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Families of small regular graphs of girth 5

M. Abreu [a], G. Araujo-Pardo [b], C. Balbuena [c,*], D. Labbate [a]

[a] *Dipartimento di Matematica, Università degli Studi della Basilicata, Viale dell'Ateneo Lucano, I-85100 Potenza, Italy*
[b] *Instituto de Matemáticas, Universidad Nacional Autónoma de México, México D. F., Mexico*
[c] *Departament de Matemàtica Aplicada III, Universitat Politècnica de Catalunya, Campus Nord, Edifici C2, C/ Jordi Girona 1 i 3, E-08034 Barcelona, Spain*

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ABSTRACT

In this paper we obtain $(q+3-u)$-regular graphs of girth 5, for $1 \le u \le q-1$ with fewer vertices than previously known ones, for each prime $q \ge 13$, performing operations of reductions and amalgams on the Levi graph B_q of an elliptic semiplane of type C. We also obtain a 13-regular graph of girth 5 on 236 vertices from B_{11} using the same technique.

1. Introduction

All graphs considered are finite, undirected and simple (without loops or multiple edges). For definitions and notations not explicitly stated the reader may refer to [11].

Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *girth* of a graph G is the length $g = g(G)$ of a shortest cycle. The *degree* of a vertex $v \in V$ is the number of vertices adjacent to v. A graph is called *k-regular* if all its vertices have the same degree k, and *bi-regular* or (k_1, k_2)*-regular* if all its vertices have either degree k_1 or k_2. A (k, g)*-graph* is a k-regular graph of girth g and a (k, g)*-cage* is a (k, g)-graph with the smallest possible number of vertices. The necessary condition obtained from the distance partition with respect to a vertex yields a lower bound $n_0(k, g)$ on the number of vertices of a (k, g)-graph, known as the Moore bound.

$$n_0(k,g) = \begin{cases} 1 + k + k(k-1) + \cdots + k(k-1)^{(g-3)/2} & \text{if } g \text{ is odd;} \\ 2(1 + (k-1) + \cdots + (k-1)^{g/2-1}) & \text{if } g \text{ is even.} \end{cases}$$

Biggs [9] calls *excess* of a (k, g)-graph G the difference $|V(G)| - n_0(k, g)$. Cages have been intensely studied since they were introduced by Tutte [30] in 1947. Erdős and Sachs [15] proved the existence of a (k, g)-graph for any value of k and g. Since then, most of the work carried out has been focused on constructing smallest (k, g)-graphs (see e.g. [1,2,4–8,12,16,19,21,26,28,29,32]). Biggs is the author of a report on distinct methods for constructing cubic cages [10]. More details about constructions of cages can be found in the surveys by Wong [32], by Holton and Sheehan [23, Chapter 6], or the recent one by Exoo and Jajcay [18].

In this paper, for each prime $q \ge 13$, we construct a family of $(q+3-u)$-regular graphs of girth 5 which ties the order of $(q+3, 5)$-graphs from [24] for $u = 0$, and improves the known bounds for $1 \le u \le q-1$ (cf. Table 1).

* Corresponding author.
E-mail addresses: marien.abreu@unibas.it (M. Abreu), garaujo@matem.unam.mx (G. Araujo-Pardo), m.camino.balbuena@upc.edu (C. Balbuena), domenico.labbate@unibas.it (D. Labbate).

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Table 1
Upper bounds for the order of $(k, 5)$-graphs.

k	Upper bound	Due to	New upper bound found in this paper
8	80	Royle	
9	96	Jørgensen	
10	126	Exoo	
11	156[a]	Jørgensen	
12	203	Exoo	
13	240[a]	Exoo	236
14	288	Jørgensen	284
15	312	Jørgensen	310
16	336	Jørgensen	[336]
17	448	Schwenk	
18	480	Schwenk	
19	512	Schwenk	
20	576	Jørgensen	572

General case: $(q + 3 - u, 5)$-graphs for $q \geq 13$ and $0 \leq u \leq q - 1$

q	u	Upper bound	Due to	New upper bound found in this paper
Prime	$u = 0$	$2(q^2 - 1)$	Jørgensen	$[2(q^2 - 1)]$
Prime	$1 \leq u \leq q - 1$	$2(q^2 - (q - 1)u - 1)$	Jørgensen	$2(q^2 - qu - 1)$
Prime power		$2(q^2 - (q - 1)u - 1)$	Jørgensen	

[a] Recently we have known that Exoo has constructed a $(11, 5)$-graph on 154 vertices and also a $(13, 5)$-graph on 230 vertices; see [17].

To construct such a family we use the following new technique that was inspired by papers from Funk [20] and Jørgensen [24]. We consider the Levi graph B_q of an elliptic semiplane of type C which is bipartite (cf. Section 2). Then we perform two reduction operations on the set of vertices of B_q (cf. Section 3) and an amalgam operation with bi-regular graphs (cf. Section 4) into the reduced graph. The novelty, with respect to [20,24], lies in performing Reduction 1 (cf. Section 3) before choosing the graphs for the amalgam.

Note that the general case presented in Section 5, holds for primes $q \geq 23$ (cf. Theorems 12 and 14). Smaller cases ($q = 13, 17, 19$) are treated with ad hoc similar constructions in Section 6, where we also obtain a 13-regular graph of girth 5 on 236 vertices from B_{11} which improves the bound found by Exoo in [17].

We conclude this section with Table 1 summarizing the state of the art regarding the new upper bounds for the order of $(k, 5)$-graphs from $k \geq 8$ (i.e. degrees k for which no cage has been constructed so far). The table is based on the values and references that appear in Table 4 of [18], and highlights the contributions of the results contained in this paper. The numbers in brackets indicate that the value found in this paper ties the previously known one.

2. Preliminaries

In this section we introduce the (bipartite) Levi graph B_q of an elliptic semiplane of type C [14,20] and we fix a labelling on its vertices which will be central for our construction since it allows us to keep track of the properties (such as regularity and girth) of the graphs obtained from B_q applying the reductions (cf. Section 3) and amalgams (cf. Section 4).

