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Nonlinear Elliptic Eigenvalue Problems with Discontinuities

Shouchuan Hu

Department of Mathematics, Southwest Missouri State University, Springfield, Missouri 65804
E-mail: shh209f@mail.smsu.edu

and

N. C. Kourogenis* and N. S. Papageorgiou

Department of Mathematics, National Technical University, Athens 15780, Greece
E-mail: npapg@math.ntua.gr

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In this paper we study the existence of solution for two different eigenvalue problems. The first is nonlinear and the second is semilinear. Our approach is based on results from the nonsmooth critical point theory. In the first theorem we prove the existence of at least two nontrivial solutions when λ is in a half-axis. In the second theorem (based on a nonsmooth variant of the generalized mountain pass theorem), we prove the existence of at least one nontrivial solution for every $\lambda \in \mathbb{R}$.

Key Words: locally Lipschitz functions; subdifferential; critical point; nonsmooth Palais–Smale condition; Mountain Pass Theorem; Rayleigh quotient; maximal monotone operators; compact embedding

1. INTRODUCTION

In this paper we study nonlinear and semilinear eigenvalue problems with discontinuities. Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a C^1 boundary Γ and f: $Z \times \mathbb{R} \Rightarrow \mathbb{R}$ be such that $f(z, \cdot)$ is locally bounded for every $z \in Z$. Since f need not be continuous in its second variable, we introduce

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the following auxiliary functions:

$$f_0(z,x) = \lim_{\epsilon \downarrow 0} \operatorname*{ess\,inf}_{|y-x|<\epsilon} f(z,y)$$

and

$$f_1(z,x) = \lim_{\epsilon \downarrow 0} \operatorname*{ess\,sup}_{|y-x|<\epsilon} f(z,y).$$

Note that if the one-sided limits $f(z,x^-)$ and $f(z,x^+)$ exist at (z,x), then

$$f_0(z,x) = \min\{f(z,x^-), f(z,x^+)\}$$

and

$$f_1(z,x) = \max\{f(z,x^-), f(z,x^+\}.$$

With the interval $[f_0(z,x), f_1(z,x)]$ filling in the gaps at the discontinuity point of $f(z,\cdot)$, we consider the following multivalued elliptic eigenvalue problems:

$$\begin{cases} -\operatorname{div}\left(\|Dx(x)\|^{p-2} Dx(z)\right) \in \lambda[f_0(z,x(x)), f_1(z,x(z))] \\ x|_\Gamma = 0, \qquad 2 \le p \end{cases} \tag{1}$$

and

$$\begin{cases} -\sum_{i,j=1}^{N} D_j\left(a_{ij}(z) D_i x(z)\right) - \lambda x(\lambda) \in [f_0(z,x(x)), f_1(z,x(z))] \\ x|_\Gamma = 0 \end{cases}. \tag{2}$$

Here $D = \operatorname{grad}$ and $D_j = \partial/\partial z_j$.

Problem (1) was studied for $p = 2$ (semilinear case) by Browder [3], Pohozaev [12] and Chang [4, 5]. In [3, 12], $f(z,\cdot)$ is assumed to be continuous, while in [4] $f(z,\cdot)$ is required to be monotone. Chang studied the problem using a nonsmooth Lagrange multiplier approach. In all these works, existence of only one nontrivial solution (eigenvalue) is proved, while in the present paper by using a variational approach we prove the existence of two nontrivial solutions. Related results for the semilinear problem can be found in Massabo [10] and Massabo–Stuart [11].

Problem (2) was studied by Rabinowitz [13], with f continuous. Our variational approach here is based on a nonsmooth version of the Palais–Smale condition introduced by Chang [5].

2. PRELIMINARIES

As mentioned earlier, our approach is variational and uses the nonsmooth critical point theory for locally Lipschitz functionals due to Chang [5]. For easy references, we recall here the main definition and results of this theory.

Let X be a Banach space and X^* its topological dual. Throughout this paper we assume that $f\colon X \to \mathbb{R}$ is locally Lipschitz, i.e., for each $x \in X$ there exists a neighborhood $\mathscr{U}$ of x and a constant $k = k(\mathscr{U})$ such that $|f(y) - f(z)| \le k\|y - z\|$ for all $y, z \in \mathscr{U}$. It is well known that a proper, convex, and lower semicontinuous function $g\colon X \to \mathbb{R} \cup \{\infty\}$ is locally Lipschitz in the interior of its effective domain: $\operatorname{dom} g = \{x \in X\colon g(x) < \infty\}$. Given $y \in X$, we can define a "generalized directional derivative" of f at x, in the direction of y, by

$$f^0(x; y) = \limsup_{h \to 0,\ \lambda \downarrow 0} \frac{f(x + h + \lambda y) - f(x + h)}{\lambda}.$$

Since $f^0(x; \cdot\,)$ is sublinear and continuous, it is the support function of a convex set $\partial f(x)$, i.e.,

$$\partial f(x) = \left\{x^* \in X^*\colon (x^*, y) \le f^0(x; y) \text{ for all } y \in X\right\}.$$

We call $\partial f(x)$ the *subdifferential* of f at x. Evidently, $\partial f(x)$ is nonempty, convex, and w^*-compact at every $x \in X$. If f is also convex, then $\partial f(x)$ coincides with the subdifferential in the sense of convex analysis, and $f^0(x; \cdot\,) = f'(x; \cdot\,)$, where $f'(x; \cdot\,)$ is the usual directional derivative of convex analysis, i.e., $f'(x; y) = \lim_{\lambda \downarrow 0}[(f(x + \lambda y) - f(x))/\lambda]$. The subdifferential operator ∂f has the following properties:

(a) $\partial(f_1 + f_2)(x) \subseteq \partial f_1(x) + \partial f_2(x)$ for all locally Lipschitz $f_1, f_2\colon X \to \mathbb{R}$.

(b) $\partial(\lambda f)(x) = \lambda\, \partial f(x)$ for all $\lambda \in \mathbb{R}$

(c) $x \to \partial f(x)$ is usc from X with the norm topology into X^* with the weak* topology.

(d) If for each y in some neighborhood of x, f admits a Gateaux derivative $Df(y)$ and $Df\colon X \to X^*$ is continuous, then $\partial f(x) = \{Df(x)\}$.

(e) $0 \in \partial f(x)$ if x is a local minimum of f.

Remark 2.1. An extension of a well-known result of Rademacher (see Christensen [6]) says that "If X, Y are separable Banach spaces with Y reflexive, and $f\colon X \to Y$ is locally Lipschitz, then f is Gateaux differentiable on a subset D_f with $X \setminus D_f$ being Haar-null in X."

