Linear Algebra and its Applications 437 (2012) 1285–1292

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

On the existence of a common eigenvector for all matrices in the commutant of a single matrix

M. Dolors Magret *, M. Eulalia Montoro

MAI, UPC, Diagonal 647, 08028 Barcelona, Spain

ARTICLE INFO

Article history:
Received 9 November 2011
Accepted 7 April 2012
Available online 17 May 2012

Submitted by P. Šemrl

AMS classification:
15A03
15A04
15A18

Keywords:
Invariant subspaces
Commutant

ABSTRACT

The main purpose of this paper is to study common invariant subspaces of any matrix in the centralizer of a given matrix $A \in M_n(\mathbb{F})$, where $\mathbb{F}$ denotes an algebraically closed field. In particular, we obtain a necessary and sufficient condition for the existence of a common eigenvector for all the matrices in this set.

1. Introduction

Though invariant subspaces were defined by von Neumann in 1935, their use did not begin until much later, with no results obtained for a long time.

Invariant subspaces are connected to many disciplines. For example, the controllability subspace of a linear dynamical system with state equation $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + Bu(t)$ is known to be the least invariant subspace under the matrix A which contains the range of matrix B.

When considering matrices with coefficients in an algebraic closed field, invariant subspaces may be deduced from the Jordan canonical form of the matrix, as shown in [3], which provide a comprehensive treatment of geometrical, algebraic, topological, and analytic properties of invariant subspaces.

Some previous related results are the following ones. It is known (see [5]) that two matrices A_1, A_2 have a common eigenvector if, and only if,

* Corresponding author.
E-mail addresses: m.dolors.magret@upc.edu (M.D. Magret), maria.eulalia.montoro@upc.edu (M.E. Montoro).

http://dx.doi.org/10.1016/j.laa.2012.04.011

$$\bigcap_{k,l=1}^{n} Ker[A_1^k, A_2^l] \neq \{0\}.$$

In the case of dimension of the invariant subspace greater than 1, George and Ikramov [2] proved that if matrices A_1, A_2 have a common invariant subspace of dimension d, then the d-th compound matrices $(A_1)_d$, $(A_2)_d$ have a common eigenvector. Conversely, if $(A_1)_d$, $(A_2)_d$ have a common eigenvector and if all eigenvalues of $(A_1)_d$ are simple and A_2 is non-singular, then there exists a common invariant subspace of dimension d for A_1 and A_2. Tsatsomeros [6] extends this result to the case where A_1 and A_2 are arbitrary.

Halmos proved that if A is a matrix and if V is an A-invariant subspace, then there exist matrices B and C such that $BA = AB$, $CA = AC$, V is the kernel of B and V is the range of C. Moreover, there exist B and C such that additionally satisfy $BC = CB = O$. See [1] for a short proof of this result.

Here we will study the existence of eigenvectors and invariant subspaces which are common to all the matrices belonging to the commutant (centralizer) of a given matrix A.

Throughout this note $\mathbb{F}$ will represent an algebraically closed field (for example, $\mathbb{F} = \mathbb{C}$). We will denote by $M_n(\mathbb{F})$ the vector subspace consisting of square matrices of order n and by $Gl_n(\mathbb{F})$ the set of invertible matrices in $M_n(\mathbb{F})$.

2. Preliminaries

Throughout the note we will consider a matrix $A \in M_n(\mathbb{F})$, such that the characteristic polynomial can be completely factored into linear factors over $\mathbb{F}$:

$$Q_A(t) = (-1)^n (t - \lambda_1)^{n_1} \cdot \ldots \cdot (t - \lambda_r)^{n_r}.$$

As it is known, under the assumption that the characteristic polynomial of A splits into linear factors over $\mathbb{F}$, there exists $S \in Gl_n(\mathbb{F})$ such that $A = SJS^{-1}$, where J is the Jordan canonical form of the matrix A, throughout the paper the notation we use is the one which can be found for example in [4].

A vector subspace $V \subseteq \mathbb{F}^n$ is called A-*invariant* when $A(V) \subseteq V$.

The commutant (centralizer) of the matrix A is the set

$$Z(A) = \{X \in M_n(\mathbb{F}) \mid AX - XA = 0\}.$$

3. Common eigenvectors and invariant subspaces

Let A be a matrix in $M_n(\mathbb{F})$, where $\mathbb{F}$ is an algebraically closed field.
In order to find common invariant subspaces for all matrices in $Z(A)$, we observe that we can reduce ourselves to the case where the matrix A is in Jordan reduced form.

