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The $\widehat{W}$-orbit of ρ, Kostant's formula for powers of the Euler product and affine Weyl groups as permutations of $\mathbb{Z}$

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Abstract

Let an affine Weyl group $\widehat{W}$ act as a group of affine transformations on a real vector space V. We analyze the $\widehat{W}$-orbit of a regular element in V and deduce applications to Kostant's formula for powers of the Euler product and to the representations of $\widehat{W}$ as permutations of the integers.

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1. Introduction

This paper stems from the attempt to deepen two seemingly unrelated topics: on one hand the combinatorial interpretation of Kostant's recent results on the powers of the Euler product suggested in type A by Tate and Zelditch, and on the other hand the problem of giving a uniform and conceptual description of certain affine Weyl groups as permutations of the integers. The common denominator of these two subjects lies in their intimate connection with the orbit of a distinguished vector under the action of an affine Weyl group. The results of the paper should be regarded for the first topic as a generalization of Tate and Zelditch's approach, and for the other as a systematic treatment of well-established results on affine Weyl groups of the classical type. To be more precise, let us fix notation. Let $(V, (\cdot,\cdot))$ be a Euclidean space, Δ a finite crystallographic irreducible root system in V, Δ^+ a fixed positive system for Δ.

Set $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$ and let θ be the highest root of Δ. We define the dual Coxeter number $h^\vee$ of Δ as $h^\vee = \frac{2(\rho,\theta)}{(\theta,\theta)} + 1$. The affine Weyl group $\widehat{W}$ of Δ is the group generated by reflections on V with respect to the set of affine hyperplanes $H_{\alpha,k} = \{x \in V \mid (x,\alpha) = k\}$, $\alpha \in \Delta^+$, $k \in \mathbb{Z}$. For each $q \in \mathbb{R}^+$, we denote by $\widehat{W}_q$ the group generated by reflections in V with respect to the set of hyperplanes $H_{\alpha,qk}$, $\alpha \in \Delta^+$, $k \in \mathbb{Z}$; thus $\widehat{W}_q$ is naturally

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isomorphic to $\widehat{W}$. We notice that scaling the inner product by $\frac{1}{q}$ changes $\widehat{W}$ into $\widehat{W}_q$ (and does not change $h^\vee$). We assume throughout the paper that

$$(\theta, \theta) = \frac{1}{h^\vee}. \tag{1.1}$$

For each $\lambda \in V$, we denote by $\widehat{W}_q \cdot \lambda$ the orbit of λ under $\widehat{W}_q$.

A basic step in our work is the analysis of $\widehat{W}_{\frac{1}{2}} \cdot \rho$. A motivation for this study occurs in the framework of Kostant's work on Dedekind's η function, which we now recall. Let $\mathfrak{g}$ be a complex finite-dimensional semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and Δ the corresponding root system. Let $V = \mathfrak{h}^*_{\mathbb{R}}$, the real span of a fixed set of simple roots, endowed with the invariant form induced by the Killing form of $\mathfrak{g}$. (It is well-known that then (1.1) holds.)

If λ is a dominant weight let χ_λ denote the character of the irreducible $\mathfrak{g}$-module V_λ with highest weight λ. Set also $a = \exp(2\pi i \cdot 2\rho)$. Working on previous results of Macdonald, Kostant found the following remarkable expansion for (certain) powers of the Euler product $\prod_{m=1}^{\infty}(1 - x^m)$.

Theorem 1.1 (*[10, Thm 3.1]*).

$$\left(\prod_{m=1}^{\infty}(1 - x^m)\right)^{\dim(\mathfrak{g})} = \sum_{\lambda\ dominant} \chi_\lambda(a)\ \dim(V_\lambda)\, x^{(\lambda+2\rho,\lambda)}. \tag{1.2}$$

Moreover, $\chi_\lambda(a) \in \{-1, 0, 1\}$.

In [11] Kostant has improved the previous formula determining the set P_{alc} of weights which give a non zero contribution in the sum (see Theorem 3.1 below). The main outcome is that

$$P_{alc} = \{\lambda \text{ dominant weight} \mid \lambda + \rho \in \widehat{W}_{\frac{1}{2}} \cdot \rho\}.$$

Moreover, he proves that the contribution of each $\lambda \in P_{alc}$ is determined by the parity of $\ell_{\frac{1}{2}}(w)$, where $w \in \widehat{W}_{\frac{1}{2}}$ is the element such that $\lambda + \rho = w(\rho)$, and $\ell_{\frac{1}{2}}$ is the length function on $\widehat{W}_{\frac{1}{2}}$.