In the variational approach one will look for the critical points of an appropriately defined "energy functional." Due to the presence of discontinuity, the energy functional is not C^1 but only locally Lipschitz. So, the notion of critical points is defined as follows.

DEFINITION 2.2. A point $x_0 \in X$ is said to be a critical point of f if $0 \in \partial f(x_0)$.

A basic tool in the variational method is a compactness type condition on the energy functional, known as the *Palais–Smale* condition, and hereafter denoted by *PS condition*. In the present situation with a locally Lipschitz energy functional, the PS condition takes the following form:

DEFINITION 2.3. A locally Lipschitz functional f: $X \to \mathbb{R}$ is said to satisfy the PS condition if any sequence $\{x_n\}$, along which $\{f(x_n)\}$ is bounded and $m(x_n) = \min[\|x^*\|_{X^*} : x^* \in \partial f(x_n)] \to 0$ as $n \to \infty$, possesses a convergent subsequence.

Remark 2.4. When X is reflexive, another equivalent definition of the nonsmooth PS condition, suggested by the variational principle of Ekeland, was introduced by Costa–Goncalves [7].

Using the compactness notion, Chang [5] proved the following nonsmooth version of the well-known "Mountain Pass Theorem," originally due to Ambrosetti–Rabinowitz [1]. Here $B_\rho = [x \in X : \|x\| < \rho\}$ and $\partial B_\rho = \{x \in X : \|x\| = \rho\}$.

THEOREM 2.5. *Assume that X is a reflexive Banach space and f: $X \to \mathbb{R}$ is a locally Lipschitz function which satisfies the PS condition and such that*

(i) $f(0) = 0$, $f|_{\partial B_\rho} \geq \xi$ *for some* $\rho, \xi > 0$, *and*

(ii) $f(y) \leq 0$ *for some* $y \in X \setminus B_\rho$.

Then f possesses a critical point x with $f(x) \geq \xi$.

In the study of Problem (1) we will use the first eigenvalue of the following quasilinear eigenvalue problem:

$$\begin{cases} -\operatorname{div}\left(\|Dx(x)\|^{p-2} Dx(z)\right) = \lambda |x(z)|^{p-2} x(z) & \text{in } Z \\ x|_\Gamma = 0 \end{cases} \tag{3}$$

By Lindqvist [9], the first eigenvalue λ_1 of (3) exists, and is positive, simple, and isolated. A corresponding eigenfunction $u_1 \in W_0^{1,p}(Z) \cap L^\infty(Z)$ can be chosen so that $u_1(z) > 0$ a.e. on Z. Moreover, λ_1 is the minimum of the Rayleigh quotient, i.e.,

$$\lambda_1 = \min\left[\frac{\|Dx\|_p^p}{\|x\|_p^p} : x \in W_0^{1,p}(Z)\right].$$

3. EXISTENCE THEOREM FOR PROBLEM (1)

For our study of the nonlinear eigenvalue problem (1), we will need the following assumptions for f. Let $F(z, x) = \int_0^x f(z, r)\, dr$ denote the potential of $f(z, \cdot)$.

H(f)$_1$: $f\colon Z \times \mathbb{R} \to \mathbb{R}$ is a Borel measurable function such that

(i) f_0, f_1 are both finite, and N-measurable (i.e., for every measurable $x: \to Z \to \mathbb{R}$, $z \to f_0(z, x(z)), f_1(z, x(z))$ is measurable, a property called the "superpositional measurability").

(ii) With some $\alpha \in L^\infty(Z)$, $c \geq 0$ and $0 < \theta < p - 1$, $|f(z, r)| \leq \alpha(z) + c|r|^\theta$ for a.e. $z \in \mathbb{Z}$ and all $r \in \mathbb{R}$.

(iii) $\lim_{r \to 0} |f(z, r)|/|r|^{p-1} < \lambda_1$ uniformly for a.e. $z \in Z$.

(iv) $F(z, r) > 0$ for a.e. $z \in Z$ and some $r \in \mathbb{R}$.

We define $J, V\colon W_0^{1,p}(Z) \to \mathbb{R}$ by

$$J(x) = \frac{1}{p}\|Dx\|_p^p \quad \text{and} \quad V(x) = \int_Z F(z, x(z))\, dz$$

It follows from [5] that V is locally Lipschitz. It is also clear that J is continuous and convex, and hence locally Lipschitz.

We say that $x \in W_0^{1,p}(Z)$ is a *solution* of Problem (1) if there exists a function $g \in L^q(Z)$ with $f_0(z, x(z)) \leq g(z) \leq f_1(z, x(z))$ a.e. on Z such that

$$-\operatorname{div}\left(\| Dx(x)\|^{p-2} Dx(z)\right) = g(z) \quad \text{a.e. on } Z.$$

Such a solution is usually referred to as a *strong solution*.

THEOREM 3.1. *If* **H(f)**$_1$ *holds, then there exists* $\lambda_0 > 0$ *such that* (1) *has at least two nontrivial solutions for every* $\lambda > \lambda_0$.

Proof. Define $A\colon W_0^{1,p}(Z) \to W^{-1,q}(Z)$ by

$$\langle A(x), y\rangle = \int_Z \| Dx(z)\|^{p-2}(Dx(z), Dy(z))_{\mathbb{R}^N}\, dz$$

for all $x, y \in W_0^{1,p}(Z)$. Here $\langle \cdot, \cdot \rangle$ is the duality bracket for the pair $(W_0^{1,p}(Z), W^{-1,q}(Z))$.

Claim 1. A is monotone and demicontinuous, hence maximal monotone. To prove this claim, for $x, y \in W_0^{1,p}(Z)$ we have

$$\begin{aligned}
&\langle A(x) - A(y), x - y\rangle \\
&= \int_Z \|Dx(z)\|^{p-2}(Dx(z), D(x-y)(z))_{\mathbb{R}^N}\, dz \\
&\qquad - \int_Z \|Dy(z)\|^{p-2}(Dy(z), D(x-y)(z))_{\mathbb{R}^N}\, dz \\
&\geq \int_Z \|Dx(z)\|^p\, dz - \int_Z \|Dx(z)\|^{p-1}\|Dy(z)\|\, dz \\
&\qquad - \int_Z \|Dx(z)\|\,\|Dy(z)\|^{p-1}\, dz + \int_Z \|Dy(z)\|^p\, dz \\
&\geq \|Dx\|_p^p + \|Dy\|_p^p - \|Dx\|_p^{p-1}\|Dy\|_p - \|Dy\|_p^{p-1}\|Dx\|_p \\
&\qquad \text{by Hölder's inequality} \\
&= \left(\|Dx\|_p^{p-1} - \|Dy\|_p^{p-1}\right)\left(\|Dx\|_p - \|Dy\|_p\right).
\end{aligned}$$

From this it follows that A is monotone and in fact, strictly monotone.