Lemma 3.1 [3]. *Let us assume that $A = SJS^{-1}$, with J the Jordan reduced form of matrix A. Then V is an A-invariant subspace if, and only if, $S^{-1}V$ is a J-invariant subspace.*

From now on, we will consider $A = J$ is a matrix in Jordan reduced form.

It is well-known (a proof can be found in [3]), that if

$$J(\lambda) = \begin{bmatrix} \lambda & 0 & \ldots & 0 & 0 \\ 1 & \lambda & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda & 0 \\ 0 & 0 & \ldots & 1 & \lambda \end{bmatrix} \in M_\alpha(\mathbb{F}) \tag{3.1}$$

then $Z(J(\lambda))$ is the set of lower triangular Toeplitz matrices:

$$\left\{ T(x_1, \dots, x_\alpha) = \begin{bmatrix} x_1 & & & & \\ x_2 & x_1 & & & \\ x_3 & x_2 & x_1 & & \\ \dots & \dots & \dots & \dots & \\ x_\alpha & x_{\alpha-1} & x_{\alpha-2} & \dots & x_1 \end{bmatrix}; x_1, x_2, \dots, x_{\alpha-1}, x_\alpha \in \mathbb{F} \right\}. \tag{2}$$

On the other hand, let us consider two matrices

$$J_1(\lambda) = \begin{bmatrix} \lambda & 0 & \dots & 0 & 0 \\ 1 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & \lambda & 0 \\ 0 & 0 & \dots & 1 & \lambda \end{bmatrix}, \quad J_2(\lambda) = \begin{bmatrix} \lambda & 0 & \dots & 0 & 0 \\ 1 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & \lambda & 0 \\ 0 & 0 & \dots & 1 & \lambda \end{bmatrix} \tag{3}$$

such that $J_1(\lambda) \in M_\alpha(\mathbb{F})$ and $J_2(\lambda) \in M_\beta(\mathbb{F})$.

(1) If $\alpha > \beta$, the set of solutions of the system $J_1(\lambda)X = XJ_2(\lambda)$ is the set of matrices of the form

$$TD(x_1, \dots, x_\beta) = \begin{bmatrix} 0 \\ \hline T(x_1, \dots, x_\beta) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & & \dots \\ 0 & 0 & 0 & \dots & 0 \\ \hline x_1 & & & & \\ x_2 & x_1 & & & \\ x_3 & x_2 & x_1 & & \\ \dots & \dots & \dots & \dots & \\ x_\beta & x_{\beta-1} & x_{\beta-2} & \dots & x_1 \end{bmatrix} \tag{4}$$

for $x_1, x_2, \dots, x_{\beta-1}, x_\beta \in \mathbb{F}$, and

(2) if $\alpha < \beta$, then the set of solutions of the system $J_1(\lambda)X = XJ_2(\lambda)$ is the set of matrices of the form

$$TL(x_1, \dots, x_\alpha) = \left[\, T(x_1, \dots, x_\alpha) \;\middle|\; 0 \,\right]$$

$$= \left[\begin{array}{ccccc|ccc} x_1 & & & & & 0 & \dots & 0 \\ x_2 & x_1 & & & & 0 & \dots & 0 \\ x_3 & x_2 & x_1 & & & \dots & & \dots \\ \dots & \dots & \dots & \dots & & 0 & \dots & 0 \\ x_\alpha & x_{\alpha-1} & x_{\alpha-2} & \dots & x_1 & & & \end{array}\right] \tag{5}$$

for $x_1, x_2, \dots, x_{\alpha-1}, x_\alpha \in \mathbb{F}$.

Let us return to our particular set-up. Let us write

$$J = \operatorname{diag}(J(\lambda_1), \ldots, J(\lambda_r))$$

with λ_i, $1 \leqslant i \leqslant r$, the distinct eigenvalues of matrix J, and

$$J(\lambda_i) = \operatorname{diag}(J_1(\lambda_i), \ldots, J_{m_i}(\lambda_i)), \quad 1 \leqslant i \leqslant r,$$

where

$$J_j(\lambda_i) = \begin{bmatrix} \lambda_i & 0 & \ldots & 0 & 0 \\ 1 & \lambda_i & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & \lambda_i & 0 \\ 0 & 0 & \ldots & 1 & \lambda_i \end{bmatrix} \in M_{\alpha(i,j)}(\mathbb{F}), \quad 1 \leqslant j \leqslant m_i. \tag{3.6}$$

We will consider that for all eigenvalue λ_i, $1 \leqslant i \leqslant r$, the Jordan blocks corresponding to $J(\lambda_i)$ are ordered in decreasing order of their sizes; that is to say, $\alpha(i, 1) \geqslant \cdots \geqslant \alpha(i, m_i)$.