Definition 1. Let $GF(q)$ be a finite field with $q \geq 2$ a prime power. Let B_q be the Levi graph of an elliptic semiplane of type C which is a bipartite graph with vertex set (V_0, V_1) where $V_r = GF(q) \times GF(q)$, $r = 0, 1$, and edge set defined as follows:

$$(x, y)_0 \in V_0 \text{ is adjacent to } (m, b)_1 \in V_1 \text{ if and only if } y = mx + b. \qquad (1)$$

This graph is also known as the incidence graph of the biaffine plane [22], and it has been used in the problem of finding extremal graphs without short cycles [13,25].

The following properties of the graph B_q are well known (see [22,25]) and they will be fundamental throughout the paper.

Proposition 2. *Let B_q be the (bipartite) Levi graph defined above. Let $P_x = \{(x, y)_0 \mid y \in GF(q)\}$, for $x \in GF(q)$ and $L_m = \{(m, b)_1 \mid b \in GF(q)\}$, for $m \in GF(q)$. Then the graph B_q has the following properties:*

(i) *it is q-regular, vertex transitive, of order $2q^2$, and has girth 6 for $q \geq 3$;*
(ii) *it admits a partition $V_0 = \bigcup_{x=0}^{q-1} P_x$ and $V_1 = \bigcup_{m=0}^{q-1} L_m$ of its vertex set;*
(iii) *each block P_x is connected to each block L_m by a perfect matching, for $x, m \in GF(q)$;*
(iv) *each vertex in P_0 and L_0 is connected* straight *to all its neighbours in B_q, meaning that $N((0, y)_0) = \{(i, y)_1 | i \in GF(q)\}$ and $N((0, b)_1) = \{(j, b)_0 | j \in GF(q)\}$;*
(v) *the other matchings between P_x and L_m are* twisted *and the rule is defined algebraically in $GF(q)$ according to (1);*
(vi) *it has diameter 4 and any two distinct points in P_x (or in L_m) are at distance exactly 4 for $x, m \in GF(q)$.*

For further information regarding these properties and for constructions of the adjacency matrix of B_q as a block (0, 1)-matrix please refer to [3,7].

3. Reductions

In this section we will describe two reduction operations that we perform on the graph B_q that together with the amalgam operation will allow us to prove our main result i.e. the construction of new $(k, 5)$-graphs of smaller order than previously known ones.

Reduction 1. Remove vertices from P_0 and L_0.

Let $T \subseteq S \subseteq GF(q)$, $S_0 = \{(0, y)_0 | y \in S\} \subseteq P_0$, $T_0 = \{(0, b)_1 | b \in T\} \subseteq L_0$ and $B_q(S, T) = B_q - S_0 - T_0$.

Lemma 3. *Let $T \subseteq S \subseteq GF(q)$. Then $B_q(S, T)$ is bi-regular with degrees $(q-1, q)$ of order $2q^2 - |S| - |T|$. Moreover, the vertices $(i, t)_0 \in V_0$ and $(j, s)_1 \in V_1$, for each $i, j \in GF(q) - \{0\}$, $s \in S$ and $t \in T$ are the only vertices of degree $q - 1$ in $B_q(S, T)$, together with $(0, s)_1 \in V_1$ for $s \in S - T$ if $T \subsetneq S$.*

Proof. It is an immediate consequence of Proposition 2(i), (v). □

The key used in our construction (cf. Sections 5 and 6) to improve the known orders for $(k, 5)$-graphs is to apply Reduction 1 before applying the amalgam (cf. Section 4) for increasing the degree of B_q.

Reduction 2. Remove pairs of blocks (P_i, L_i) from B_q or from $B_q(S, T)$.

Let $u \in \{1, \dots, q-1\}$. Define $B_q(u) = B_q - \bigcup_{i=1}^{u}(P_{q-i} \cup L_{q-i})$ the graph obtained from B_q by deleting the last u pairs of blocks of vertices P_i, L_i, and let $B_q(S, T, u) = B_q - S_0 - T_0 - \bigcup_{i=1}^{u}(P_{q-i} \cup L_{q-i})$. For $u = 0$, $B_q(0) = B_q$ and $B_q(S, T, 0) = B_q(S, T)$.

Lemma 4. *Let $u \in \{0, \dots, q-1\}$. Then, the graph $B_q(u)$ is $(q-u)$-regular of order $2(q^2 - qu)$ and the graph $B_q(S, T, u)$ is bi-regular with degrees $(q-u-1, q-u)$ and order $2(q^2 - qu) - |S| - |T|$. Moreover, the vertices $(i, t)_0 \in V_0$ and $(j, s)_1 \in V_1$, for each $i, j \in GF(q)$, $s \in S$ and $t \in T$ are the only vertices of degree $q - u - 1$ in $B_q(S, T, u)$, together with $(0, s)_1 \in V_1$ for $s \in S - T$ if $T \subsetneq S$.*

Proof. It is immediate from Proposition 2(i), (iv) and Lemma 3. □

Reduction 2 has been widely used by several authors and with different names (cf. e.g. [1,2,6,5,13,20,24]) and we use it together with Reduction 1 and the amalgam for our construction of $(k, 5)$-graphs. Note that, to our purpose, it is indifferent when to apply Reduction 2 with respect to the other two operations.

4. Amalgams

In this section we will describe an *amalgam* operation inspired by Funk [20] and Jørgensen [24] where regular bipartite graphs were transformed into (no longer bipartite) regular graphs of higher degree adding *weighted* edges with different weights on opposite sides of the bipartition.

Since we apply Reduction 1 before increasing the degree of B_q, we describe the amalgam operation performed on the reduced graph $B_q(S, T, u)$ for $0 \le u \le q - 1$. The labelling for B_q introduced in Section 2, will be essential, in the choice of the graphs used for the amalgam, to guarantee regularity and girth 5 of the final graph.

Let Γ_1 and Γ_2 be two graphs of the same order and with the same labels on their vertices. In general, an *amalgam of Γ_1 into Γ_2* is a graph obtained adding all the edges of Γ_1 to Γ_2.

Let P_i and L_i be defined as in Section 2. Consider the graph $B_q(S, T, u)$, for some $T \subseteq S \subseteq GF(q)$ and $u \in \{0, \dots, q-1\}$. Let $S_0 \subseteq P_0$, $T_0 \subseteq L_0$ as in Reduction 1, and let $P'_0 := P_0 - S_0$ and $L'_0 := L_0 - T_0$ be the blocks in $B_q(S, T, u)$ of orders $q - |S|$ and $q - |T|$, respectively.