Next we check the demicontinuity of A. To this end, let $x_n \to x$ in $W_0^{1,p}(Z)$. Then for every $y \in W_0^{1,p}(Z)$ we have

$$\begin{aligned}
|\langle A(x_n) - A(x), y\rangle| = \Big| \int_Z \Big(&\|Dx_n(z)\|^{p-2}(Dx_n(z), Dy(z))_{\mathbb{R}^N} \\
&- \|Dx(z)\|^{p-2}(Dx(z), Dy(z))_{\mathbb{R}^N}\Big)\, dz \Big|
\end{aligned}$$

Since $x_n \to x$ in $W_0^{1,p}(Z)$, $Dx_n \to Dx$ in $L^q(Z)$ and we may assume that $DX_n(z) \to Dx(z)$ a.e. on Z. Using the generalized Lebesgue dominated convergence theorem we have, as $n \to \infty$,

$$\begin{aligned}
&\int_Z \|Dx_n(z)\|^{p-2}(Dx_n(z), Dy(z))_{\mathbb{R}^N}\, dz \\
&\qquad \to \int_Z \|Dx(z)\|^{p-2}(Dx(z), Dy(z))_{\mathbb{R}^N}\, dz,
\end{aligned}$$

which implies that

$$|\langle A(x_n) - A(x), y\rangle| \to 0.$$

As $y \in W_0^{1,p}(Z)$ was arbitrary, it follows that $A(x_n) \xrightarrow{w} A(x)$ in $W^{-1,q}(Z)$, and so A is demicontinuous (actually it is clear that from this argument that A is continuous). Finally, recall that a monotone demicontinuous and everywhere defined operator is maximal monotone (see, e.g., Zeidler [15]). This proves Claim 1.

Now we define $R_\lambda\colon W_0^{1,p}(Z) \to \mathbb{R}$ by

$$R_\lambda(x) = J(x) - \lambda V(x) = \frac{1}{p}\|Dx\|_p^p - \lambda\int_Z F(z, x(z))\,dz.$$

Then R_λ is locally Lipschitz, and therefore

$$\partial R_\lambda(x) \subseteq \partial J(x) - \lambda\,\partial V(x) = A(x) - \lambda\,\partial V(x). \tag{4}$$

Claim 2. R_λ satisfies the nonsmooth PS condition. To this end, let $\{x_n\} \subseteq W_0^{1,p}(Z)$ be any sequence such that, with some $M > 0$ and $m(x) = \inf\{\|x^*\|_*\colon x^* \in \partial R_\lambda(x)]$,

$$|R_\lambda(x_n)| \le M \quad \text{for all } n \ge 1, \qquad \text{and} \quad m(x_n) \to 0 \quad \text{as } n \to \infty.$$

Here $\|\cdot\|$ (resp., $\|\cdot\|$ resp., $\|\cdot\|_*$) denotes the norm of $W_0^{1,p}(Z)$ (resp., of $W^{-1,q}(Z)$). Recall that $\partial R_\lambda(x_n)$ is nonempty, w-compact, and convex. Since $\|\cdot\|_*$ is w^*-lower semicontinuous, we can find $x_n^* \in \partial R_\lambda(x_n)$ such that $m(x_n) = \|x_n^*\|_*$. By (4), $x_n^* = A(x_n) - \lambda g_n$ with some $g_n \in \partial V(x_n)$. For every $n \ge 1$, we have from $\mathbf{H(f)}_1\mathbf{(ii)}$

$$M \ge R_\lambda(x_n) = \frac{1}{p}\|Dx_n\|_p^p - \lambda\int_Z F(z, x_n(z))\,dz$$
$$\ge \frac{1}{p}\|Dx_n\|_p^p - \lambda\|\alpha\|_q\|x_n\|_p - \lambda c_1\|x_n\|_p^{\theta+1}.$$

Let $\beta = p/(\theta+1) > 1$ and β' its conjugate exponent. By the Young's inequality with any $\epsilon > 0$, we have

$$\lambda c_1\|x_n\|_p^{\theta+1} \le |\lambda|c_1\|x_n\|_p^{\theta+1} \le \frac{(|\lambda|c_1)^{\beta'}}{\epsilon^{\beta'}\beta'} + \frac{\epsilon^\beta}{\beta}\|x_n\|_p^p.$$

Therefore,

$$R_\lambda(x_n) \ge \frac{1}{p}\|Dx_n\|_p^p - \lambda\|\alpha\|_q\|x_n\|_p - \frac{(|\lambda|c_1)^{\beta'}}{\epsilon^{\beta'}\beta'} - \frac{\epsilon^\beta}{\beta}\|x_n\|_p^p.$$

Now since $\lambda_1\|x_n\|_p^p \le \|Dx_n\|_p^p$ for all $n \ge 1$, we get

$$R_\lambda(x_n) \ge \left(\frac{1}{p} - \frac{\epsilon^\beta}{\lambda_1 \beta}\right)\|Dx_n\|_p^p - \lambda\|\alpha\|_q\|x_n\|_p - \frac{\left(|\lambda|c_1\right)^{\beta'}}{\epsilon^{\beta'}\beta'}. \qquad (5)$$

Choose $\epsilon > 0$ so that $1/p > \epsilon^\beta/(\lambda_1 \beta)$. By (5), $\{x_n\}$ is a bounded sequence in $W_0^{1,p}(Z)$. Hence we may assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$ and hence $x_n \to x$ in $L^p(Z)$ (recall that $W_0^{1,p}(Z)$ is embedded compactly in $L^p(Z)$). We now show that $f_0(z, x_n(z)) \le g_n(z) \le f_1(z, x_n(z))$ a.e. on Z for all $n \ge 1$. So, let $\hat{V}: L^p(Z) \to \mathbb{R}$ be defined by

$$\hat{V}(x) = \int_Z F(z, x(z))\, dz.$$

Evidently, $\hat{V}|_{W_0^{1,p}(Z)} = V$. From Chang [5] it follows that $\hat{V}$ is locally Lipschitz. By Theorem 2.2 of [5], we see that $\partial V(x) \subseteq \partial\hat{V}(x) \subseteq L^q(Z)$ for all $x \in W_0^{1,p}(Z)$. By the definition of subdifferentials we have

$$\partial\hat{V}(x) = \left\{v \in L^q(Z) : \int_Z v(z)u(z)\, dz \le \hat{V}^0(x; u) \text{ for all } u \in L^p(Z)\right\}.$$