Lemma 3.2.

(a) A matrix $X \in M_n(\mathbb{F})$ belongs to $Z(J)$ if, and only if, $X = \operatorname{diag}(X_1, \ldots, X_r)$ with $X_i \in Z(J(\lambda_i))$, $1 \leqslant i \leqslant r$.
(b) For $1 \leqslant i \leqslant r$, consider the partition of matrix $X_i \in Z(J(\lambda_i))$ according to the block partition of matrix $J(\lambda_i)$,

$$X_i = \begin{bmatrix} X_{i,1}^1 & \ldots & X_{i,m_i}^1 \\ \vdots & & \vdots \\ X_{i,1}^{m_i} & \ldots & X_{i,m_i}^{m_i} \end{bmatrix}. \tag{3.7}$$

Then all matrix blocks $X_{i,j}^k \in M_{\alpha(i,j)\times\alpha(i,k)}$, $1 \leqslant j \leqslant r$, $1 \leqslant k \leqslant m_i$ are of the form $T(x_1, \ldots, x_\alpha)$, $TD(x_1, \ldots, x_\alpha)$ or $TL(x_1, \ldots, x_\alpha)$ for some values of the parameters $x_1, \ldots, x_\alpha$.

Proof. Then matrices $X_{i,j}^k \in M_{\alpha(i,j)\times\alpha(i,k)}$ satisfy the linear system of matrix equations:

$$J_j(\lambda_i) \cdot X_{i,j}^k = X_{i,j}^k \cdot J_k(\lambda_i); \; 1 \leqslant j, k \leqslant m_i \tag{3.8}$$

and therefore all of them are of one of the types in the statement. □

Theorem 3.3. *Matrices X in $Z(J)$ have a common eigenvector if, and only if, there exist $i \in \{1, \ldots, r\}$ such that $m_i = 1$ or the Jordan blocks in $J(\lambda_i)$ have orders*

$$\alpha(i, 1) > \alpha(i, 2) \geqslant \alpha(i, 3) \geqslant \cdots \geqslant \alpha(i, m_i).$$

Proof. We write $\alpha_i = \sum_{1\leqslant j\leqslant m_i} \alpha(i, j)$. If we denote by $e_1, \ldots, e_n$ the vectors in the natural basis of $\mathbb{F}^n$, it is obvious that if there exists $i \in \{1, \ldots, r\}$ such that $m_i = 1$ or $m_i > 1$ and $\alpha(i, 1) > \alpha(i, 2)$, then all matrices in $Z(J)$ have as a common eigenvector $e_{\alpha_1+\ldots\alpha_{i-1}+\alpha(i,1)}$.

Conversely, if we assume $m_i > 1$ and $\alpha(i, 1) = \alpha(i, 2)$, for all $i \in \{1, \ldots, r\}$, we can consider the following matrices in $Z(J)$:

- Y a lower bidiagonal matrix with all non-zero entries equal to 1.
- $Z = \operatorname{diag}(Z_1, \ldots, Z_r)$ with

$$Z_i = \begin{bmatrix} Z_{i,1}^1 & \ldots & Z_{i,m_i}^1 \\ \vdots & & \vdots \\ Z_{i,1}^{m_i} & \ldots & Z_{i,m_i}^{m_i} \end{bmatrix} \tag{3.9}$$

with block matrices $Z_{i,k}^l$ of the form $T(x_1, \ldots, x_\alpha)$, $TD(y_1, \ldots, y_\beta)$, $TL(z_1, \ldots, z_\gamma)$ with all the values of the parameters different (for example, satisfying an arithmetic recurrence).

