_i\gamma_i\left|u_i(t)\right|\left|G_i\bigl(u_i(t)\bigr)\right|^{\kappa-1}\right.\\
&\quad+\sum_{j=1}^{p}\kappa\alpha_i p_{ji}^{+}\left|G_i\bigl(u_i(t)\bigr)\right|^{\kappa-1}\int_0^{\infty}K_{ji}(u)\left|F_j\bigl(v_j(t-u)\bigr)\right|du\\
&\quad+\sum_{j=1}^{p}p_{ji}^{+}\alpha_i\int_0^{\infty}K_{ji}(u)e^{\alpha u}\,du\left|F_j\bigl(v_j(t)\bigr)\right|^{\kappa}\\
&\quad\left.-\sum_{j=1}^{p}p_{ji}^{+}\alpha_i\int_0^{\infty}K_{ji}(u)\left|F_j\bigl(v_j(t-u)\bigr)\right|^{\kappa}du\right\}.
\end{aligned}$$

Further, we have

$$\left.\frac{dV_1}{dt}\right|_{(2.2)}\leqslant e^{\alpha t}\sum_{i=1}^{n}\lambda_i\kappa\alpha_i(\alpha-a_i\gamma_i)\left|u_i(t)\right|\left|G_i\bigl(u_i(t)\bigr)\right|^{\kappa-1}$$

$$
\begin{aligned}
&+ e^{\alpha t}\sum_{i=1}^{n}\sum_{j=1}^{p}\lambda_i\alpha_i p_{ji}^{+}\int_0^{\infty}K_{ji}(u)\big|F_j\big(v_j(t-u)\big)\big|^{\kappa}\,du\\
&+ e^{\alpha t}\sum_{i=1}^{n}\sum_{j=1}^{p}\lambda_i\alpha_i p_{ji}^{+}k_{ji}(\kappa-1)\big|G_i\big(u_i(t)\big)\big|^{\kappa}\\
&+ e^{\alpha t}\sum_{i=1}^{n}\sum_{j=1}^{p}\lambda_i p_{ji}^{+}\alpha_i\int_0^{\infty}K_{ji}(u)e^{\alpha u}\,du\big|F_j\big(v_j(t)\big)\big|^{\kappa}\\
&- e^{\alpha t}\sum_{i=1}^{n}\sum_{j=1}^{p}\lambda_i p_{ji}^{+}\alpha_i\int_0^{\infty}K_{ji}(u)\big|F_j\big(v_j(t-u)\big)\big|^{\kappa}\,du\\
\leqslant{}& e^{\alpha t}\sum_{i=1}^{n}\lambda_i\Bigg[\kappa(\alpha-a_i\gamma_i)+(\kappa-1)\alpha_i\sum_{j=1}^{p}p_{ji}^{+}k_{ji}\Bigg]\big|G_i\big(u_i(t)\big)\big|^{\kappa}\\
&+ e^{\alpha t}\sum_{j=1}^{p}\Bigg(\sum_{i=1}^{n}\lambda_i\alpha_i p_{ji}^{+}\int_0^{\infty}K_{ji}(u)e^{\alpha u}\,du\Bigg)\big|F_j\big(v_j(t)\big)\big|^{\kappa}. \qquad (4.5)
\end{aligned}
$$

Similar to (4.5), we get

$$
\begin{aligned}
\frac{dV_2}{dt}\bigg|_{(2.2)} \leqslant{}& e^{\alpha t}\sum_{j=1}^{p}\lambda_{n+j}\Bigg[\kappa(\alpha-b_j\mu_j)+\beta_j(\kappa-1)\sum_{i=1}^{n}q_{ij}^{+}s_{ij}\Bigg]\big|F_j\big(v_j(t)\big)\big|^{\kappa}\\
&+ e^{\alpha t}\sum_{i=1}^{n}\Bigg(\sum_{j=1}^{p}\lambda_{n+j}\beta_j q_{ij}^{+}\int_0^{\infty}S_{ij}(u)e^{\alpha u}\,du\Bigg)\big|G_i\big(u_i(t)\big)\big|^{\kappa}. \qquad (4.6)
\end{aligned}
$$

