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Partitioning regular graphs into equicardinal linear forests

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Abstract

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It is a well-known fact that the linear arboricity of a k-regular graph is $\lceil (k+1)/2 \rceil$ for $k = 3, 4$. In this paper, we prove that if the number of edges of a k-regular graph is divisible by $\lceil (k+1)/2 \rceil$, then its edge set can be partitioned into $\lceil (k+1)/2 \rceil$ linear forests, all of which have the same number of edges ($k = 3, 4$).

In this paper, we consider finite simple graphs.

A *linear forest* is a forest whose components are paths. Akiyama, Exoo and Harary [1] proved that the edge set of a 3-regular graph can be partioned into two linear forests.

Theorem A (Akiyama, Exoo and Harary [1]). *Every* 3-*regular graph G has a partition (F_1, F_2) of $E(G)$ such that both $(V(G), F_1)$ and $(V(G), F_2)$ are linear forests.*

Akiyama, Exoo and Harary [2] also proved that the edge set of a 4-regular graph can be partitioned into three linear forests.

Theorem B (Akiyama, Exoo and Harary [2]). *Every 4-regular graph G has a partition (F_1, F_2, F_3) of $E(G)$ such that every $(V(G), F_i)$ is a linear forest $(i = 1, 2, 3)$.*

On the other hand, in [4] there is a conjecture that if the number of edges of a 3-regular graph is even (i.e. its order is divisible by 4), then $E(G)$ can be partitioned into two isomorphic linear forests.

Conjecture A ([4]). Let G be a 3-regular graph with $|E(G)| \equiv 0 \pmod 2$. Then there exists a partition (F_1, F_2) of $E(G)$ such that $(V(G), F_1)$ and $(V(G), F_2)$ are isomorphic linear forests.

The purpose of this paper is to prove the following extensions of Theorem A and Theorem B.

Theorem 1. *Let G be a 3-regular graph with $|E(G)| \equiv 0 \pmod 2$. Then there exists a partition (F_1, F_2) of $E(G)$ such that both $(V(G), F_1)$ and $(V(G), F_2)$ are linear forests and $|F_1| = |F_2|$.*

Theorem 2. *Let G be a 4-regular graph with $|E(G)| \equiv 0 \pmod 3$. Then there exists a partition (F_1, F_2, F_3) of $E(G)$ such that every $(V(G), F_i)$ is a linear forest $(i = 1, 2, 3)$ and $|F_1| = |F_2| = |F_3|$.*

Note that Theorem 1 lies between Theorem A and Conjecture A.

Let G be a graph and $F \subset E(G)$. When there is no fear of confusion, we identify F with the subgraph of G induced by F (which is the subgraph obtained from $(V(G), F)$ by removing isolated vertices). We denote the number of components of F by $w(F)$. For $x \in V(G)$ we denote the degree of x in G by $\deg_G(x)$, and the set of vertices adjacent to x in G by $N_G(x)$. The maximum degree and the minimum degree of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. For a nonnegative integer k and $F \subset E(G)$, we say that F is a k-factor if F induces a k-regular spanning subgraph of G. We denote by P_k a path of k vertices. For other basic graph-theoretic notation, we refer the reader to [3].

In order to prove Theorem 1 and Theorem 2, we first consider connected graphs G with $\Delta(G) \leqslant 3$ and $\delta(G) = 2$.

Definition. Let G be a connected graph with maximum degree at most 3 and minimum degree (exactly) 2. Let (F_1, F_2) be a partition of $E(G)$.

(1) Suppose $|E(G)|$ is even. Then (F_1, F_2) is said to be an *admissible partition* if it satisfies:

(1-1) Both F_1 and F_2 are linear forests,

(1-2) $w(F_1) = w(F_2)$, and

(1-3) $\deg_{F_1}(x) = \deg_{F_2}(x) = 1$ for every vertex $x \in V(G)$ of degree 2.

(2) Suppose $|E(G)|$ is odd. Then (F_1, F_2) is said to be an *admissible partition* if it satisfies:

(2-1) Both F_1 and F_2 are linear forests,

(2-2) $w(F_1) = w(F_2)$, and

(2-3) $\deg_{F_1}(x) = \deg_{F_2}(x) = 1$ for every vertex $x \in V(G)$ of degree 2 except for one vertex x_0. Furthermore, $\deg_{F_1}(x_0) = 2$ and $\deg_{F_2}(x_0) = 0$.