Then the eigenvectors of matrix Y, corresponding are:

$$e_{\alpha(1,1)}, e_{\alpha(1,1)+\alpha(1,2)}, \ldots, e_{\alpha_1}; e_{\alpha_1+\alpha(2,1)}, \ldots, e_{\alpha_1+\alpha_2}; \ldots, e_n$$

and none of them is an eigenvector of matrix Z. Therefore matrices Y and Z, both of them belonging to $Z(J)$, have no common eigenvector. □

Remark 3.4. Let us assume that $X, Y \in Z(J)$ share an eigenvector v (that is to say, $Xv = \lambda v, Yv = \mu v$ for some $\lambda, \mu \in \mathbb{F}$). If $J^k v \neq 0, k \in \mathbb{N}$, then X and Y also have as common eigenvectors $Jv, J^2v, \ldots, J^kv, \ldots$

Example 3.5. We can consider the following matrix:

$$A = \begin{bmatrix} 3 & -1 & 1 & -2 & 0 & 0 \\ 1 & 1 & -2 & 4 & 0 & 0 \\ 0 & 0 & 11 & -18 & 0 & 0 \\ 0 & 0 & 6 & -10 & 0 & 0 \\ 2 & -1 & 8 & -14 & -2 & -1 \\ -3 & 4 & -17 & 30 & 2 & 1 \end{bmatrix} \tag{3.10}$$

with Jordan form J and Jordan basis S:

$$J = \left[\begin{array}{cc|c|cc|c} 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad S = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 & 1 & 2 \end{bmatrix}. \tag{3.11}$$

The commutant of J is:

$$Z(J) = \left\{ \left[\begin{array}{cc|c|cc|c} x_1 & 0 & 0 & 0 & 0 & 0 \\ x_2 & x_1 & x_3 & 0 & 0 & 0 \\ \hline x_4 & 0 & x_5 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & x_6 & 0 & 0 \\ 0 & 0 & 0 & x_7 & x_6 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & x_8 \end{array}\right] ; x_i \in \mathbb{F} \right\}. \tag{3.12}$$

In this case, $r = 3, m_1 = 2, m_2 = 1, m_3 = 1$ and:

$$\alpha(1,1) = 2, \alpha(1,2) = 1, \alpha_1 = 3,$$

$$\alpha(2,1) = 2, \alpha_2 = 2,$$

$$\alpha(3,1) = 1, \alpha_3 = 1.$$

It is immediate that e_2, e_5, e_6 are the common eigenvectors of $Z(J)$ and thus

$$v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad v_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad v_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 2 \end{bmatrix} \tag{3.13}$$

are the common eigenvectors of $Z(A)$.

Corollary 3.6. *The number of common eigenvectors of all matrices in $Z(J)$ is*

$$\sharp\{i \in \{1, \dots, r\} \mid m_i = 1 \text{ or } m_i > 1 \text{ and } \alpha(i,1) > \alpha(i,2)\}.$$

Corollary 3.7. *Let V be a d-dimensional invariant common subspace for all matrices in $Z(J)$, with $d \geqslant 2$. Then, all matrices in $Z(J)$ have a common eigenvector belonging to V if, and only if, the restriction of J to V fulfill the conditions in the Theorem* 3.3.

Proof. Given any basis $\{u_1, \dots, u_d\}$ of the vector subspace V, the matrices in $Z(J)$, in a basis of $\mathbb{F}^n$ of the form $\{u_1, \dots, u_d, u_{d+1}, \dots, u_n\}$ are of the form:

$$\begin{bmatrix} X_1 & X_2 \\ 0 & X_3 \end{bmatrix} \qquad X_1 \in M_d(\mathbb{F}), X_3 \in M_{n-d}(\mathbb{F}) \tag{3.14}$$

with $X_1 \in Z(J_{|V})$ and $X_3 \in Z(J_{|G})$, where G represents a complementary vector subspace of V and the statement follows. $\square$

Example 3.8. We can consider the following matrix:

$$J = \left[\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]. \tag{3.15}$$

The commutant of J is:

$$Z(J) = \left\{ \left[\begin{array}{cc|cc} x_1 & 0 & x_2 & 0 \\ x_3 & x_1 & x_4 & x_2 \\ \hline x_5 & 0 & x_6 & 0 \\ x_7 & x_5 & x_8 & x_6 \end{array}\right]; x_i \in \mathbb{F}, 1 \leqslant i \leqslant 8 \right\}. \tag{3.16}$$

We have that $V = \langle e_2, e_4 \rangle$ is a 2-invariant subspace for all matrices in $Z(J)$, which are of the form:

$$\left[\begin{array}{c|c} X_1 & X_2 \\ \hline 0 & X_3 \end{array}\right] \tag{3.17}$$

in basis $\{e_2, e_4, e_1, e_3\}$.