From (4.5) and (4.6), we obtain

$$
\begin{aligned}
\frac{dV}{dt}\bigg|_{(2.2)} \leqslant{}& e^{\alpha t}\sum_{i=1}^{n}\Bigg\{\lambda_i\Bigg[\kappa(\alpha-a_i\gamma_i)+(\kappa-1)\alpha_i\sum_{j=1}^{p}p_{ji}^{+}k_{ji}\Bigg]\\
&+\sum_{j=1}^{p}\lambda_{n+j}\beta_j q_{ij}^{+}\int_0^{\infty}S_{ij}(u)e^{\alpha u}\,du\Bigg\}\big|G_i\big(u_i(t)\big)\big|^{\kappa}\\
&+ e^{\alpha t}\sum_{j=1}^{p}\Bigg\{\lambda_{n+j}\Bigg[\kappa(\alpha-b_j\mu_j)+(\kappa-1)\beta_j\sum_{i=1}^{n}q_{ij}^{+}s_{ij}\Bigg]\\
&+\sum_{i=1}^{n}\lambda_i\alpha_i p_{ji}^{+}\int_0^{\infty}K_{ji}(u)e^{\alpha u}\,du\Bigg\}\big|F_j\big(v_j(t)\big)\big|^{\kappa}\\
\leqslant{}& 0. \qquad (4.7)
\end{aligned}
$$

This shows that $V(t)\leqslant V(0)$, for $t\geqslant 0$. From (4.4) and Lemma 4, we have

$$
\begin{aligned}
V(t) &\geqslant e^{\alpha t}\left\{\sum_{i=1}^{n}\kappa\lambda_i\alpha_i\int_0^{u_i(t)}\operatorname{sgn}(s)\left|G_i(s)\right|^{\kappa-1}ds \right.\\
&\quad\left.+\sum_{j=1}^{p}\kappa\lambda_{n+j}\beta_j\int_0^{v_j(t)}\operatorname{sgn}(s)\left|F_j(s)\right|^{\kappa-1}ds\right\}\\
&\geqslant e^{\alpha t}\min_{1\leqslant l\leqslant n+p}\{\lambda_l\}\left\{\sum_{i=1}^{n}\kappa\alpha_i\int_0^{u_i(t)}\operatorname{sgn}(s)\left|G_i(s)\right|^{\kappa-1}ds\right.\\
&\quad\left.+\sum_{j=1}^{p}\kappa\beta_j\int_0^{v_j(t)}\operatorname{sgn}(s)\left|F_j(s)\right|^{\kappa-1}ds\right\}\\
&\geqslant e^{\alpha t}\min_{1\leqslant l\leqslant n+p}\{\lambda_l\}\left\{\sum_{i=1}^{n}\left|G_i\big(u_i(t)\big)\right|^{\kappa}+\sum_{j=1}^{p}\left|F_j\big(v_j(t)\big)\right|^{\kappa}\right\}\\
&= e^{\alpha t}M_1\left\{\sum_{i=1}^{n}\left|G_i\big(u_i(t)\big)\right|^{\kappa}+\sum_{j=1}^{p}\left|F_j\big(v_j(t)\big)\right|^{\kappa}\right\},
\end{aligned}
\tag{4.8}
$$

where $M_1=\min_{1\leqslant l\leqslant n+p}\{\lambda_l\}$. On the other hand, from (4.4), Lemma 3 and (R_5), we have