On the other hand, in [1], Adin and Frumkin made explicit, by using the well-known connection between dominant weights and partitions, the combinatorial content of Kostant's result in type A. Their result also makes it easy to determine the sign of $\chi_\lambda(a)$. After the appearance of Kostant's paper, a simple approach to the combinatorial interpretation of Kostant's result in type A using the affine Weyl group was explained by Tate and Zelditch in [15]. We shall obtain results analogous to those of [15] for all classical types and for G_2. The exposition of these results is the content of Section 3.

The crucial observation is that ρ is the *unique* element in the weight lattice of Δ lying in the fundamental alcove of $\widehat{W}_{\frac{1}{2}}$. By the basic properties of the action of the affine group on V, this implies that $\widehat{W}_{\frac{1}{2}} \cdot \rho$ is the set of weights which lie in some alcove of $\widehat{W}_{\frac{1}{2}}$, or, equivalently, which do not belong to any of the reflecting hyperplanes. Once the root systems are explicitly described in coordinates, this allows us to easily describe P_{alc} by purely arithmetical conditions, for all types.

We shall write down this description only for the classical types and for G_2. For each of these cases, we shall also give a simple rule for recovering the parity of $\ell(w)$ from $w(\rho)$. For type A, we re-obtain the rule of [1]. The affine Weyl group is the semidirect product of the finite Weyl group W of $\mathfrak{g}$ and the group $Q^\vee$ acting on V by translations, hence $\widehat{W}_{\frac{1}{2}} \cong \frac{1}{2}Q^\vee \rtimes W$. Moreover, if $w = t_\tau v$, where t_τ is the translation by $\tau \in \frac{1}{2}Q^\vee$, and $v \in W$, then $\ell(w) \equiv \ell(v) \bmod 2$. Our rule is in fact a sort of Euclidean algorithm which produces v and τ from $w(\rho)$.

The last section of the paper deals with affine Weyl groups regarded as permutation groups of the set of integers. This point of view was introduced by Lusztig [12] for type $\tilde{A}$, and generalized to the other classical cases by his students (and other people). A thorough and systematic account of the combinatorial aspects of the theory can now be found in Chapter 8 of [3].

From the explicit description of $\widehat{W}_{\frac{1}{2}} \cdot \rho$, we see that in cases $\tilde{A}$ and $\tilde{C}$ we can quite naturally associate to each $w \in \widehat{W}_{\frac{1}{2}}$ a permutation of $\mathbb{Z}$, uniquely determined by $w(\rho)$. In this way, we obtain an injective homomorphism of $\widehat{W}_{\frac{1}{2}}$ into $S(\mathbb{Z})$, the group of permutations of $\mathbb{Z}$, which agrees with the usual permutation representation. This suggests that the known permutation representations of all classical affine Weyl groups can be obtained from the explicit description of the orbit $\widehat{W}_q \cdot \lambda$, for an appropriate choice of q and λ. In fact, the final outcome of our study is a uniform and concise treatment of the known permutation representations of classical Weyl groups. Our point of view is also successful for type $\tilde{G}_2$. To our knowledge, a similar unified approach does not appear in the literature, even if the existence of a connection between the orbit of a regular vector and the permutation representation of $\widehat{W}$ is noted in Eriksson's unpublished Ph.D. Thesis [6].

We have already explained the content of Sections 3 and 4. The results of Section 2 are a kind of "context free" preparation for the next Sections, and rely on the standard theory of the geometric action of affine Weyl groups. The main contribution is Proposition 2.1.

2. Preliminary results

We retain the notation set at the beginning of the Introduction: V is an n-dimensional Euclidean space with inner product $(\cdot,\cdot)$, Δ is a finite crystallographic irreducible root system of rank n in V. Denote by W the corresponding finite reflection group. Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be a set of simple roots for Δ (with positive system Δ^+). Denote by Q the root lattice. For $\beta \in Q$ set $\beta^\vee = \frac{2\beta}{(\beta,\beta)}$, and let

$$Q^\vee = \sum_{i=1}^{n} \mathbb{Z}\alpha_i^\vee,$$
$$P = \left\{\lambda \in \mathfrak{h} \mid (\lambda, \alpha^\vee) \in \mathbb{Z}\, \forall \alpha \in \Delta\right\},$$

be the coroot and weight lattices. Denote by P^+ the set of dominant weights:

$$P^+ = \{\lambda \in P \mid (\lambda, \alpha^\vee) \geq 0\, \forall \alpha \in \Pi\}.$$

Let $\omega_1, \ldots, \omega_n$ be the fundamental weights, so that $P = \sum_{i=1}^n \mathbb{Z}\omega_i$ and $\rho = \sum_{i=1}^n \omega_i$. Remark that if $\theta^\vee = \sum_{i=1}^n m_i \alpha_i^\vee$ then $h^\vee = 1 + \sum_{i=1}^n m_i$.