Since $J_{|V} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $Z(J_{|V}) = \left\{ \left[\begin{array}{c|c} x_1 & x_2 \\ \hline x_3 & x_4 \end{array}\right]; x_i \in \mathbb{K} \right\}$, there is no common eigenvector for matrices in $Z(J_{|V})$ and thus matrices in $Z(J)$ have no common eigenvector belonging to V.

We can actually generalize, with an analogous reasoning as that in the proof of Theorem 3.3, the statement in Theorem 3.3 to (non-trivial) invariant subspaces of dimension greater than 1.

Theorem 3.9. *Let us assume that for some $i \in \{1, \ldots, r\}$, one of the following conditions hold:*

(i) $m_i = 1$ and $\alpha(i, 1) \geqslant d \geqslant 2$,
(ii) $m_i > 1$ and $\alpha(i, 1) \geqslant \alpha(i, 2) + d$, $d \geqslant 2$.

Then all matrices in $Z(J)$ have as a d-dimensional common invariant subspace the vector subspace spanned by

$$e_{\alpha_1+\cdots+\alpha_{i-1}+\alpha(i,1)-d+1}, e_{\alpha_1+\cdots+\alpha_{i-1}+\alpha(i,1)-d+2}, \ldots, e_{\alpha_1+\cdots+\alpha_{i-1}+\alpha(i,1)}.$$

Proof. The vector subspace spanned by vectors above is an X_i-invariant subspace and thus an X-invariant subspace for all $X \in Z(J)$. Its dimension is clearly d. □

Remark 3.10. Note that the vector subspaces above are not necessarily the only d-dimensional common invariant subspaces, since the sum of invariant subspaces (like vector subspaces above and subspaces spanned by eigenvectors) is again an invariant subspace for all matrices in $Z(J)$. Theorem above provides a lower bound for the number of common invariant subspaces for all $X \in Z(J)$ of dimension $d \geqslant 2$ (though in general this is not a tight bound, as can be seen in next example).

Example 3.11. Let us return to matrix A in Example 3.5. Condition 1 in Theorem 3.9. is satisfied by eigenvalue $\lambda_2 = -1$. Therefore all matrices in $Z(J)$ have a 2-dimensional common invariant subspace:

$$\langle e_4, e_5 \rangle.$$

Directly, the non-trivial X-invariant subspaces for all $X \in Z(J)$, can be found and are the following ones.

Dimension $d = 2$: $\langle e_2, e_3\rangle, \langle e_2, e_5\rangle, \langle e_2, e_6\rangle, \langle e_4, e_5\rangle, \langle e_5, e_6\rangle$
Dimension $d = 3$: $\langle e_1, e_2, e_3\rangle, \langle e_2, e_3, e_5\rangle, \langle e_2, e_3, e_6\rangle, \langle e_2, e_5, e_6\rangle, \langle e_4, e_5, e_6\rangle$
Dimension $d = 4$: $\langle e_1, e_2, e_3, e_5\rangle, \langle e_1, e_2, e_3, e_6\rangle, \langle e_2, e_3, e_4, e_5\rangle, \langle e_2, e_3, e_5, e_6\rangle, \langle e_2, e_4, e_5, e_6\rangle$
Dimension $d = 5$: $\langle e_1, e_2, e_3, e_4, e_5\rangle, \langle e_1, e_2, e_3, e_5, e_6\rangle, \langle e_2, e_3, e_4, e_5, e_6\rangle$.

Acknowledgments

The authors wish to thank Prof. F. Puerta for his detailed revision of the manuscript and helpful comments.

This work was partially supported by Grant MTM2010-19356-C02-02.

References

[1] I. Domanov, On invariant subspaces of matrices: a new proof of a theorem of Halmos, Linear Algebra Appl. 433 (2010) 2255–2256.
[2] A. George, Kh.D. Ikramov, Common invariant subspaces of two matrices, Linear Algebra Appl. 287 (1999) 171–179.
[3] I. Gohberg, P. Lancaster, L. Rodman, Invariant Subspaces of Matrices with Applications, SIAM, 1986.
[4] F. Puerta, Àlgebra lineal, Ed. ETSEIB-UPC, 1990.
[5] D. Shemesh, Common eigenvectors of two matrices, Linear Algebra Appl. 62 (1984) 11–18.
[6] M. Tsatsomeros, A criterion for the existence of common invariant subspaces of matrices, Linear Algebra Appl. 322 (2001) 51–59.