Let H_1, H_2, G_i, for $i = 1, 2$, be graphs of girth at least 5 and orders $q - |S|$, $q - |T|$ and q, respectively. Let H_1 be a k-regular graph. If $|S| = |T|$, let H_2 be k-regular and otherwise let it be $(k, k+1)$-regular with $|S - T|$ vertices of degree $k+1$. If $T = \emptyset$, let G_1 be a k-regular graph and otherwise let it be $(k, k+1)$-regular with $|T|$ vertices of degree $k+1$. Finally, let G_2 be a $(k, k+1)$-regular with $|S|$ vertices of degree $k+1$.

We define $B_q^*(S, T, u)$ to be the *amalgam* of H_1 into P'_0, H_2 into L'_0, G_1 into P_i and G_2 into L_i, for $i \in \{1, \dots, q-u-1\}$ and $u \in \{0, \dots, q-2\}$. We also define $B_q^*(S, T, q-1)$ to be the amalgam of H_1 into P'_0, H_2 into L'_0.

To simplify notation in our results, we label P_i and L_i as in Section 2, but assume that the labellings of H_1, H_2, G_1 and G_2, correspond to the second coordinates of P'_0, L'_0, P_i and L_i respectively for $i \in \{1, \dots, q-u-1\}$ and $u \in \{0, \dots, q-2\}$. Suppose also that the vertices of degree $k+1$, if any, in H_2, G_1 and G_2 are labelled in correspondence with the elements of $S - T$, T and S, respectively.

With such a labelling, let $\alpha\beta$ be an edge in H_1, H_2, G_1 or G_2, and define the *weight* or the *Cayley Colour* of $\alpha\beta$ to be $\pm(\beta - \alpha) \in GF(q) - \{0\}$. Let $\mathcal{P}_\omega$ be the set of weights in H_1 and G_1, and let $\mathcal{L}_\omega$ be the set of weights in H_2 and G_2.

Cayley Colours have been used by Funk in [20] to construct $(k, 5)$-graphs from elliptic semiplanes. The following result is a special case of [20, Theorem 2.8] for the labelling we have chosen for B_q (cf. Section 2). On the other hand, it generalizes such a theorem since we delete vertices from P_0 and L_0, pairs of blocks P_i, L_i and amalgam with graphs which are not regular, but chosen in such a way that the obtained amalgam is regular.

Theorem 5. *Let $T \subseteq S \subseteq GF(q)$, $u \in \{0, \ldots, q-1\}$. Let H_1, H_2, G_1 and G_2 be defined as above and suppose that the weights $\mathcal{P}_\omega \cap \mathcal{L}_\omega = \emptyset$. Then the amalgam $B_q^*(S, T, u)$ is a $(q+k-u)$-regular graph of girth at least 5 and order $2q(q-u) - |S| - |T|$.*

Proof. The order and the regularity of $B_q^*(S, T, u)$ follow from Lemma 4 and the choice of H_1, H_2, G_1 and G_2. Note that the vertices of L_i, with degree $q-u-1$ in $B_q(S, T, u)$, have degree $k+1$ in G_2, which add up to degree $q+k-u$ in $B_q^*(S, T, u)$, for $i \in \{1, \ldots, q-u-1\}$. Similarly for the vertices in L_0 and for those in P_i, for $i \in \{1, \ldots, q-u-1\}$.

Let C be a shortest cycle in $B_q^*(S, T, u)$ and suppose, by contradiction, that $|C| \leq 4$. Therefore, $C = (xyz)$ or $C = (wxyz)$. Since B_q has girth 6 and H_1, H_2, G_1, G_2 have girth at least 5, then C cannot be completely contained in B_q or in some H_i or G_i for $i = 1, 2$. Then, w.l.o.g. the path xyz in C is such that $x, y \in P_i$ and $z \in L_m$ for some $i, m \in GF(q)$. Since the edges between P_i and L_m form a matching, then $xz \notin E(B_q)$ and hence $xz \notin E(B_q^*(S, T, u))$. Thus $|C| > 3$ and we can assume $|C| = 4$ and $C = (wxyz)$.

If $w \in P_i$, by the same argument, $wz \notin E(B_q^*(S, T, u))$ and we have a contradiction. There are no edges between P_i and P_j in $B_q^*(S, T, u)$, so $w \notin P_j$ for $j \in GF(q) - \{i\}$, which implies that $w \in L_n$ for some $n \in GF(q)$. If $n \neq m$, we have a contradiction since there are no edges between L_m and L_n in $B_q^*(S, T, u)$. Therefore $x, y \in P_i$ and $w, z \in L_m$. Let $x = (i, \alpha)_0$, $y = (i, \beta)_0$, $z = (m, \gamma)_1$ and $w = (m, \delta)_1$ as in the labelling chosen in Section 2. Then $wx, yz \in E(B_q^*(S, T, u))$ imply that $\alpha = m \cdot i + \delta$ and $\beta = m \cdot i + \gamma$, respectively, which give $\beta - \alpha = \gamma - \delta$. On the other hand $xy, wz \in E(B_q^*(S, T, u))$ imply that $\alpha\beta \in E(H_1) \cup E(G_1)$ and $\gamma\delta \in E(H_2) \cup E(G_2)$, so $\pm(\alpha - \beta) \in \mathcal{P}_\omega$ and $\pm(\gamma - \delta) \in \mathcal{L}_\omega$, a contradiction, since by hypothesis $\mathcal{P}_\omega \cap \mathcal{L}_\omega = \emptyset$. $\square$

Remark 6. The graph $B_q^*(S, T, u)$ has girth exactly 5 since B_q has diameter 4 (cf. Proposition 2(vi)) and any edge in G_i, $i = 1, 2$, creates a 5-cycle between vertices at distance 4.

5. New regular graphs of girth 5

In this section we will construct new $(q+3)$-regular graphs of girth 5, for any prime $q \geq 23$, applying Reductions 1, 2 and amalgam to the graph B_q as previously described. In each case we will specify the sets S and T of vertices to be deleted from P_0 and L_0 and the graphs H_1, H_2, G_1, G_2 to be used for the amalgam into $B_q^*(S, T, u)$.