Making a substitution $r(\eta) = x(z) + h(z) + \eta\lambda u(z)$ in

$$\begin{aligned}\hat{V}^0(x; u) &= \limsup_{h \to 0,\ \lambda \downarrow 0} \frac{1}{\lambda}\left[\hat{V}(x + h + \lambda u) - \hat{V}(x + h)\right]\\ &= \limsup_{h \to 0,\ \lambda \downarrow 0} \frac{1}{\lambda}\int_Z \int_{(x+h)(z)}^{(x+h+\lambda u)(z)} f(z, r)\, dr\, dz,\end{aligned}$$

we obtain that

$$\begin{aligned}\hat{V}^0(x; u) &= \limsup_{h \to 0,\ \lambda \downarrow 0} \frac{1}{\lambda}\int_Z \int_0^1 f(z, x(z) + h(z) + \eta\lambda u(z))\\ &\qquad \times \lambda u(z)\, d\eta\, dz\\ &\le \int_Z \limsup_{h \to 0,\ \lambda \downarrow 0} \int_0^1 f(z, x(z) + h(z) + \eta\lambda u(z))u(z)\, d\eta\, dz\\ &\qquad \text{(Fatou's lemma)}\\ &\le \int_{\{u>0\}} f_1(z, x(z))u(z)\, dz + \int_{\{u<0\}} f_0(z, x(z))u(z)\, dz.\end{aligned}$$

Thus, for $v \in \partial \hat{V}(x)$ and all $y \in L^p(Z)$ we get

$$\int_Z v(z)y(z)\,dz \le \int_{\{y>0\}} f_1(z, x(z))y(z)\,dz + \int_{\{y<0\}} f_0(z, x(z))y(z)\,dz.$$

Therefore, $v(z) \in [f_0(z, x(z)), f_1(z, x(z))]$ a.e. on Z, which yields

$$f_0(z, x_n(z)) \le g_n(z) \le f_1(z, x_n(z)) \quad \text{a.e. on } Z \quad \text{for all } n \ge 1.$$

By **H(f)$_1$(ii)**, $\{g_n\}$ is a bounded sequence in $L^q(Z)$ and so, we may assume that $g_n \xrightarrow{w} g$ in $L^q(Z)$. Now

$$\limsup\left[\langle A(x_n), x_n - x\rangle - \lambda\langle g_n, x_n - x\rangle\right] = \limsup\langle x_n^*, x_n - x\rangle,$$

where $(\cdot,\cdot)_{pq}$ denotes the duality bracket for the pair $(L^p(Z), L^q(Z))$

$$\begin{aligned}\limsup\langle A(x_n), x_n - x\rangle - \limsup \lambda(g_n, x_n - x)_{pq}\\ \le \limsup\|x_n^*\|_*\|x_n - x\| = 0.\end{aligned}$$

Consequently, $\limsup\langle A(x_n), x_n - x\rangle \le 0$.

Note that $\{A(x_n)\} \subseteq W^{-1,q}(Z)$ is bounded and so we may assume that $A(x_n) \xrightarrow{w} W$ in $W^{-1,q}(Z)$. Since A is maximal monotone, it has property (M) (see Zeidler [15]). Therefore, $w = A(x)$ and $\langle A(x_n), x_n\rangle \to \langle A(x), x\rangle$, and thus $\|Dx_n\|_p \to \|Dx\|_p$. As $Dx_n \xrightarrow{w} Dx$ in the uniformly convex space $L^p(Z, \mathbb{R}^N)$, we have $Dx_n \to Dx$ in $L^p(Z, \mathbb{R}^N)$. Hence $x_n \to x$ in $W_0^{1,p}(Z)$, and this proves Claim 2.

Note by (5) that R_λ is bounded below, which together with Claim 2 allows the application of theorem 3.5 of [5] to show that

$$\xi_\lambda = \inf\left[R_\lambda(x) : x \in W_0^{1,p}(Z)\right]$$

is a critical value of R_λ. So there exists $x_\lambda \in W_0^{1,p}(Z)$ such that $0 \in \partial R_\lambda(x_\lambda)$. Thus $A(x_\lambda) \in \lambda\,\partial V_\lambda(x_\lambda)$, and consequently $\langle A(x_\lambda) - \lambda g, y\rangle = 0$ for all $y \in W_0^{1,p}(Z)$ and some $g \in \partial V_\lambda(x_\lambda)$. Therefore, $A(x_\lambda) = \lambda g$ with $g \in L^q(Z)$. Using a generalized Green's formula due to Kenmochi [8, Prop. 1.4], we have

$$\begin{cases} -\operatorname{div}\left(\|Dx_\lambda(z)\|^{p-2} Dx_\lambda(z)\right) = g(z) & \text{a.e. on } Z \\ x|_\Gamma = 0 \end{cases}$$

with $g \in L^q(Z)$ satisfying $f_0(z, x_\lambda(z))) \le g(z) \le f_1(z, x_\lambda(z))$ a.e. on Z. Therefore, x_λ is a solution of Problem (1). We now show that $x_\lambda \neq 0$. By **H(f)**$_1$**(iv)** and the fact that $W_0^{1,p}(Z)$ is dense in $L^q(Z)$, there exists some $x \in W_0^{1,p}(Z)$ such that $V(x) > 0$. So we can find $\lambda_0 > 0$ such that $R_\lambda(x) < 0$ for all $\lambda > \lambda_0$. Hence, $\xi_\lambda < 0 = R_\lambda(0)$, which implies that $x_\lambda \neq 0$.