We use the following lemma to prove Theorem 1 and Theorem 2.

Lemma 1. *Let G be a connected graph with $\Delta(G) \leqslant 3$ and $\delta(G) = 2$.*

(1) *If $|E(G)|$ is even, then G has an admissible partition of $E(G)$.*

(2) *If $|E(G)|$ is odd, then for every vertix x_0 of degree 2 there exists an admissible partition (F_1, F_2) such that* $\deg_{F_1}(x_0) = 2$.

Proof. We proceed by induction on $|E(G)|$. The lemma clearly holds when G is a cycle. Hence we may assume $|E(G)| \geqslant 4$ and G is not a cycle.

Case 1: *$|E(G)|$ is even.*

Since G is not a cycle, G has a vertex x of degree 3. Then G has a path $x = a_0, a_1, \ldots, a_l = y$ such that $x \neq y$, $\deg_G(x) = \deg_G(y) = 3$ and $\deg_G(a_i) = 2$ for $1 \leqslant i \leqslant l-1$ $(l \geqslant 1)$. Let

$$H = \begin{cases} G - \{a_1, \ldots, a_{l-1}\}, & \text{if } l \geqslant 2; \\ G - a_0a_1, & \text{if } l = 1. \end{cases}$$

Then $\delta(H) = 2$ and $\Delta(H) \leqslant 3$. We consider two subcases.

Subcase 1.1: *H is connected.* Suppose $|E(H)|$ is even. Then H has an admissible partition (F'_1, F'_2) by the induction hypothesis. Let $w(F'_1) = w(F'_2) = w$. Since $|E(H)| = |E(G)| - l$, l is even. Let $F_1 = F'_1 \cup \{a_0a_1, a_2a_3, \ldots, a_{l-2}a_{l-1}\}$ and $F_2 = F'_2 \cup \{a_1a_2, a_3a_4, \ldots, a_{l-1}a_l\}$. It is easy to see that (F_1, F_2) is an admissible partition with $w(F_1) = w(F_2) = w + l/2 - 1$.

Suppose $|E(H)|$ is odd. This implies that l is odd. By the induction hypothesis H has an admissible partition (F'_1, F'_2) such that $\deg_{F'_1}(a_0) = 2$. Let $F_1 = F'_1 \cup \{a_1a_2, a_3a_4, \ldots, a_{l-2}a_{l-1}\}$ and $F_2 = F'_2 \cup \{a_0a_1, a_2a_3, \ldots, a_{l-1}a_l\}$. Then (F_1, F_2) is an admissible partition of G satisfying $w(F_1) = w(F_2) = w(F'_1) + (l-1)/2$.

Subcase 1.2: *H is disconnected.* In this case H has exactly two components H_1, H_2. We may assume that $a_0 \in V(H_1)$ and $a_l \in V(H_2)$. First, suppose l is even. This implies $|E(H_1)| \equiv |E(H_2)| \pmod 2$. If both $|E(H_1)|$ and $|E(H_2)|$ are even, let $(F_{i,1}, F_{i,2})$ be the admissible partition of H_i $(i = 1, 2)$. Let

$$F_1 = F_{1,1} \cup F_{2,1} \cup \{a_0a_1, a_2a_3, \ldots, a_{l-2}a_{l-1}\}$$

and

$$F_2 = F_{1,2} \cup F_{2,2} \cup \{a_1a_2, a_3a_4, \ldots, a_{l-1}a_l\}.$$

Then (F_1, F_2) is an admissible partition of G. If both $|E(H_1)|$ and $|E(H_2)|$ are odd, let $(F_{1,1}, F_{1,2})$ be an admissible partition of H_1 such that $\deg_{F_{1,1}}(a_0) = 2$, and let $(F_{2,1}, F_{2,2})$ be an admissible partition of H_2 such that $\deg_{F_{2,1}}(a_l) = 2$. Let

$$F_1 = F_{1,1} \cup F_{2,2} \cup \{a_1a_2, a_3a_4, \ldots, a_{l-1}a_l\}$$

and

$$F_2 = F_{1,2} \cup F_{2,1} \cup \{a_0a_1, a_2a_3, \ldots, a_{l-2}a_{l-1}\}.$$

Then (F_1, F_2) is an admissible partition of G.