For each prime $q \geq 23$, we construct a family of $(q+3-u)$-regular graphs of girth 5 which ties the order of $(q+3, 5)$-graphs from [24] for $u = 0$, and improves the known bounds for $1 \leq u \leq q-1$ (cf. Table 1). As mentioned in the Introduction, smaller cases ($q = 13, 17, 19$) are treated with ad hoc similar constructions in Section 6, where we also obtain a 13-regular graph of girth 5 on 236 vertices from B_{11} which improves the bound found by Exoo in [17].

Recall that every prime q is either congruent to 1 or 5 modulo 6. We will now treat these two cases separately, when $q = 6n+1$ or $q = 6n+5$ is a prime.

5.1. Construction for primes $q = 6n+1$

Throughout this subsection we will consider $n \geq 5$. The smaller cases will be treated in Section 6 since some of the graphs in this section have girth smaller than 5 when $n < 5$.

Let H_1 and H_2 be two graphs of order $q-1$ with the vertices labelled from 1 through $6n$, and partitioned into $W_1 = \{1, 2, \ldots, 3n\}$ and $W_2 = \{3n+1, \ldots, 6n\}$.

Define the set of edges $E(H_1) = A_1 \cup B_1 \cup C_1$ as follows.

Set	Edges	Description
A_1	$\{(i, i+1) \mid i = 1, \ldots, 3n-1\} \cup \{(3n, 1)\}$	$(3n)$-cycle with weights 1 and $3n-1$
B_1	$\{(i, i+2) \mid i = 3n+1, \ldots, 6n-2\} \cup \{(6n-1, 3n+1), (6n, 3n+2)\}$	One or two cycles according to the parity of n, with weights 2 and $3n-2$
C_1	$\{(i, 3n+i) \mid i = 1, \ldots, 3n\}$	Prismatic edges between W_1 and W_2 of weight $3n$

The graph H_1 is cubic and has weights $\pm\{1, 2, 3n-2, 3n-1, 3n\}$.

Lemma 7. *The graph H_1 has girth 5.*

Proof. Let C be a shortest cycle in H_1. If C is a subgraph of either the induced subgraph $H_1[W_1]$ or $H_1[W_2]$ then $|C| \geq 5$, since $H_1[W_1]$ has girth at least 15 and $H_1[W_2]$ has girth at least 9. Otherwise, there is a path xyz in C is such that either $x, y \in W_1$ and $z \in W_2$ or $x \in W_1$ and $y, z \in W_2$. The first case has the following subcases:

(i) $x = 1, y = 3n, z = 6n$;
(ii) $x = i, y = i-1, z = 3n+i-1$, for $i = 2, \ldots, 3n$;
(iii) $x = i, y = i+1, z = 3n+i+1$, for $i = 1, \ldots, 3n-1$;
(iv) $x = 3n, y = 1, z = 3n+1$.

The second case has similar subcases. If we show that $z \notin N_{H_1}(x)$ then $|C| \neq 3$, and if $\{y\} = N_{H_1}(x) \cap N_{H_1}(z)$ then $|C| \neq 4$. In subcase (i) the neighbourhoods of x and z in H_1 are $N_{H_1}(x) = \{2, 3n, 3n+1\}$ and $N_{H_1}(z) = \{3n, 3n+2, 6n-2\}$, respectively. Thus, $z \notin N_{H_1}(x)$ and $\{y\} = N_{H_1}(x) \cap N_{H_1}(z)$. Hence, $|C| \geq 5$. All the other cases are analogous. The cycle $(1, 2, 3, 3n+3, 3n+1)$ is a 5-cycle in H_1. □

Define the set of edges $E(H_2) = A_2 \cup B_2 \cup C_2$ as follows.

Set	Edges	Description
A_2	$\{(i, i+3) \mid i = 1, \ldots, 3n-3\} \cup \{(3n-2, 1), (3n-1, 2), (3n, 3)\}$	Three n-cycles with weights 3 and $3n-3$
B_2	$\{(i, i+4) \mid i = 3n+1, \ldots, 6n-4\} \cup \{(6n-3, 3n+1), (6n-2, 3n+2), (6n-1, 3n+3), (6n, 3n+4)\}$	One, two or four cycles according to the congruency of $3n$ modulo 4, with weights 4 and $3n-4$
C_2	$\{(i, 3n+4+i) \mid i = 1, \ldots, 3n-4\} \cup \{(3n-3, 3n+1), (3n-2, 3n+2), (3n-1, 3n+3), (3n, 3n+4)\}$	Prismatic edges between W_1 and W_2 of weights 4 and $3n+4 \equiv 3n-3 \mod q$

The graph H_2 is cubic and has weights $\pm\{3, 4, 3n-4, 3n-3\}$.

Lemma 8. *The graph H_2 has girth at least 5.*

Proof. Similar to the proof of Lemma 7. □

Lemma 9. *Let G be a graph of girth at least 5. Let $x_1x_2, x_3x_4 \in E(G)$ be two independent edges of G such that $N_G(x_i) \cap N_G(x_j) = \emptyset$, for all $i, j \in \{1, 2, 3, 4\}$, $i \neq j$. Let v be a vertex such that $v \notin V(G)$ and let G' be a graph with $V(G') = V(G) \cup \{v\}$ and $E(G') = E(G) - \{x_1x_2, x_3x_4\} \cup \{(v, x_i) \mid i = 1, 2, 3, 4\}$. Then G' has girth at least 5.*

Proof. Let C be a shortest cycle in G'. If $E(C) \subset E(G)$ then, by hypothesis, $|C| > 4$. Otherwise $v \in V(C)$ and $x_i v x_j$ is a path in C for some $i, j \in \{1, 2, 3, 4\}$, $i \neq j$. In G' the set $\{x_i \mid i = 1, 2, 3, 4\}$ is independent, so $|C| > 3$. By hypothesis, $N_{G'}(x_i) \cap N_{G'}(x_j) = \{v\}$ in G' and hence $|C| > 4$. □

Let G_1 be a graph on q vertices labelled from 0 through $q-1$ and defined as follows: $G_1 := H_1 - \{(1, 3n), (\lfloor \frac{3n+1}{2} \rfloor, 3n + \lfloor \frac{3n+1}{2} \rfloor)\} + \{(0, 1), (0, \lfloor \frac{3n+1}{2} \rfloor), (0, 3n), (0, 3n + \lfloor \frac{3n+1}{2} \rfloor)\}$.