Next we establish the existence of a second nontrivial solution $y_\lambda \in W_0^{1,p}(Z)$. Due to **H(f)**$_1$**(iii)**, there exist $\mu < \lambda_1$ and $\delta > 0$ such that for almost all $z \in Z$ we have $|f(z,r)| \le \mu |r|^{p-1}$ for all $|r| \le \delta$. Hence, $|F(z,r)| \le (\mu/p)|r|^p$ for all $|r| \le \delta$. From **H(f)**$_1$**(ii)**, with $\alpha_1(\cdot) = \alpha(\cdot) + c/p \in L^\infty(Z)$, we have

$$|F(z,r)| \le \left(\alpha(z) + \frac{c}{p}\right)|r| \le \alpha_1(z)$$

for all $\delta \le |r| \le 1$ and a.e. $z \in Z$ and with $p < p_1 \le p^* = Np/(N-p)$

$$|F(z,r)| \le \alpha_1(z)|r|^{p_1} \quad \text{for all } |r| \ge 1.$$

Let $c_1 > 0$ be large enough so that $\alpha_1(z) \le c_1 |r|^{p_1}$ for a.e. $z \in Z$ and all $\delta \le |r| \le 1$. Thus, with $c_2 = c_1 + c/p$

$$|F(z,r)| \le \frac{\mu}{p}|r|^p + c_2|r|^{p_1} \quad \text{a.e. on } Z.$$

Therefore,

$$R_\lambda(x) \ge \frac{1}{p}\left(1 - \frac{\mu}{\lambda_1}\right)\|Dx\|_p^p - \frac{c_2}{\lambda^{p_1/p}}\|Dx\|_p^{p_1} \tag{6}$$

Since $\mu < \lambda_1$, $p < p_1$ and $\|Dx\|_p$ is an equivalent norm on $W_0^{1,p}(Z)$, by (6) we see that there exists $\rho > 0$ small enough such that $R_\lambda(x) \ge \beta > 0$ for some β and all x satisfying $\|Dx\|_p = \rho$. This allows the application of Theorem 2.5 to obtain $y_\lambda \in W_0^{1,p}(Z)$ such that $0 \in \partial R_\lambda(y_\lambda)$. Now as above, from this inclusion we can conclude that y_λ is a solution of Problem (1). Moreover, $x_\lambda \neq y_\lambda \neq 0$, since $R_\lambda(x_\lambda) < 0 = R_\lambda(0) < R_\lambda(y_\lambda)$. ∎

Remark 3.2. Theorem 3.1 provides the existence of multiple nontrivial solutions of the eigenvalue Problem (1) with $\lambda \in \mathbb{R}$ running on a half line. In contrast, Chang [5, theorem 5.5] deals with a semilinear eigenvalue problem (i.e., $p = 2$) and proves existence of one nontrivial solution for some value of λ. Also, the present theorem extends to quasilinear eigenvalue problems with discontinuities Theorem 3.35 of Ambrosetti–Rabinowitz [1].

4. EXISTENCE THEOREM FOR (2)

In this section we prove an existence theorem for the semilinear eigenvalue problem (2). For this purpose, we introduce the following assumptions on the discontinuous term f.

H(f)$_2$: $f: Z \times \mathbb{R} \to \mathbb{R}$ is a Borel measurable function such that

(i) f_0, f_1 are both finite and N-measurable;

(ii) $|f(z,r)| \le a_1(z) + c_1|r|^s$ for a.e. $z \in Z$, all $r \in \mathbb{R}$, and some $a_1 \in L^\infty(Z)$ and $0 \le s < (N+2)/(N-2)$, if $N > 2$;

(iii) $\lim_{r\to 0}|f(z,r)|/|r| = 0$ uniformly for a.e. $z \in Z$;

(iv) $0 < \theta F(z,r) \le rf(z,r)$ for a.e. $z \in Z$, all $|r| \ge \xi$, and some $\theta > 2$ and $\xi \ge 0$;

(v) $rf(z,r) \ge 0$ for a.e. $z \in Z$ and all $r \in \mathbb{R}$.

The assumptions on the coefficients of the partial differential operators are the following, with $0 < c < c_1$ and $1/\theta < c/2c_1$,

H(a): With $a_{ij} \in L^\infty(Z)$ and $a_{ij} = a_{ji}$ for all $1 \le i, j \le N$, for a.e. $z \in Z$ and all $\xi \in \mathbb{R}^N$ we have

$$c\|\xi\|^2 \le \sum_{i,j=1}^{N} a_{ij}(z)\xi_i\xi_j \le c_1\|\xi\|^2.$$

By a *solution* of (2) we mean a function $x \in H_0^1(Z)$ such that with some $g \in L^2(Z)$ satisfying $f_0(z,x(z)) \le g(z) \le f_1(z,x(z))$ a.e. on Z, we have

$$-\sum_{i,j=1}^{N} D_j\big(a_{ij}(z)D_i x(z)\big) - \lambda x(z) = g(z) \quad \text{a.e. on } Z.$$

Our variational approach will be based on the following nonsmooth version of Theorem 5.3 of Rabinowitz [13], whose proof can be carried out the same way, except we need to use the nonsmooth deformation theorem, Theorem 3.1, of Chang [5].

PROPOSITION 4.1. *Let $Y = V \oplus X$ be a reflexive Banach space with $\dim V < \infty$, $G: Y \to \mathbb{R}$ locally Lipschitz satisfying the nonsmooth PS condition and*

(i) *there exist $\rho, \alpha > 0$ such that $G(x) \ge \alpha$ for all $x \in X$ with $\|x\| = \rho$;*

(ii) *there exist $u \in X$ and $R > \rho$ such that $G(x) \le 0$ for all $x \in \partial Q$, where $Q = \{v \in V: \|v\| \le 1\} \oplus \{\beta u : 0 < \beta < R\}$ and ∂Q is the boundary of Q in $V \oplus \operatorname{span}\{u\}$.*

Then, $c = \inf_{h \in \Gamma} \max_{x \in Q} G(h(x)) \geq \alpha$ *is a critical value, where* $\Gamma = \{h \in C(\overline{Q}, Y)$: *h is the identity on* $Q\}$.

Remark 4.2. (a) If $G|_V \leq 0$, and there exist $u \in \partial B_1 \cap X$ and $R > \rho$ such that $G(y) \leq 0$ for all $y \in V \oplus \operatorname{span}\{u\}$ with $\|y\| \geq R_1$, then $G|_Q \leq 0$ for large R and Q defined as before. (b) A special case of this proposition is the nonsmooth mountain pass theorem of Chang [5].

Using this proposition we can prove the following existence theorem for Problem (2).

THEOREM 4.3. *Assume that* $\mathbf{H(f)}_2$ *and* $\mathbf{H(a)}$ *hold. Then Problem* (2) *has a nontrivial solution for every* $\lambda \in \mathbb{R}$.