Next, suppose l is odd. We may assume $|E(H_1)|$ is odd and $|E(H_2)|$ is even. Let $(F_{1,1}, F_{1,2})$ be an admissible partition of H_1 such that $\deg_{F_{1,1}}(a_0) = 2$, and let $(F_{2,1}, F_{2,2})$ be an admissible partition of H_2. Let $F_1 = F_{1,1} \cup F_{2,1} \cup \{a_1a_2, a_3a_4, \ldots, a_{l-2}a_{l-1}\}$ and $F_2 = F_{1,2} \cup F_{2,2} \cup \{a_0a_1, a_2a_3, \ldots, a_{l-1}a_l\}$. Then (F_1, F_2) is an admissible partition of G.

Case 2: *$|E(G)|$ is odd.*

Since G is not a cycle, there exists a trail $T = a_0, a_1, \ldots, a_l$ containing x_0 such that $\deg_G(a_0) = \deg_G(a_l) = 3$ and $\deg_G(a_i) = 2$ $(1 \leq i \leq l-1, l \geq 2)$. If $a_0 \neq a_l$, then T is a path, otherwise T is a cycle. Suppose $x_0 = a_k$.

Subcase 2.1: *T is a path.* Let $H = G - \{a_1, \ldots, a_{l-1}\}$. We consider two subcases.

Subcase 2.1.1: *H is connected.* Note $|E(H)| = |E(G)| - l$. First, suppose l is odd. Then $|E(H)|$ is even. We may assume that k is even. By the induction hypothesis, H has an admissible partition (F_1', F_2'). Let

$$F_1 = F_1' \cup \left\{a_{2i-1}a_{2i}: 1 \leq i \leq \frac{k}{2}\right\} \cup \left\{a_{2i}a_{2i+1}: \frac{k}{2} \leq i \leq \frac{l-1}{2}\right\}$$

and

$$F_2 = F_2' \cup \left\{a_{2i}a_{2i+1}: 0 \leq i \leq \frac{k-2}{2}\right\} \cup \left\{a_{2i-1}a_{2i}: \frac{k+2}{2} \leq i \leq \frac{l-1}{2}\right\}.$$

Then (F_1, F_2) is an admissible partition of G such that $\deg_{F_1}(x_0) = 2$.

Next, suppose that both k and l are even. Then $|E(H)|$ is odd. By the induction hypothesis, H has an admissible partition (F_1', F_2') such that $\deg_{F_1'}(a_l) = 2$. Let

$$F_1 = F_1' \cup \left\{a_{2i-1}a_{2i}: 1 \leq i \leq \frac{k}{2}\right\} \cup \left\{a_{2i}a_{2i+1}: \frac{k}{2} \leq i \leq \frac{l-2}{2}\right\}$$

and

$$F_2 = F_2' \cup \left\{a_{2i}a_{2i+1}: 0 \leq i \leq \frac{k-2}{2}\right\} \cup \left\{a_{2i-1}a_{2i}: \frac{k+2}{2} \leq i \leq \frac{l}{2}\right\}.$$

Then (F_1, F_2) is an admissible partition of G such that $\deg_{F_1}(x_0) = 2$.

Finally, suppose l is even and k is odd. By the induction hypothesis H has an admissible partition (F_1', F_2') such that $\deg_{F_1'}(a_l) = 2$. Let

$$F_1 = F_2' \cup \left\{a_{2i}a_{2i+1}: 0 \leq i \leq \frac{k-1}{2}\right\} \cup \left\{a_{2i-1}a_{2i}: \frac{k+1}{2} \leq i \leq \frac{l}{2}\right\}$$

and

$$F_2 = F_1' \cup \left\{a_{2i-1}a_{2i}: 1 \leq i \leq \frac{k-1}{2}\right\} \cup \left\{a_{2i}a_{2i+1}: \frac{k+1}{2} \leq i \leq \frac{l-2}{2}\right\}.$$

Then (F_1, F_2) is an admissible partition of G such that $\deg_{F_1}(x_0) = 2$.