Lemma 10. *The graph G_1 has girth at least 5.*

Proof. The edges $e_1 = (1, 3n)$ and $e_2 = (\lfloor \frac{3n+1}{2} \rfloor, 3n + \lfloor \frac{3n+1}{2} \rfloor)$ are independent in H_1. The neighbourhoods of the endvertices of e_1 and e_2 are:

$$N(1) = \{2, 3n, 3n+1\};$$
$$N\left(\left\lfloor \frac{3n+1}{2} \right\rfloor\right) = \left\{\left\lfloor \frac{3n+1}{2} \right\rfloor - 1, \left\lfloor \frac{3n+1}{2} \right\rfloor + 1, 3n + \left\lfloor \frac{3n+1}{2} \right\rfloor\right\};$$
$$N(3n) = \{1, 3n-1, 6n\};$$
$$N\left(3n + \left\lfloor \frac{3n+1}{2} \right\rfloor\right) = \left\{3n + \left\lfloor \frac{3n+1}{2} \right\rfloor - 1, 3n + \left\lfloor \frac{3n+1}{2} \right\rfloor + 1, \left\lfloor \frac{3n+1}{2} \right\rfloor\right\};$$

which satisfy the hypothesis of Lemma 9. Since G_1 is constructed from H_1 in the same way as G' from G in Lemma 9, we can conclude that G_1 has girth at least 5. □

All together the weights of H_1 and G_1 modulo q give

$$\mathcal{P}_\omega := \begin{cases} \pm\left\{1, 2, \dfrac{3n+1}{2}, 3n-2, 3n-1, 3n\right\} & \text{if } n \text{ is odd;} \\ \pm\left\{1, 2, \dfrac{3n}{2}, \dfrac{3n+2}{2}, 3n-2, 3n-1, 3n\right\} & \text{if } n \text{ is even.} \end{cases} \tag{2}$$

Let G_2 be a graph on q vertices labelled from 0 through $q-1$ and defined as follows:

$$G_2 := \begin{cases} H_2 - \{(3, 22), (5, 24)\} + \{(0, 3), (0, 5), (0, 22), (0, 24)\} & \text{if } n = 5; \\ H_2 - \{(3, 3n+7), (4, 3n+8)\} + \{(0, 3), (0, 4), (0, 3n+7), (0, 3n+8)\} & \text{if } n \geq 6. \end{cases}$$

Note that for $n = 5$ the edge $(0, 3n+8) = (0, 23)$ has weight -8 which lies already in $\mathcal{P}_\omega$ and Theorem 5 cannot be applied. This is why, in the definition of G_2, we choose to delete the edge $(5, 24)$ from H_2, instead of $(4, 3n+8) = (4, 23)$.

Lemma 11. *The graph G_2 has girth at least 5.*

Proof. First suppose $n \geq 6$. As in Lemma 10, the edges $(3, 3n+7)$, $(4, 3n+8)$ are independent in H_2 and the neighbourhoods

$$N(3) = \{6, 3n, 3n+7\};$$
$$N\left(\left\lfloor \frac{3n+1}{2} \right\rfloor\right) = \left\{\left\lfloor \frac{3n+1}{2} \right\rfloor - 1, \left\lfloor \frac{3n+1}{2} \right\rfloor + 1, 3n + \left\lfloor \frac{3n+1}{2} \right\rfloor\right\};$$
$$N(3n) = \{1, 3n-1, 6n\};$$
$$N\left(3n + \left\lfloor \frac{3n+1}{2} \right\rfloor\right) = \left\{3n + \left\lfloor \frac{3n+1}{2} \right\rfloor - 1, 3n + \left\lfloor \frac{3n+1}{2} \right\rfloor + 1, \left\lfloor \frac{3n+1}{2} \right\rfloor\right\}$$

satisfy the hypothesis of Lemma 9. Since G_2 is constructed from H_2 as G' from G in Lemma 9, G_2 has girth at least 5.

Similarly for $n = 5$. □

All together the weights of H_2 and G_2 modulo q give

$$\mathcal{L}_\omega := \begin{cases} \pm\{3, 4, 7, 9, 11, 12\} & \text{if } n = 5; \\ \pm\{3, 4, 3n-7, 3n-6, 3n-4, 3n-3\} & \text{if } n \geq 6. \end{cases} \tag{3}$$

Theorem 12. *Let q be a prime such that $q = 6n+1$, $n \geq 2$. Then, there is a $(q+3-u)$-regular graph of girth 5 and order $2(q^2 - qu - 1)$, for each $0 \leq u \leq q-1$.*

Proof. We treat the cases $n = 2, 3$ in Section 6. For $n = 4$, $q = 6n+1 = 25$ is not a prime, therefore we can assume that $n \geq 5$.

Let $S = T = \{0\}$ and choose H_i, G_i for $i = 1, 2$ as previously described in this subsection. Lemmas 7, 8, 10 and 11 together with (2) and (3) imply that the hypothesis of Theorem 5 is satisfied. Therefore, the graphs $B_q^*(S, T, u)$ are $(q+3-u)$-regular of girth 5 and order $2(q^2 - u - 1)$ for each $0 \leq u \leq q-1$. The girth of $B_q^*(S, T, u)$ is exactly 5 by Remark 6. □

5.2. Construction for primes $q = 6n+5$

We consider $n \geq 3$ throughout this subsection and we treat smaller cases in Section 6 since some of the graphs in this section have girth smaller than 5 when $n < 3$.

Let H_1 and H_2 be two graphs of order $q-1$ with the vertices labelled from 1 through $6n+4$, and partitioned into $W_1 = \{1, 2, \ldots, 3n+2\}$ and $W_2 = \{3n+3, \ldots, 6n+4\}$.

Define the set of edges $E(H_1) = A_1 \cup B_1 \cup C_1$ as follows.

Set	Edges	Description
A_1	$\{(i, i+1) \mid i = 1, \ldots, 3n+1\} \cup \{(3n+2, 1)\}$	$(3n+2)$-cycle with weights 1 and $3n+1$
B_1	$\{(i, i+2) \mid i = 3n+3, \ldots, 6n+2\} \cup \{(6n+3, 3n+3), (6n+4, 3n+4)\}$	One or two cycles according to the parity of n, with weights 2 and $3n$
C_1	$\{(i, 3n+i+2) \mid i = 1, \ldots, 3n+2\}$	Prismatic edges between W_1 and W_2 of weight $3n+2$

The graph H_1 is cubic and has weights $\pm\{1, 2, 3n, 3n+1, 3n+2\}$.