Proof. Let a: $H_0^1(Z) \times H_0^1(Z) \to \mathbb{R}$ be the bilinear Dirichlet form defined by

$$a(x, y) = \int_Z \sum_{i,j=1}^{N} a_{ij}(z)\, D_i x(z)\, D_j y(z)\, dz$$

By $\mathbf{H(a)}$, $a(\cdot, \cdot)$ is symmetric, bounded and coercive (i.e., $a(x,x) \geq c\|x\|^2$). Let $A \in \mathscr{L}(H_0^1(Z), H^{-1}(Z))$ be the strongly monotone operator defined by $\langle A(x), y\rangle = a(x, y)$, where $\langle \cdot , \cdot \rangle$ denotes the duality bracket for the pair $(H_0^1(Z), H^{-1}(Z))$. Invoking Theorem 7.C of Showalter [14] (see also Theorem 22.G of Zeidler [15]), we see that A has a sequence $\{\lambda_n\}$ of positive eigenvalues with $\lambda_n \to \infty$. The corresponding eigenvectors $\{\phi_n\}$ form an orthonormal basis of $H_0^1(Z)$, furnished with the equivalent inner product $(x, y) \to a(x, y)$. So $a(\phi_k, \phi_m) = 0$ for $k \neq m$, and $a(\phi_m, \phi_m) = \lambda_m \|\phi_m\|^2 = 1$.

We first assume that $\lambda_k \leq \lambda < \lambda_{k+1}$. Let $V = \operatorname{span}\{\phi_m\}_{m=1}^k$ and $X = V^{\perp}$ in $H_0^1(Z)$. Also, define R_λ: $H_0^1(Z) \to \mathbb{R}$ by

$$R_\lambda(x) = \frac{1}{2} \sum_{i,j=1}^{N} \int_Z a_{ij}(z)\, D_i x(z)\, D_j x(z)\, dz - \frac{1}{2} \lambda \int_Z [x(z)]^2\, dz - \int_Z F(z, x(z))\, dz$$

For $x \in X$, we have $x = \sum_{m \leq k+1} \beta_m \phi_m$, with $\beta_m \in \mathbb{R}$. So we have

$$R_\lambda(x) = \frac{1}{2} \sum_{m \geq k+1} \left(1 - \frac{\lambda}{\lambda_m}\right) \beta_m^2 - V(x)$$

where as before $V(x) = \int_Z F(z, x(z))\,dz$. By $\mathbf{H(f)_2(iii)}$, for any given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|F(z,r)| \le \epsilon |r|^2 \quad \text{for a.e. } z \in Z \text{ and all } |r| \le \delta.$$

On the other hand, from $\mathbf{H(f)_2(ii)}$ we have, for a.e. $z \in Z$ and all $|r| > \delta$,

$$|F(z,r)| \le a_1(z)|r| + \frac{c_1}{s}|r|^{s+1} = \left(\frac{a_1(z)}{|r|^s} + \frac{c_1}{s}\right)|r|^{s+1}$$

$$\le \left(\frac{\|a_1\|_\infty}{\delta^s} + \frac{c_1}{s}\right)|r|^{s+1} = \gamma |r|^{s+1}.$$

Thus, for a.e. $z \in Z$ and all $r \in \mathbb{R}$ we obtain $|F(z,r)| \le \epsilon|r|^2 + \gamma|r|^{s+1}$, which implies that

$$|V(x)| \le \int_Z |F(z,x(z))|\,dz \le \epsilon \int_Z |x(z)|^2\,dz + \gamma \int_Z |x(z)|^{s+1},dz.$$

Note that $s + 1 < 2N/(N-2)$. Thus, by the Sobolev embedding theorem, we conclude that $H_0^1(Z)$ is embedded continuously in $L^{s+1}(Z)$. Hence for each $x \in H_0^1(Z)$, with $\gamma_1, \gamma_2 \ge 0$,

$$|V(x)| \le \gamma_1 \epsilon \|x\|^2 + \gamma_2 \|x\|^{s+1}.$$

If $\|x\| \le (\gamma_1 \epsilon / \gamma_2)^{1/(s-1)}$, then $|V(x)| \le 2\gamma_1 \epsilon \|x\|^2$.

Since $\epsilon > 0$ was arbitrary, we deduce that $V(x) = o(\|x\|^2)$. Therefore, we can write

$$R_\lambda(x) = \frac{1}{2} \sum_{m \ge k+1} \left(1 - \frac{\lambda}{\lambda_m}\right)\beta_m^2 - o\left(\|x\|^2\right) \tag{7}$$

Since $\lambda_k \le \lambda < \lambda_{k+1}$, we conclude that $\lambda/\lambda_m < 1$ for all $m \ge k+1$, and thus from (7) it follows that there exist $\rho > 0$ and $\alpha > 0$ such that, for all $x \in X$ with $\|x\| = \rho$,

$$R_\lambda(x) \ge \alpha > 0.$$

Next we show that there exists $R > \rho$ such that $R_\lambda(x) \le 0$ for all $x \in \text{span}\{\phi_m\}_{m=1}^{k+1}$ with $\|x\| \ge R$. To this end, let $x \in V$. Then $x = \sum_{m=1}^k \beta_m \phi_m$, with $\beta_m \in \mathbb{R}$, and so

$$R_\lambda(x) = \frac{1}{2} \sum_{m=1}^{k} \left(1 - \frac{\lambda}{\lambda_m}\right)\beta_m^2 - \int_Z F(z, x(z))\,dz.$$

We have

$$\int_Z F(z,x(z))\,dz = \int_{\{x>0\}}\int_0^{x(z)} f(z,r)\,dr\,dz - \int_{\{x<0\}}\int_{x(z)}^0 f(z,r)\,dr\,dz.$$

By **H(f)**$_2$(**v**), $\int_Z F(z,x(z))\,dz \geq 0$. Hence

$$R_\lambda(x) \leq \frac{1}{2}\sum_{m=1}^{k}\left(1-\frac{\lambda}{\lambda_m}\right)\beta_m^2 \leq 0$$

since $\lambda_i \leq \lambda < \lambda_{k+1}$. Therefore, $R_\lambda|_V \leq 0$. Now let $x \in \operatorname{span}\{\phi_m\}_{m=1}^{k+1}$ and assume that $\|x\| = 1$. Then $x = \sum_{m=1}^{k+1}\beta_m\phi_m$ with $\beta_m \in \mathbb{R}$. Hence for $\eta > 0$ we have

$$\begin{aligned} R_\lambda(\eta x) &= \frac{1}{2}\sum_{m=1}^{k+1}\left(1-\frac{\lambda}{\lambda_m}\right)\beta_m^2\eta^2 - V(\eta x) \\ &\leq \frac{1}{2}\left(1-\frac{\lambda}{\lambda_{k+1}}\right)\beta_{k+1}^2\eta^2 - V(\eta x) \end{aligned}$$