Subcase 2.1.2: *H is disconnected.* In this case, H has exactly two components H_1 and H_2. We may assume that H_1 contains a_0 and H_2 contains a_l.

First, suppose l is even. Then $|E(H)|$ is odd. We may assume $|E(H_1)|$ is odd and $|E(H_2)|$ is even. Let $(F'_{1,1}, F'_{1,2})$ be an admissible partition of H_1 such that $\deg_{F'_{1,1}}(a_0) = 2$, and let $(F'_{2,1}, F'_{2,2})$ be an admissible partition of H_2. If k is even, then let

$$F_1 = F'_{1,1} \cup F'_{2,1} \cup \left\{a_{2i-1}a_{2i}\colon 1 \leqslant i \leqslant \frac{k}{2}\right\} \cup \left\{a_{2i}a_{2i+1}\colon \frac{k}{2} \leqslant i \leqslant \frac{l-2}{2}\right\}$$

and

$$F_2 = F'_{1,2} \cup F'_{2,2} \cup \left\{a_{2i}a_{2i+1}\colon 0 \leqslant i \leqslant \frac{k-2}{2}\right\} \cup \left\{a_{2i-1}a_{2i}\colon \frac{k+2}{2} \leqslant i \leqslant \frac{l}{2}\right\}.$$

If k is odd, then let

$$F_1 = F'_{1,2} \cup F'_{2,1} \cup \left\{a_{2i}a_{2i+1}\colon 0 \leqslant i \leqslant \frac{k-1}{2}\right\} \cup \left\{a_{2i-1}a_{2i}\colon \frac{k+1}{2} \leqslant i \leqslant \frac{l}{2}\right\}$$

and

$$F_2 = F'_{1,1} \cup F'_{2,2} \cup \left\{a_{2i-1}a_{2i}\colon 1 \leqslant i \leqslant \frac{k-1}{2}\right\} \cup \left\{a_{2i}a_{2i+1}\colon \frac{k+1}{2} \leqslant i \leqslant \frac{l-2}{2}\right\}.$$

Then in either case (F_1, F_2) is an admissible partition such that $\deg_{F_1}(x_0) = 2$.

Next suppose that l is odd. Then we may assume k is even. In this case $|E(H)|$ is even. Therefore, both $|E(H_1)|$ and $|E(H_2)|$ are even, or both $|E(H_1)|$ and $|E(H_2)|$ are odd. First, suppose both $|E(H_1)|$ and $|E(H_2)|$ are even. Let $(F'_{i,1}, F'_{i,2})$ be an admissible partition of H_i $(i = 1, 2)$. Let

$$F_1 = F'_{1,1} \cup F'_{2,1} \cup \left\{a_{2i-1}a_{2i}\colon 1 \leqslant i \leqslant \frac{k}{2}\right\} \cup \left\{a_{2i}a_{2i+1}\colon \frac{k}{2} \leqslant i \leqslant \frac{l-1}{2}\right\}$$

and

$$F_2 = F'_{1,2} \cup F'_{2,2} \cup \left\{a_{2i}a_{2i+1}\colon 0 \leqslant i \leqslant \frac{k-2}{2}\right\} \cup \left\{a_{2i-1}a_{2i}\colon \frac{k+2}{2} \leqslant i \leqslant \frac{l-1}{2}\right\}.$$

Then (F_1, F_2) is an admissible partition such that $\deg_{F_1}(x_0) = 2$.

Next suppose both $|E(H_1)|$ and $|E(H_2)|$ are odd. Let $(F'_{1,1}, F'_{1,2})$ be an admissible partition of H_1 such that $\deg_{F'_{1,1}}(a_0) = 2$ and let $(F'_{2,1}, F'_{2,2})$ be an admissible partition of H_2 such that $\deg_{F'_{2,1}}(a_l) = 2$. Let

$$F_1 = F'_{1,1} \cup F'_{2,2} \cup \left\{a_{2i-1}a_{2i}\colon 1 \leqslant i \leqslant \frac{k}{2}\right\} \cup \left\{a_{2i}a_{2i+1}\colon \frac{k}{2} \leqslant i \leqslant \frac{l-1}{2}\right\}$$

and

$$F_2 = F'_{1,2} \cup F'_{2,1} \cup \left\{a_{2i}a_{2i+1}\colon 0 \leqslant i \leqslant \frac{k-2}{2}\right\} \cup \left\{a_{2i-1}a_{2i}\colon \frac{k+2}{2} \leqslant i \leqslant \frac{l-1}{2}\right\}.$$

Then (F_1, F_2) is an admissible partition of G such that $\deg_{F_1}(x_0) = 2$.