Define the set of edges $E(H_2) = A_2 \cup B_2 \cup C_2$ as follows.

Set	Edges	Description
A_2	$\{(i, i+3) \mid i = 1, \ldots, 3n-1\} \cup \{(3n, 1), (3n+1, 2), (3n+2, 3)\}$	One $3n+2$-cycle with weights 3 and $3n-1$
B_2	$\{(i, i+4) \mid i = 3n+3, \ldots, 6n\} \cup \{(6n+1, 3n+3), (6n+2, 3n+4), (6n+3, 3n+5), (6n+4, 3n+6)\}$	One, two or four cycles according to the congruency of n modulo 4, with weights 4 and $3n-2$
C_2	$\{(i, 3n+i+6) \mid i = 1, \ldots, 3n-2\} \cup \{(3n-1, 3n+3), (3n, 3n+4), (3n+1, 3n+5), (3n+2, 3n+6)\}$	Prismatic edges between W_1 and W_2 of weights 4 and $3n+6 \equiv 3n-1 \bmod q$

The graph H_2 is cubic and has weights $\pm\{3, 4, 3n-2, 3n-1\}$.

Let G_1 be a graph on q vertices labelled from 0 through $q-1$ and defined as follows:

$$G_1 := \begin{cases} H_1 - \{(1, 12), (6, 17)\} + \{(0, 1), (0, 6), (0, 12), (0, 17)\} & \text{if } n = 3; \\ H_1 - \left\{(1, 3n+3), \left(\left\lfloor \frac{3n+1}{2} \right\rfloor, 3n+2+\left\lfloor \frac{3n+1}{2} \right\rfloor\right)\right\} & \\ \quad + \left\{(0, 1), \left(0, \left\lfloor \frac{3n+1}{2} \right\rfloor\right), (0, 3n+3), \left(0, 3n+2+\left\lfloor \frac{3n+1}{2} \right\rfloor\right)\right\} & \text{if } n \geq 4. \end{cases}$$

Note that for $n = 3$ the independent edges $(1, 3n + 3) = (1, 12)$ and $(\lfloor \frac{3n+1}{2} \rfloor, 3n + 2 + \lfloor \frac{3n+1}{2} \rfloor) = (5, 6)$ of H_1 have a common neighbour, namely $N_{H_1}(12) \cap N_{H_1}(16) = \{14\}$, and Lemma 9 cannot be applied. This is why we choose the independent edges (1, 12) and (6, 17) in H_1 with pairwise disjoint neighbourhoods to define G_1.

All together the weights of H_1 and G_1 modulo q give

$$\mathcal{P}_\omega := \begin{cases} \pm\{1, 2, 6, 9, 10, 11\} & \text{if } n = 3; \\ \pm\left\{1, 2, \dfrac{3n+1}{2}, \dfrac{3n+5}{2}, 3n, 3n+1, 3n+2\right\} & \text{if } n \text{ is odd and } n \geq 5; \\ \pm\left\{1, 2, \dfrac{3n}{2}, \dfrac{3n+6}{2}, 3n, 3n+1, 3n+2\right\} & \text{if } n \text{ is even.} \end{cases} \quad (4)$$

Let G_2 be a graph on q vertices labelled from 0 through $q - 1$ and defined as follows: $G_2 := H_2 - \{(3, 3n + 9), (4, 3n + 10)\} + \{(0, 3), (0, 4), (0, 3n + 9), (0, 3n + 10)\}$.

All together the weights of H_2 and G_2 modulo q give

$$\mathcal{L}_\omega := \pm\{3, 4, 3n - 5, 3n - 4, 3n - 2, 3n - 1\}. \quad (5)$$

Lemma 13. *The graphs H_1, H_2, G_1 and G_2 have girth at least 5.*

Proof. Similar to Lemmas 7, 8, 10 and 11. □

Note that in general, the girth of H_1 is exactly 5, since $(1, 2, 3, 3n + 5, 3n + 3)$ is a 5-cycle in H_1.

Theorem 14. *Let q be a prime such that $q = 6n + 5$, for $n \geq 3$. Then, there is a $(q + 3 - u)$-regular graph of girth 5 and order $2(q^2 - qu - 1)$ for each $0 \leq u \leq q - 1$.*

Proof. Let $S = T = \{0\}$ and choose H_i, G_i for $i = 1, 2$ as previously described in this subsection. By (4) and (5) and Lemma 13, all the hypothesis of Theorem 5 are satisfied. Thus, the graphs $B_q^*(S, T, u)$ are $(q + 3 - u)$-regular of girth 5 and order $2(q^2 - u - 1)$ for each $0 \leq u \leq q - 1$. The girth of $B_q^*(S, T, u)$ is exactly 5 by Remark 6. □

6. Small cases

We now present some ad hoc constructions of graphs $B_q^*(S, T, u)$ for small prime values of q. The first two constructions complete the proof of Theorem 12 and we treat them here separately since some of the graphs in Section 5 have girth smaller than 5 when $n < 5$. Furthermore, we also obtain a 13-regular graph of girth 5 on 236 vertices from B_{11} which improves the bound found by Exoo in [17] as well as a 20-regular graph of girth 5 of order 572 from B_{17} which improves the bound found by Jørgensen [24] (cf. Table 1).

6.1. $q = 13$

In this case, let $S = T = \{0\}, H_1, H_2, G_1$ and G_2 be as in Fig. 1. The graphs G_i are obtained from H_i deleting two independent edges satisfying the hypothesis of Lemma 9 and joining all their end-vertices to a new vertex, say 0, for $i = 1, 2$. Specifically $G_1 = H_1 - \{(1, 10), (3, 12)\} + \{(0, 1), (0, 3), (0, 10), (0, 12)\}$, $G_2 = H_2 - \{(2, 8), (5, 11)\} + \{(0, 2), (0, 8), (0, 5), (0, 11)\}$ and as unlabelled graphs G_1 is isomorphic to G_2. Hence, the graphs G_1 and G_2 have order 13, girth 5 and are bi-regular with one vertex of degree four and all other vertices of degree three.