With **H(f)**$_2$(**iv**), we obtain (see also Remark 2.13 of Rabinowitz [13])

$$F(z,r) \geq c_3|r|^\theta - c_4, \qquad \text{with } c_3, c_4 > 0. \tag{8}$$

which implies that $V(x) \geq c_3\|x\|_\theta^\theta - c_4|Z|$. Thus we have

$$\begin{aligned} R_\lambda(\eta x) &\leq \frac{1}{2}\left(1-\frac{\lambda}{\lambda_{k+1}}\right)\beta_{k+1}^2\eta^2 - c_3\eta^\theta\|x\|_\theta^\theta + c_4|Z| \\ &= \frac{1}{2}\left(1-\frac{\lambda}{\lambda_{k+1}}\right)\beta_{k+1}^2\eta^2 - c_3\eta^\theta + c_4|Z|. \end{aligned}$$

Since $\theta > 2$, it follows from the above inequality that $R_\lambda(\eta x) \to -\infty$ as $\eta \to \infty$. So, there exists $R > \rho$ such that $R_\lambda(x) \leq 0$ for all $x \in \operatorname{span}\{\phi_m\}_{m=1}^{k+1}$ with $\|x\| \geq R$.

Next we will check that R_λ satisfies the nonsmooth PS condition. To this end, let $\{x_n\} \subseteq H_0^1(Z)$ be such that $|R_\lambda(x_n)| \leq M$ for all $n \geq 1$, and $m(x_n) \to 0$ as $n \to \infty$. Let $v_n \in \partial R_\lambda(x_n)$ with $\|v_n\|_* = m(x_n)$. We have $v_n = Ax_n - \lambda x_n - u_n$ with $u_n \in \partial V(x_n)$ for all $n \geq 1$. Let $1/\theta < \gamma < \min\{\frac{1}{2}, c/(2c_1)\}$. Then for large $n \geq 1$, $-\gamma\langle v_n, x_n\rangle \leq \gamma\|x_n\|$ which implies that

$$-\gamma\langle A(x_n), x_n\rangle + \gamma\lambda\|x_n\|_2^2 + \gamma(u_n, x_n)_2 \leq \gamma\|x_n\|,$$

where $(\cdot,\cdot)_2$ denotes the inner product in $L^2(Z)$. So, we have for some μ_1

$$M + \gamma\|x_n\| \geq R_\lambda(x_n) - \gamma\langle A(x_n), x_n\rangle + \gamma\lambda\|x_n\|_2^2 + \gamma(u_n, x_n)_2$$
$$\geq \left(\frac{c}{2} - c_1\gamma\right)\|Dx_n\|_2^2 + \mu_1\lambda\left(\gamma - \frac{1}{2}\right)\|x_n\|_\theta^2$$
$$-\int_Z \big(F(z, x_n(z)) - \gamma u_n(z)x_n(z)\big)\,dz. \tag{9}$$

From the proof of Theorem 3.1 we know that $u_n \in \partial V(x_n)$ if and only if $f_0(z, x_n(z)) \leq u_n(z) \leq f_1(z, x(z))$ a.e. on Z. By $\mathbf{H(f)}_2\mathbf{(iv)}$ we see that $\theta F(z,r) \leq rf_0(z,r)$ and $\theta F(z,r) \leq rf_1(z,r)$ for almost all $z \in Z$ and all $|r| \geq \xi$. Thus

$$\int_Z \big(\gamma u_n(z)x_n(z) - F(z, x_n(z))\big)\,dz$$
$$= \int_{\{|x_n|\geq\xi\}} \big(\gamma u_n(z)x_n(z) - F(z, x_n(z))\big)\,dz$$
$$+ \int_{\{|x_n|<\xi\}} \gamma u_n(z)x_n(z)\,dz - \int_{\{|x_n|<\xi\}} F(z, x_n(z))\,dz$$
$$\geq \int_{\{|x_n|\geq\xi\}} (r\theta - 1)F(z, x_n(z))\,dz - \int_{\{|x_n|<\xi\}} F(z, x_n(z))\,dz. \tag{10}$$

Using (8) we have

$$\int_{\{|x_n|\geq\xi\}} (\gamma\theta - 1)F(z, x_n(z))\,dz \geq \int_{\{|x_n|\geq\xi\}} (\gamma\theta - 1)\big(c_3|x_n(z)|^\theta - c_4\big)\,dz$$
$$= (\gamma\theta - 1)\int_Z \big(c_3|x_n(z)|^\theta - c_4\big)\,dz$$
$$-(\gamma\theta - 1)\int_{\{|x_n|<\xi\}} \big(c_3|x_n(z)|^\theta - c_4\big)\,dz$$
$$\geq (\gamma\theta - 1)\big(c_3\|x_n\|_\theta^\theta - c_4|Z|\big) - (\gamma\theta - 1)\big(c_3\xi^n|Z| - c_4|Z|\big). \tag{11}$$

Moreover, by **H(f)**$_2$(**ii**) we see that

$$\int_{\{|x_n|<\xi\}} F(z, x_n(z))\,dz \le c_5 \quad \text{for some } c_5 > 0. \tag{12}$$

By combiing (9)–(12) we obtain, for some $c_6 > 0$,

$$M + \gamma\|x_n\| \ge \left(\frac{c}{2} - c_1\gamma\right)\|Dx_n\|_2^2 + \mu_1\lambda\left(\gamma - \frac{1}{2}\right)\|x_n\|_\theta^2$$
$$+ (\gamma\theta - 1)c_3\|x_n\|_\theta^\theta - c_6.$$

Since $\theta > 2$, Young's inequality with $\epsilon > 0$ implies

$$\|x_n\|_\theta^2 \le \delta(\epsilon) + \epsilon\|x_n\|_\theta^\theta$$

with $\delta(\epsilon) \to \infty$ as $\epsilon \downarrow 0$. Therefore,

$$M + \gamma\|x_n\| \ge \left(\frac{c}{2} - c_1\gamma\right)\|Dx_n\|_2^2 - \mu_1\lambda\left(\frac{1}{2} - \gamma\right)\epsilon\|x_n\|_\theta^\theta$$
$$+ (\gamma\theta - 1)c_3\|x_n\|_\theta^\theta - c_7(\epsilon)$$

with $c_7(\epsilon) \to \infty$ as $\epsilon \downarrow 0$. Choose $\epsilon > 0$ such that $\gamma\theta - 1 > \mu_1\lambda(\frac{1}{2} - \lambda)\epsilon$; then for some $c_8 > 0$

$$M + \gamma\|x_n\| \ge \left(\frac{c}{2} - c_1\gamma\right)\|Dx_n\|_2^2 - c_8. \tag{13}$$