Subcase 2.2: *T is a cycle*. In this case, we extend the trail T into $T' = a_0, a_1, \ldots, a_l(=a_0), a_{l+1}, a_{l+2}, \ldots, a_m$ so that $\deg_G(a_i) = 2$ if $1 \leqslant i \leqslant l-1$ or $l+1 \leqslant i \leqslant m-1$ and $\deg_G(a_0) = \deg_G(a_m) = 3$. Obviously $a_m \neq a_0$. Let $x_0 = a_k$.

First, suppose k is even. Let

$$F_1'' = \left\{a_{2i-1}a_{2i}\colon 1 \leqslant i \leqslant \frac{k}{2}\right\} \cup \left\{a_{2i}a_{2i+1}\colon \frac{k}{2} \leqslant i \leqslant \left\lceil \frac{m}{2} \right\rceil - 1\right\},$$

$$F_2'' = \left\{a_{2i}a_{2i+1}\colon 0 \leqslant i \leqslant \frac{k}{2} - 1\right\} \cup \left\{a_{2i-1}a_{2i}\colon \frac{k}{2} + 1 \leqslant i \leqslant \left\lfloor \frac{m}{2} \right\rfloor\right\},$$

and $H' = G - \{a_1, \ldots, a_{m-1}\}$. Note $|E(H')| = |E(G)| - m$ and H' is connected. If m is odd, then $|E(H')|$ is even. By the induction hypothesis H' has an admissible partition (F_1', F_2'). Let $F_1 \cup F_1' \cup F_1''$ and $F_2 = F_2' \cup F_2''$. Then (F_1, F_2) is an admissible partition of G such that $\deg_{F_1}(x_0) = 2$.

If m is even, then $|E(H')|$ is odd. Let (F_1', F_2') be an admissible partition of G such that $\deg_{F_1'}(a_m) = 2$. Let $F_1 = F_1' \cup F_1''$, $F_2 = F_2' \cup F_2''$. Then (F_1, F_2) is an admissible partition of G such that $\deg_{F_1}(x_0) = 2$.

Next assume k is odd. Let

$$F_1'' = \left\{a_{2i}a_{2i+1}\colon 0 \leqslant i \leqslant \frac{k-1}{2}\right\} \cup \left\{a_{2i-1}a_{2i}\colon \frac{k+1}{2} \leqslant i \leqslant \left\lfloor \frac{m}{2} \right\rfloor\right\}$$

and

$$F_2'' = \left\{a_{2i-1}a_{2i}\colon 1 \leqslant i \leqslant \frac{k-1}{2}\right\} \cup \left\{a_{2i}a_{2i+1}\colon \frac{k+1}{2} \leqslant i \leqslant \left\lceil \frac{m}{2} \right\rceil - 1\right\}.$$

If m is odd, then $|E(H')|$ is even. Let (F_1', F_2') be an admissible partition of H'. Then $(F_1' \cup F_1'', F_2' \cup F_2'')$ is an admissible partition of G such that $\deg_{F_1}(x_0) = 2$. If m is even, then $|E(H')|$ is odd. Let (F_1', F_2') be an admissible partition of H' such that $\deg_{F_1'}(a_m) = 2$. Then $(F_2' \cup F_1'', F_1' \cup F_2'')$ is an admissible partition of G such that $\deg_{F_1}(x_0) = 2$. □

The assumption that G is connected in Lemma 1 cannot be removed. A disjoint union of two or more odd cycles has no admissible partition.

We now establish Theorem 1 by proving the following stronger result.