Note that as unlabelled graphs H_1 is isomorphic to H_2 and they are both isomorphic to one of the two cubic graphs on 12 vertices of girth 5, specifically #84 *in the list of cubic graphs of order* 12 from [27,31].

Lemma 15. *Let $S = T = \{0\}, H_1, H_2, G_1$ and G_2 be as described above. Then the graph $B_{13}^*(0, 0, u)$ is a $(16 - u)$-regular graph of girth 5 and order $336 - 26u$, for $0 \leq u \leq 12$.*

Proof. The weights of these graphs are $\mathcal{P}_\omega = \pm\{1, 3, 4\}$ and $\mathcal{L}_\omega = \pm\{2, 5, 6\}$. Thus, by Theorem 5, the graph $B_{13}^*(0, 0, u)$ is a $(16 - u)$-regular graph of girth 5 and order $26(13 - u) - 2 = 336 - 26u$, for $0 \leq u \leq 12$. □

- For $u = 0$, we obtain a 16-regular graph of girth 5 and order 336, with exactly the same order as the (16, 5)-graph that appears in [24].
- For $u = 1$, we obtain a 15-regular graph of girth 5 and 310 vertices which has two vertices less than the (15, 5)-graph that appears in [24].
- For $u = 2$ we obtain a 14-regular graph of girth 5 and 284 vertices which has four vertices less than the (14, 5)-graph that appears in [24].

6.2. $q = 19$

Let $S = T = \{0\}$ and let H_1, H_2, G_1 and G_2 be as in Fig. 2. The graphs G_i are obtained from H_i deleting two independent edges satisfying the hypothesis of Lemma 9 and joining all their end-vertices to a new vertex, say 0, for $i = 1, 2$. Specifically $G_1 = H_1 - \{(1, 10), (9, 16)\} + \{(0, 1), (0, 9), (0, 10), (0, 16)\}$ and $G_2 = H_2 - \{(8, 13), (11, 15)\} +$

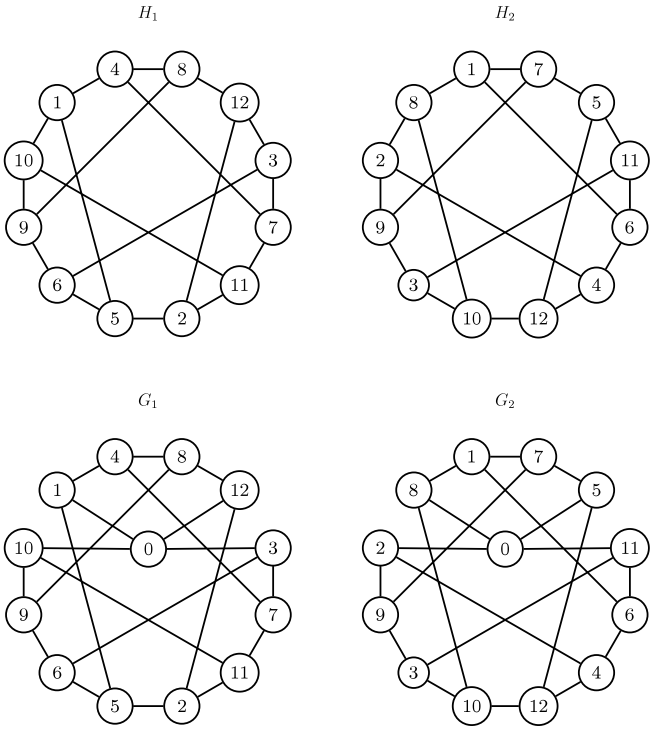

Fig. 1. The graphs H_i and G_i for $i = 1, 2$ and $q = 13$.

$\{(0, 8), (0, 13), (0, 11), (0, 15)\}$. Hence, the graphs G_1 and G_2 have order 19, girth 5 and are bi-regular with one vertex of degree four and all other vertices of degree 3.

Lemma 16. *Let $S = T = \{0\}$, H_1, H_2, G_1 and G_2 be as described above. Then the graph $B^*_{19}(0, 0, u)$ is a $(22 - u)$-regular graph of girth 5 and order $720 - 38u$, for $0 \leq u \leq 18$.*

Proof. The weights of these graphs are $\mathcal{P}_w = \pm\{1, 2, 3, 7, 9\}$ and $\mathcal{L}_w = \pm\{4, 5, 6, 8\}$. Thus, by Theorem 5, the graph $B^*_{19}(0, 0, u)$ is a $(22 - u)$-regular graph of girth 5 and order $38(19 - u) - 2 = 720 - 38u$, for $0 \leq u \leq 18$. □

- For $u = 0$, we obtain a 22-regular graph of girth 5 and order 720, with exactly the same order as the $(22, 5)$-graph that appears in [24].
- For $u = 1$, we obtain a 21-regular graph of girth 5 and 682 vertices which has two vertices less than the $(21, 5)$-graph that appears in [24].

6.3. $q = 11$

For $q = 11$ we are going to remove 6 vertices from B_{11} instead of 2, but we will construct a $(q + 2)$-regular graph instead of a $(q + 3)$-regular one.

Lemma 17. *Let $S = \{0, 1, 2, 4, 6, 8\}$ and $T = \emptyset$. Let $H_1 = (3, 5, 10, 7, 9)$ be a 5-cycle with weights $\pm\{2, 3, 5\}$, $G_1 = (0, 2, 4, 6, 8, 10, 1, 3, 5, 7, 9)$ be a 11-cycle with weight $\{\pm 2\}$, and $H_2 = G_2 = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10) + (0, 4) + (2, 6) + (1, 8)$ be a 11-cycle with three chords and weights $\pm\{1, 4\}$ (see Fig. 3). Then the graph $B^*_{11}(S, T, u)$ is a $(13 - u)$-regular graph of girth 5 and order $22(11 - u) - 6 = 236 - 22u$, for $u \leq 10$. In particular, we obtain a 13-regular graph of girth 5 and order 236 for $u = 0$.*

Proof. Since $\mathcal{P}_\omega = \pm\{2, 3, 5\}$ and $\mathcal{L}_\omega = \pm\{1, 4\}$, the thesis follows by Theorem 5. □

Note that the graph $B^*_{11}(S, T, 0)$ has four vertices less than the one constructed in [17].