$$
\begin{aligned}
V(0) &= \sum_{i=1}^{n}\kappa\lambda_i\alpha_i\int_0^{u_i(0)}\operatorname{sgn}(s)\left|G_i(s)\right|^{\kappa-1}ds+\sum_{j=1}^{p}\kappa\lambda_{n+j}\beta_j\int_0^{v_j(0)}\operatorname{sgn}(s)\left|F_j(s)\right|^{\kappa-1}ds\\
&\quad+\sum_{i=1}^{n}\sum_{j=1}^{p}\lambda_i p_{ji}^{+}\alpha_i\int_0^{\infty}\int_{-u}^{0}K_{ji}(u)\left|F_j\big(v_j(s)\big)\right|^{\kappa}e^{\alpha(u+s)}\,ds\,du\\
&\quad+\sum_{j=1}^{p}\sum_{i=1}^{n}\lambda_{n+j}q_{ij}^{+}\beta_j\int_0^{\infty}\int_{-u}^{0}S_{ij}(u)\left|G_i\big(u_i(s)\big)\right|^{\kappa}e^{\alpha(u+s)}\,ds\,du\\
&\leqslant \max_{1\leqslant l\leqslant n+p}\{\lambda_l\}\left\{\sum_{i=1}^{n}\kappa\alpha_i\left|u_i(0)\right|\left|G_i\big(u_i(0)\big)\right|^{\kappa-1}+\sum_{j=1}^{p}\kappa\beta_j\left|v_j(0)\right|\left|F_j\big(v_j(0)\big)\right|^{\kappa-1}\right.\\
&\quad+\sum_{i=1}^{n}\sum_{j=1}^{p}p_{ji}^{+}\alpha_i\int_0^{\infty}\int_{-u}^{0}K_{ji}(u)\beta_j^{\kappa}\left|v_j(s)\right|^{\kappa}e^{\alpha u}\,ds\,du\\
&\quad\left.+\sum_{j=1}^{p}\sum_{i=1}^{n}q_{ij}^{+}\beta_j\int_0^{\infty}\int_{-u}^{0}S_{ij}(u)\alpha_i^{\kappa}\left|u_i(s)\right|^{\kappa}e^{\alpha u}\,ds\,du\right\}\\
&\leqslant \max_{1\leqslant l\leqslant n+p}\{\lambda_l\}\left\{\sum_{i=1}^{n}\kappa\alpha_i^{\kappa}+\sum_{j=1}^{p}\kappa\beta_j^{\kappa}+\sum_{i=1}^{n}\sum_{j=1}^{p}p_{ji}^{+}\alpha_i\beta_j^{\kappa}\int_0^{\infty}uK_{ji}(u)e^{\alpha u}\,du\right.
\end{aligned}
$$

$$+\sum_{j=1}^{p}\sum_{i=1}^{n} q_{ij}^{+}\beta_j\alpha_i^{\kappa}\int_0^{\infty} uS_{ij}(u)e^{\alpha u}\,du\Bigg\}\big\|\big(\phi^T,\psi^T\big)^T-\big(x^{*T},y^{*T}\big)^T\big\|^{\kappa}$$

$$\triangleq \max_{1\leqslant l\leqslant n+p}\{\lambda_l\}M_2\big\|\big(\phi^T,\psi^T\big)^T-\big(x^{*T},y^{*T}\big)^T\big\|^{\kappa}, \tag{4.9}$$

where

$$M_2 \triangleq \Bigg\{\sum_{i=1}^{n}\kappa\alpha_i^{\kappa}+\sum_{j=1}^{p}\kappa\beta_j^{\kappa}+\sum_{i=1}^{n}\sum_{j=1}^{p} p_{ji}^{+}\alpha_i\beta_j^{\kappa}\int_0^{\infty} uK_{ji}(u)e^{\alpha u}\,du$$

$$+\sum_{j=1}^{p}\sum_{i=1}^{n} q_{ij}^{+}\beta_j\alpha_i^{\kappa}\int_0^{\infty} uS_{ij}(u)e^{\alpha u}\,du\Bigg\}.$$

Set

$$M_3=\max_{1\leqslant l\leqslant n+p}\{\lambda_l\}\frac{M_2}{M_1}.$$

It follows from (4.8) and (4.9) that

$$\sum_{i=1}^{n}\big|G_i\big(u_i(t)\big)\big|^{\kappa}+\sum_{j=1}^{p}\big|F_j\big(v_j(t)\big)\big|^{\kappa}\leqslant M_3\big\|\big(\phi^T,\psi^T\big)^T-\big(x^{*T},y^{*T}\big)^T\big\|^{\kappa}e^{-\alpha t}. \tag{4.10}$$

Therefore,

$$\begin{aligned}\big|G_i\big(u_i(t)\big)\big| &\leqslant \sqrt[\kappa]{M_3}\big\|\big(\phi^T,\psi^T\big)^T-\big(x^{*T},y^{*T}\big)^T\big\|e^{-\frac{\alpha}{\kappa}t},\\ \big|F_j\big(v_j(t)\big)\big| &\leqslant \sqrt[\kappa]{M_3}\big\|\big(\phi^T,\psi^T\big)^T-\big(x^{*T},y^{*T}\big)^T\big\|e^{-\frac{\alpha}{\kappa}t},\end{aligned} \tag{4.11}$$

for $t\geqslant 0$, $i=1,2,\ldots,n$, $j=1,2,\ldots,p$.