Theorem 3. *Let G be a 3-regular graph.*

(1) *If $|E(G)|$ is even, then there exists a partition (F_1, F_2) of $E(G)$ such that $|F_1| = |F_2|$ and both F_1 and F_2 are linear forests.*

(2) *If $|E(G)|$ is odd, then there exists a partition (F_1, F_2) such that $|F_1| = |F_2| + 1$ and both F_1 and F_2 are linear forests.*

Proof. We proceed by induction on $|E(G)|$. If $|E(G)| = 6$, then G is a complete graph of order 4 and the theorem follows. So suppose $|E(G)| \geqslant 9$.

First, suppose G is disconnected. Let G_1 be one of the components of G. Let $G_2 = G - G_1$. If both $|E(G)|$ and $|E(G_1)|$ are even, the theorem follows easily by the induction hypothesis. If $|E(G)|$ is even and $|E(G_1)|$ is odd, then $|E(G_2)|$ is odd. Then G_i has a partition $(F_{i,1}, F_{i,2})$ such that $|F_{i,1}| = |F_{i,2}| + 1$ and both $F_{i,1}$ and $F_{i,2}$ are linear forests $(i = 1, 2)$. Then $(F_{1,1} \cup F_{2,2}, F_{1,2} \cup F_{2,1})$ is a partition which satisfies (1). We can prove (2) similarly.

Now we may assume that G is connected. Take $x \in V(G)$ so that $G - x$ is connected. Let $N_G(x) = \{x_1, x_2, x_3\}$.

First, we prove (1). The graph G' satisfies the assumption of Lemma 1, and $|E(G')|$ is odd. Let (F_1', F_2') be an admissible partition of $E(G')$ such that $\deg_{F_1'}(x_1) = 2$. Let $F_1 = F_1' \cup \{xx_3\}$ and $F_2 = F_2' \cup \{xx_1, xx_2\}$. Then regardless of adjacency among $\{x_1, x_2, x_3\}$, both F_1 and F_2 are linear forests and $w(F_1) = w(F_2) = w(F_1')$. Since both F_1 and F_2 are spanning subgraphs of G, we have $|F_1| = |F_2|$.

Next, we prove (2). Let (F_1', F_2') be an admissible partition of $E(G')$. Since each x_i is an endvertex of F_1' $(1 \leqslant i \leqslant 3)$, all of $\{x_1, x_2, x_3\}$ cannot be contained in the same path of F_1'. We may assume x_2 and x_3 are contained in different paths in F_1'. Let $F_1 = F_1' \cup \{xx_2, xx_3\}$ and $F_2 = F_2' \cup \{xx_1\}$. Then both F_1 and F_2 are linear forests of G, $w(F_1) = w(F_1') - 1$, $w(F_2) = w(F_2')$ and hence $w(F_1) = w(F_2) - 1$. Again, since both F_1 and F_2 are spanning, $|F_1| = |F_2| + 1$. Therefore, the theorem holds. □

Next, we consider 4-regular graphs. We use the following lemma, which is an immediate corollary of Lemma 1.

Lemma 2. *Let G be a (possibly disconnected) graph with $\Delta(G) \leqslant 3$ and $\delta(G) = 2$. If $|E(G)|$ is even, then G has a partition (F_1, F_2) of $E(G)$ such that both F_1 and F_2 are linear forests and $|F_1| = |F_2|$.*

Now we prove the following theorem, which is stronger than Theorem 2.

Theorem 4. *Let G be a 4-regular graph. Then G has a partition (F_1, F_2, F_3) of $E(G)$ such that every F_i is a linear forest $(i = 1, 2, 3)$ and $||F_i| - |F_j|| \leqslant 1$ for all i, j, $1 \leqslant i, j \leqslant 3$.*

Proof. Note that $|E(G)| = 2|V(G)|$. We first consider the case $|E(G)| \equiv 0 \pmod 3$. Then $|V(G)| \equiv 0 \pmod 3$, say $|V(G)| = 3k$. Then $|E(G)| = 6k$.

First, we claim that G has a spanning linear forest F of G such that $|F| \geqslant 2k$. Since G is 4-regular, G has a 2-factor F_0. Trivially, $|F_0| = |V(G)| = 3k$. Let the components of F_0 be $C_1, \ldots, C_r$. Then $|V(G)| = \sum_{i=1}^{r} |V(C_i)| \geqslant 3r$, and hence $r \leqslant k$. Choose one edge e_i from C_i $(1 \leqslant i \leqslant r)$ and let $F = F_0 - \{e_1, e_2, \ldots, e_r\}$. Then F is a spanning linear forest and $|F| = |F_0| - r = 3k - r \geqslant 2k$.