Since $\|Dx\|_2$ is an equivalent norm on $H_0^1(Z)$, we infer that $\{x_n\} \subseteq H_0^1(Z)$ is bounded. Hence we may assume that $x_n \xrightarrow{w} x$ in $H_0^1(Z)$. Thus by the Sobolev embedding theorem $H_0^1(Z)$ is embedded compactly in $L^{s+1}(Z)$. Therefore we also have that $x_n \to x$ in $L^{s+1}(Z)$ since $s + 1 < 2N/(N-2)$. Recall that $u_n \in \partial V(x_n)$ for all $n \ge 1$, and V is locally Lipschitz on $L^{s+1}(Z)$. So $\{u_n\} \subseteq L^\nu(Z)$ is bounded, where $1/\nu + 1/(s+1) = 1$. But $L^\nu(Z)$ is embedded compactly in $H^{-1}(Z) = H_0^1(Z)^*$. Therefore, $\{u_n\} \subseteq H^{-1}(Z)$ is relatively compact. Hence we may assume that $u_n \to u$ in $H^{-1}(Z)$ as $n \to \infty$. With $v_n = A(x_n) - \lambda x_n - u_n$ and $\|v_n\|_* \to 0$, we have

$$A(x_n) - \lambda x_n \to u = A(x) - \lambda x \quad \text{in } H^{-1}(Z) \quad \text{as } n \to \infty.$$

Let j denote the embedding of $H_0^1(Z)$ into $H^{-1}(Z)$ (duality map). If $\lambda \notin \sigma(A)$, then $x_n = (A - \lambda j)^{-1}(v_n + u_n) \to (A - \lambda j)^{-1}(u) = x$ in $H_0^1(Z)$ as $n \to \infty$. If $\lambda \in \sigma(A)$, then $\lambda = \lambda_m$ for some $m \ge 1$. Hence $x_n - p_m(x_n) \to y$ in $H_0^1(Z)$ as $n \to \infty$, where p_m denotes the orthogonal projection to the eigenspace corresponding to λ_m. As the eigenspace is finite dimensional, we conclude that $\{x_n\}$ has a strongly convergent subsequence in $H_0^1(Z)$. In any case, R_λ satisfies the PS condition. Thus, when $\lambda_k \leq \lambda < \lambda_{k+1}$ we may apply Proposition 4.1 to obtain a critical point of R_λ.

Now it remains to check the case when $\lambda < \lambda_1$. To this end, set $|x|_\lambda^2 = \langle A(x), x \rangle - \lambda \|x\|_2^2$. Then $|\cdot|_\lambda$ is a norm on $H_0^1(Z)$ equivalent to the usual norm, and

$$R_\lambda(x) = \tfrac{1}{2}|x|_\lambda^2 - V(x).$$

As $V(x) = o(\|x\|^2)$, which in turn implies $V(x) = o(|x|_\lambda^2)$, it follows that there exist $\rho, \alpha > 0$ such that $R_\lambda(x) \geq \alpha > 0$ for all $|x|_\lambda = \rho$. Let $|x|_\lambda = 1$. By (8) we see that for every $\eta > 0$ we have

$$R_\lambda(\eta x) \leq \frac{\eta^2}{2}|x|_\lambda^2 - c_3\|x\|_\theta^\theta \eta^\theta - c_4 = \frac{\eta^2}{2} - c_3\|x\|_\theta^\theta \eta^\theta - c_4$$

which implies that $R_\lambda(\eta x) \to -\infty$ as $\eta \to \infty$.

To verify the PS condition, let $\{x_n\} \subseteq H_0^1(Z)$ be a sequence such that $|R_\lambda(x_n)| \leq M$ and $m(x_n) \to 0$. As before, let $v_n \in \partial R_\lambda(x_n)$ so that $\|v_n\|_* = m(x_n)$. Then $v_n = A(x_n) - \lambda j(x_n) - u_n$ with $u_n \in \partial V(x_n)$. Arguing as before, for $n \geq 1$ large enough and some $c_9 > 0$ we have

$$\begin{aligned} M + \frac{1}{\theta}|x_n|_\lambda &\geq R_\lambda(x_n) - \frac{1}{\theta}|x_n|_\lambda^2 + \frac{1}{\theta}\int_Z u_n(z)x_n(z)\,dz \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta}\right)|x_n|_\lambda^2 + \frac{1}{\theta}\int_Z \big(u_n(z)x_n(z) - F(z, x_n(z))\big)\,dz \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta}\right)|x_n|_\lambda^2 - c_9. \end{aligned}$$

from which we deduce that $\{x_n\} \subseteq H_0^1(Z)$ is bounded. So we can extract a strongly convergent subsequence, which implies that R_λ satisfies the PS condition and thus all the condition of Proposition 4.1, when $\lambda < \lambda_1$.

Therefore, for any $\lambda \in \mathbb{R}$, we can apply Proposition 4.1 on R_λ and deduce that it has a critical value $c > 0$ with $x_\lambda \in H_0^1(Z)$, a corresponding critical point, i.e., $0 \in \partial R_\lambda(x_\lambda)$. Obviously, $x_\lambda \neq 0$. Also, since $0 = A(x_\lambda) - \lambda x_\lambda - u_\lambda$ with $u_\lambda \in \partial V(x_\lambda)$, we have for all $\psi \in C_0^\infty(Z)$

$$\sum_{i,j=1}^{n} \int_Z a_{ij}(z)\, D_i x(z)\, D_j \psi(z)\,dz = \int_Z (\lambda x_\lambda(z) + u_\lambda(z))\psi(z)\,dz.$$

Since $\lambda x_\lambda + u_\lambda \in L^2(z)$, from the definition of the distribution derivative it follows that, with $f_0(z, x_\lambda(z)) \le u_\lambda(z) \le f_1(z, x_\lambda(z))$ a.e. on Z, we have

$$\begin{cases} -\sum_{i,j=1}^{N} D_j\big(a_{ij}(z)\, D_i x(z)\big) - \lambda x_\lambda(z) = u_\lambda(z) & \text{a.e. on } Z \\ x|_\Gamma = 0 \end{cases}$$

So x_λ is a nontrivial solution of Problem (2). ∎

Remark 4.4. Theorem 4.3 extends Theorem 5.16 of Rabinowitz [13] where $f(z, r)$ is assumed to be continuous and the differential operator is the Laplacian.

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