Again it follows from (2.2a) that

$$\begin{aligned}
&D^+\big|u_i(t)\big|\\
&\quad\leqslant -a_i\gamma_i\big|u_i(t)\big|+\sum_{j=1}^{p}p_{ji}^{+}\int_0^{\infty}K_{ji}(u)\big|F_j\big(v_j(t-u)\big)\big|\,du\\
&\quad\leqslant -a_i\gamma_i\big|u_i(t)\big|+\sum_{j=1}^{p}p_{ji}^{+}\sqrt[\kappa]{M_3}\big\|\big(\phi^T,\psi^T\big)^T-\big(x^{*T},y^{*T}\big)^T\big\|\int_0^{\infty}K_{ji}(u)e^{-\frac{\alpha}{\kappa}(t-u)}\,du\\
&\quad\leqslant -a_i\gamma_i\big|u_i(t)\big|+\sum_{j=1}^{p}p_{ji}^{+}\sqrt[\kappa]{M_3}\big\|\big(\phi^T,\psi^T\big)^T-\big(x^{*T},y^{*T}\big)^T\big\|\int_0^{\infty}K_{ji}(u)e^{\frac{\alpha u}{\kappa}}\,du\,e^{-\frac{\alpha}{\kappa}t},
\end{aligned}$$

which shows that

$$\big|u_i(t)\big|\leqslant e^{-a_i\gamma_i t}\big|u_i(0)\big|+e^{-a_i\gamma_i t}\sum_{j=1}^{p}p_{ji}^{+}\sqrt[\kappa]{M_3}\big\|\big(\phi^T,\psi^T\big)^T-\big(x^{*T},y^{*T}\big)^T\big\|$$

$$
\begin{aligned}
&\times \int_0^\infty K_{ji}(u) e^{\frac{\alpha}{\kappa}u}\,du \int_0^t e^{(a_i\gamma_i - \frac{\alpha}{\kappa})s}\,ds \\
&\leqslant e^{-a_i\gamma_i t} \left\| \left(\phi^T, \psi^T\right)^T - \left(x^{*T}, y^{*T}\right)^T \right\| \\
&\quad \times \left[1 + \sum_{j=1}^p p_{ji}^+ \sqrt[\kappa]{M_3} \int_0^\infty K_{ji}(u) e^{\frac{\alpha}{\kappa}u}\,du \frac{\kappa}{a_i\gamma_i\kappa - \alpha} \left(e^{\frac{\kappa a_i\gamma_i - \alpha}{\kappa}t} - 1 \right) \right] \\
&\leqslant \left\| \left(\phi^T, \psi^T\right)^T - \left(x^{*T}, y^{*T}\right)^T \right\| \\
&\quad \times \left[e^{-a_i\gamma_i t} + \sum_{j=1}^p p_{ji}^+ \sqrt[\kappa]{M_3} \int_0^\infty K_{ji}(u) e^{\frac{\alpha}{\kappa}u}\,du \frac{\kappa}{a_i\gamma_i\kappa - \alpha} e^{-\frac{\alpha}{\kappa}t} \right] \\
&\leqslant \left\| \left(\phi^T, \psi^T\right)^T - \left(x^{*T}, y^{*T}\right)^T \right\| \\
&\quad \times \left[1 + \sum_{j=1}^p p_{ji}^+ \sqrt[\kappa]{M_3} \int_0^\infty K_{ji}(u) e^{\frac{\alpha}{\kappa}u}\,du \frac{\kappa}{a_i\gamma_i\kappa - \alpha} \right] e^{-\frac{\alpha}{\kappa}t}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\left|v_j(t)\right| &\leqslant \left\| \left(\phi^T, \psi^T\right)^T - \left(x^{*T}, y^{*T}\right)^T \right\| \\
&\quad \times \left[1 + \sum_{i=1}^n q_{ij}^+ \sqrt[\kappa]{M_3} \int_0^\infty S_{ij}(u) e^{\frac{\alpha}{\kappa}u}\,du \frac{\kappa}{b_j\mu_j\kappa - \alpha} \right] e^{-\frac{\alpha}{\kappa}t}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\sum_{i=1}^n \left|u_i(t)\right| + \sum_{j=1}^p \left|v_j(t)\right| \leqslant \left\| \left(\phi^T, \psi^T\right)^T - \left(x^{*T}, y^{*T}\right)^T \right\| \\
&\quad \times \left[n + p + \kappa \sqrt[\kappa]{M_3} \sum_{i=1}^n \sum_{j=1}^p \int_0^\infty \left(\frac{p_{ji}^+ K_{ji}(u)}{\kappa a_i\gamma_i - \alpha} + \frac{q_{ij}^+ S_{ij}(u)}{\kappa b_j\mu_j - \alpha} \right) e^{\frac{\alpha}{\kappa}u}\,du \right] e^{-\frac{\alpha}{\kappa}t},
\end{aligned}
$$