Now we take a spanning linear forest F_1 of G with $|F_1| \geqslant 2k$ so that $|F_1|$ is minimum. We claim $|F_1| = 2k$. Assume, to the contrary, that $|F_1| \geqslant 2k + 1$. If F_1 has a path P of length at least 3 as its component, say $P = a_0, a_1, \ldots, a_l$ $(l \geqslant 3)$, then $F_1 - a_1a_2$ is also a spanning linear forest. This contradicts the minimality of F_1. So every component of F_1 is isomorphic to P_2 or P_3. So we can write $F_1 = xP_2 \cup yP_3$ for some nonnegative integer x and y. Since F_1 is spanning, $3k = |V(G)| = 2x + 3y$. Then

$$|F_1| = x + 2y = \tfrac{2}{3}(\tfrac{3}{2}x + 3y) = \tfrac{2}{3}(3k - \tfrac{1}{2}x) = 2k - \tfrac{1}{3}x \leqslant 2k.$$

This is a contradiction, and the claim follows.

Now we consider $G' = G - F_1$. Then $\Delta(G') \leqslant 3$ and $|E(G')| = 4k$. Since the number of edges of a 1-factor of G is $\frac{3}{2}k$, F_1 is not a 1-factor. Hence $\delta(G') = 2$. Then by Lemma 2, G' has a partition (F_2, F_3) such that both F_2 and F_3 are linear forests and $|F_2| = |F_3| = 2k$. Then (F_1, F_2, F_3) satisfies the desired condition.

If $|E(G)| \equiv 1 \pmod 3$, then we can write $|V(G)| = 3k - 1$ and $|E(G)| = 6k - 2$, for some integer k. Then by the argument similar to those in the previous paragraphs, it is proved that G has a spanning linear forest F with $|F| = 2k$. Then by Lemma 2 we find a desired partition. The proof is similar for the case $|E(G)| \equiv 2 \pmod 3$. □

Finally, we give an alternative proof to Theorem 3, which uses Theorem A.

Alternative Proof of Theorem 3. We give the proof of (1). The proof of (2) is similar and hence is omitted. By Theorem A, G has a partition (F_1, F_2) of $E(G)$ such that both F_1 and F_2 are linear forests. We may assume that $|F_1| \geqslant |F_2|$. We take this partition (F_1, F_2) so that $|F_1| - |F_2|$ is minimum. We claim that $|F_1| = |F_2|$.

Assume, to the contrary, that $|F_1| > |F_2|$. Then $|F_1| \geqslant |F_2| + 2$ since $|E(G)|$ is even. If F_1 has a path P of length at least 4 as its component, say $P = a_0, a_1, a_2, \ldots, a_l$ $(l \geqslant 4)$, then consider $(F_1 - \{a_1a_2\}, F_2 \cup \{a_1a_2\})$. Trivially, $F_1 - \{a_1a_2\}$ is a linear forest. Hence $F_2 \cup \{a_1a_2\}$ is not a linear forest because of the minimality of $|F_1| - |F_2|$. Since $\deg_{F_2}(a_1) = \deg_{F_2}(a_2) = 1$, this implies that a_1 and a_2 are endvertices of the same path of F_2. Then $F_1 - \{a_2a_3\}$ and $F_2 \cup \{a_2a_3\}$ are linear forests, which contradicts the minimality of $|F_1| - |F_2|$. Therefore, all components of F_1 are paths of length at most 3. Hence $|F_1| \leqslant 3w(F_1)$. Since F_1 is a spanning (linear) forest of G, $w(F_1) = |V(G)| - |F_1|$. Hence

$$|F_1| \leqslant 3w(F_1) = 3|V(G)| - 3|F_1|.$$

However, since G is 3-regular, $2\,|E(G)| = 3\,|V(G)|$, and $|F_1| \leqslant 2\,|E(G)| - 3\,|F_1|$, or $|F_1| \leqslant \frac{1}{2}\,|E(G)|$. This is a contradiction since $|F_2| < |F_1|$ and $|F_1| + |F_2| = |E(G)|$. Therefore, the theorem follows. □

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