6.4. $q = 17$

For $q = 17$ we are going to remove 6 vertices instead of 2 and construct a $(q + 3)$-regular graph, obtaining a better result than the one obtained in [10].

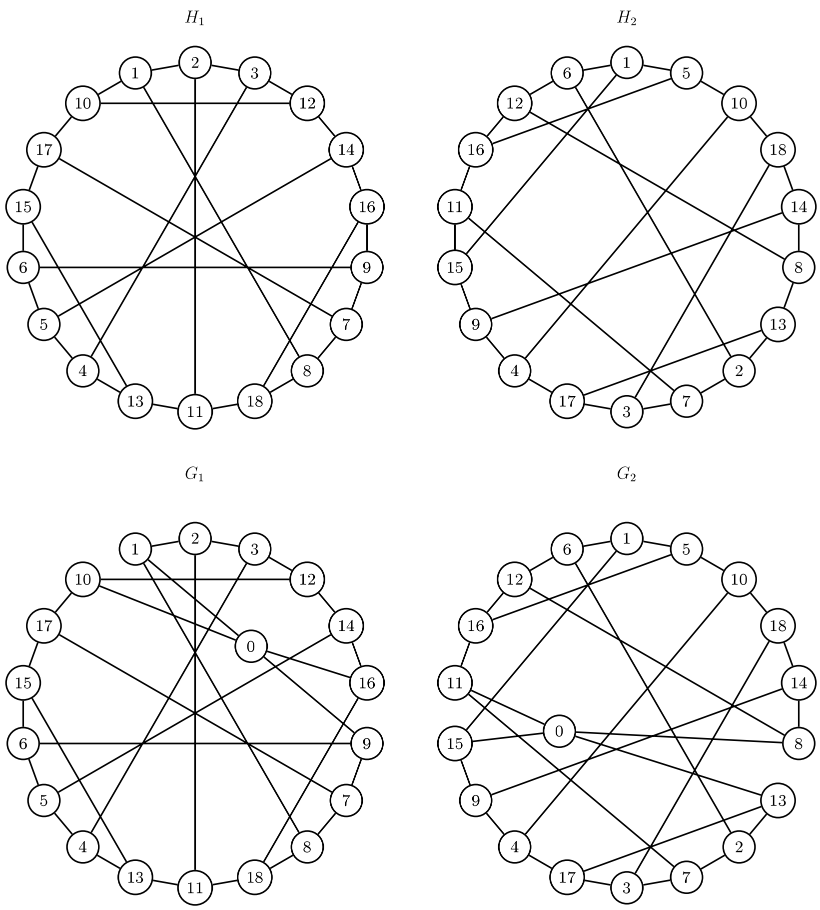

Fig. 2. The graphs H_i and G_i for $i = 1, 2$ and $q = 19$.

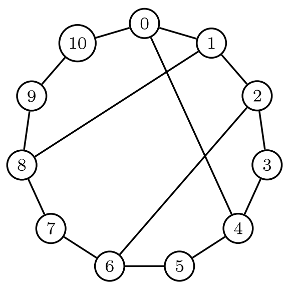

Fig. 3. The graphs $H_2 = G_2$ for $q = 11$.

Lemma 18. *Let $S = T = \{7, 10, 12\}$, H_1, H_2, G_1 and G_2 be as in Fig. 4. The graphs G_1 and G_2 have order 17, girth 5 and are bi-regular with three vertices of degree four and all other vertices of degree 3. Then the graph $B^*_{17}(S, T, u)$ is a $(20 - u)$-regular graph of girth 5 and order $572 - 34u$, for $u \geq 16$.*

Proof. In this case $\mathcal{P}_w = \pm\{1, 3, 4, 5\}$ and $\mathcal{L}_w = \pm\{2, 6, 7, 8\}$, thus, by Theorem 5, the graph $B^*_{17}(S, T, u)$ is a $(20 - u)$-regular graph of girth 5 and order $34(17 - u) - 6 = 572 - 34u$ for $u \leq 16$. $\square$

In [24] the author constructs $(k, 5)$-graphs of order $32(k - 2)$, while we have constructed $(k, 5)$-graphs of order $34(k - 3) - 6$ which have $44 - 2k$ fewer vertices, for $k \in \{4, \ldots, 20\}$. In particular, we obtain a 20-regular graph of girth 5 and order 572 which has four vertices less than the one constructed in [24]. Note also that as unlabelled graphs $H_1 \cong H_2$ and they are both isomorphic to the Heawood graph.

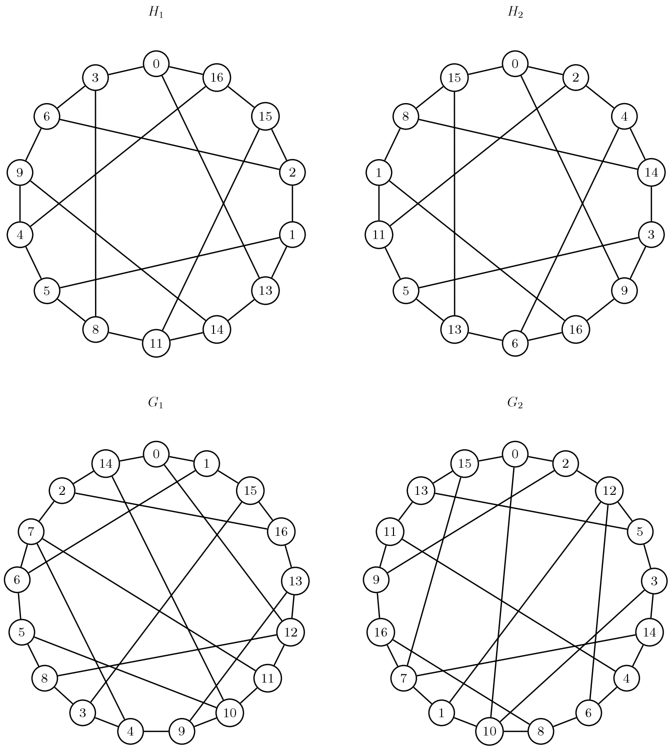

Fig. 4. The graphs H_i and G_i for $i = 1, 2$ and $q = 17$.

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