i.e.,

$$
\sum_{i=1}^n \left|u_i(t)\right| + \sum_{j=1}^p \left|v_j(t)\right| \leqslant M \left\| \left(\phi^T, \psi^T\right)^T - \left(x^{*T}, y^{*T}\right)^T \right\| e^{-\frac{\alpha}{\kappa}t},
$$

where

$$
M = \left[n + p + \kappa \sqrt[\kappa]{M_3} \sum_{i=1}^n \sum_{j=1}^p \int_0^\infty \left(\frac{p_{ji}^+ K_{ji}(u)}{\kappa a_i\gamma_i - \alpha} + \frac{q_{ij}^+ S_{ij}(u)}{\kappa b_j\mu_j - \alpha} \right) e^{\frac{\alpha}{\kappa}u}\,du \right] > 1,
$$

for $t \geqslant 0$. Thus, according to Definition 1, the proof is completed. □

5. An example

In this section, we give an example to illustrate our results. Consider the following simple BAM networks with periodic coefficients and distributed delays:

$$\begin{cases} x_i'(t) = -a_i h_i(x_i(t)) + \sum_{j=1}^{2} p_{ji}(t) \int_0^\infty K_{ji}(u) f_j(y_j(t-u))\,du + I_i(t), \\ y_j'(t) = -b_j c_j(y_j(t)) + \sum_{i=1}^{2} q_{ij}(t) \int_0^\infty S_{ij}(u) g_i(x_i(t-u))\,du + J_j(t), \end{cases} \tag{5.1}$$

where

$$I_i(t) = \sin 40\pi t, \qquad J_j(t) = \cos 40\pi t, \qquad \omega = \frac{1}{20},$$
$$K_{ji}(u) = S_{ij}(u) = e^{-u} \quad (i, j = 1, 2),$$
$$h_i(x) = 2x + \sin x, \qquad c_j(y) = 2y - \sin y \quad (i, j = 1, 2).$$

Then

$$k_{ji} = \int_0^\infty K_{ji}(u)\,du = 1, \qquad s_{ij} = \int_0^\infty S_{ij}(u)\,du = 1,$$
$$\gamma_i = \mu_j = 1.$$

For $\mu^* < 1$, we have

$$\int_0^\infty s K_{ji}(s) e^{\mu^* s}\,ds = \int_0^\infty s S_{ij}(s) e^{\mu^* s}\,ds = \frac{1}{1-\mu^*}.$$

Taking $g_i(x) = f_j(x) = \frac{1}{2}(|x+1| - |x-1|)$, we have $\alpha_i = \beta_j = 1$ $(i, j = 1, 2)$. Take

$$(a_1, a_2)^T = (1, 1)^T, \qquad (b_1, b_2)^T = (1, 1)^T,$$

and let

$$\begin{pmatrix} q_{11}(t) & q_{12}(t) \\ q_{21}(t) & q_{22}(t) \end{pmatrix} = \begin{pmatrix} \frac{2}{21}\cos 40\pi t & \frac{2}{21}\sin 40\pi t \\ \frac{2}{21}\sin 40\pi t & \frac{2}{21}\cos 40\pi t \end{pmatrix},$$
$$\begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix} = \begin{pmatrix} \frac{2}{21}\sin 40\pi t & \frac{2}{21}\cos 40\pi t \\ \frac{2}{21}\cos 40\pi t & \frac{2}{21}\sin 40\pi t \end{pmatrix}.$$

Then

$$\begin{pmatrix} q_{11}^+ & q_{12}^+ \\ q_{21}^+ & q_{22}^+ \end{pmatrix} = \begin{pmatrix} \frac{2}{21} & \frac{2}{21} \\ \frac{2}{21} & \frac{2}{21} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} p_{11}^+ & p_{12}^+ \\ p_{21}^+ & p_{22}^+ \end{pmatrix} = \begin{pmatrix} \frac{2}{21} & \frac{2}{21} \\ \frac{2}{21} & \frac{2}{21} \end{pmatrix}.$$

Moreover,

$$A = \begin{pmatrix} 1 & 0 & -0.1 & -0.1 \\ 0 & 1 & -0.1 & -0.1 \\ -0.1 & -0.1 & 1 & 0 \\ -0.1 & -0.1 & 0 & 1 \end{pmatrix}, \qquad A^{-1} = \begin{pmatrix} \frac{49}{48} & \frac{1}{48} & \frac{5}{48} & \frac{5}{48} \\ \frac{1}{48} & \frac{49}{48} & \frac{5}{48} & \frac{5}{48} \\ \frac{5}{48} & \frac{5}{48} & \frac{49}{48} & \frac{1}{48} \\ \frac{5}{48} & \frac{5}{48} & \frac{1}{48} & \frac{49}{48} \end{pmatrix} > 0,$$

hence, A is an M-matrix. Setting $\lambda_i = 1$ $(i = 1, 2, 3, 4)$, $\kappa = 2$, we get

$$\begin{pmatrix} \tilde{p}_{11} & \tilde{p}_{12} \\ \tilde{p}_{21} & \tilde{p}_{22} \end{pmatrix} = \begin{pmatrix} \tilde{q}_{11} & \tilde{q}_{12} \\ \tilde{q}_{21} & \tilde{q}_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\min\left[\min_{1\leqslant i\leqslant 2}\left(a_i\gamma_i - \alpha_i\sum_{j=1}^{2}|\widetilde{q}_{ij}|s_{ij}\right), \min_{1\leqslant j\leqslant 2}\left(b_j\mu_j - \beta_j\sum_{i=1}^{2}|\widetilde{p}_{ji}|k_{ji}\right)\right] = 1 > 0,$$

$$\lambda_i\left(-\kappa a_i\gamma_i + (\kappa-1)\alpha_i\sum_{j=1}^{2}p_{ji}^{+}k_{ji}\right) + \sum_{j=1}^{2}\lambda_{2+j}q_{ij}^{+}s_{ij}\beta_j = -\frac{34}{21} < 0,$$

$$\lambda_{2+j}\left(-\kappa b_j\mu_j + (\kappa-1)\beta_j\sum_{i=1}^{2}q_{ij}^{+}s_{ij}\right) + \sum_{i=1}^{2}\lambda_i p_{ji}^{+}k_{ji}\beta_j = -\frac{34}{21} < 0,$$

$$0 \leqslant \frac{g_i(x)-g_i(y)}{x-y} \leqslant 1, \quad g_i(0)=0, \ i=1,2,$$

$$0 \leqslant \frac{f_j(x)-f_j(y)}{x-y} \leqslant 1, \quad f_j(0)=0, \ j=1,2,$$

for $i, j = 1, 2$. Thus, it follows from Theorems 1 and 2 that the unique $\frac{1}{20}$-periodic solution of (5.1) is a global exponential periodic attractor.

Remark 1. When $h_i(x_i) = x_i$, $i = 1, 2, \ldots, n$; $c_j(y_j) = y_j$, $j = 1, 2, \ldots, p$, $p_{ji}(t) = p_{ji}$, $q_{ij}(t) = q_{ij}$ and the delay kernels are δ-functions, i.e.,

$$K_{ji}(u) = \delta(u - \tau_{ji}), \qquad S_{ij}(u) = \delta(u - \sigma_{ij}),$$

system (1.1) reduces to the following system:

$$\begin{cases} x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{p} p_{ji} f_j(y_j(t-\tau_{ji})) + I_i(t), \\ y_j'(t) = -b_j y_j(t) + \sum_{i=1}^{n} q_{ij} f_i(x_i(t-\sigma_{ij})) + J_j(t). \end{cases} \tag{5.2}$$

In [4], system (5.2) was considered and some sufficient conditions were derived guaranteeing the existence and exponential stability of the periodic solution. Clearly, (5.2) is just a special case of our mode (1.1).

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