Fix $q \in \mathbb{R}^+$. Recall the group $\widehat{W}_q$ defined in the Introduction. Then $\widehat{W}_q = T_q \rtimes W$ where T_q is the group of translations of V by elements in $qQ^\vee$. It is clear that $\widehat{W}_1$ is the usual affine Weyl group. Ours is a slight extension of the usual definition which turns out to be very useful for our goals.

For $\alpha \in V \setminus \{0\}$, $\beta \in V$ denote by s_α, t_β the reflection in α and the translation by β, respectively.

Recall that $\widehat{W}_q$ is a Coxeter group with generators $s_i = s_{\alpha_i}$ for $i = 1, \ldots, n$ and $s_0 = t_{q\theta^\vee} s_\theta$. We denote by ℓ_q the length function with respect to this choice of generators. Set $H_{rq,\alpha} = \{x \in V \mid (x, \alpha) = rq\}$ for $r \in \mathbb{Z}$ and $\alpha \in \Delta^+$. The *alcoves* of $\widehat{W}_q$ are the connected components of $V \setminus \bigcup_{\substack{\alpha\in\Delta^+\\ r\in\mathbb{Z}}} H_{rq,\alpha}$. The *fundamental alcove* is the alcove

$$C_q = \left\{x \in V \mid (x, \alpha) > 0\, \forall \alpha \in \Delta^+,\ (x, \theta) < q\right\}.$$

It is well-known that $\widehat{W}_q$ acts on the set of alcoves and this action is simply transitive. This means that wC_q is an alcove and for each alcove C'_q there exists a unique $w \in \widehat{W}_q$ such that $C'_q = w(C_q)$. Moreover, $\overline{C_q}$ is a fundamental domain for the action of $\widehat{W}_q$ on V. In particular if y belongs to some alcove, then there exist unique $w \in \widehat{W}_q$ and $x \in C_q$ such that $w(x) = y$. We shall tacitly use these standard properties in the following.

Definition 2.1. We say that $v \in V$ is q-regular if it belongs to some alcove, or, equivalently,

$$v \in V \setminus \bigcup_{\substack{\alpha\in\Delta^+\\ r\in\mathbb{Z}}} H_{rq,\alpha}.$$

Any alcove can be expressed as an intersection (ranging over Δ^+) of strips $H_\alpha^{rq} = \{x \in V \mid rq < (x, \alpha) < (r + 1)q\}$, $(r \in \mathbb{Z})$. Denote by $k(w, \alpha)$ the integers such that

$$wC_q = \bigcap_{\alpha\in\Delta^+} H_\alpha^{k(w,\alpha)q}.$$

The collection $\{k(w, \alpha)\}_{\alpha\in\Delta^+}$ has been introduced by Shi and called the *alcove form* of w.

Remark 2.2. Suppose that μ is q-regular. If $\mu \in wC_q$, then

$$k(w,\alpha) = \left\lfloor \frac{(\mu,\alpha)}{q} \right\rfloor \tag{2.1}$$

and

$$\ell_q(w) = \sum_{\alpha\in\Delta^+} \left| \left\lfloor \frac{(\mu,\alpha)}{q} \right\rfloor \right|. \tag{2.2}$$

To obtain (2.1), it suffices to remark that the r.h.s. counts the number of hyperplanes $H_{rq,\alpha}$ separating C_q from wC_q. Since the total number of separating hyperplanes $H_{rq,\alpha}$ when α ranges over Δ^+, gives $\ell_q(w)$ (see [8, 4.5]), (2.2) follows.

We state as a proposition the following elementary observation, which will play a prominent role in the sequel.

Proposition 2.1. *Fix $\lambda \in V$. Let L be a lattice in V such that $\lambda + L$ is $\widehat{W}_q$-stable and $(\lambda + L) \cap C_q = \{\lambda\}$. Then*

$$\widehat{W}_q \cdot \lambda = \{\mu \in \lambda + L \mid \text{for all } \alpha \in \Delta,\ (\mu,\alpha) \notin q\mathbb{Z}\}.$$

Proof. Assume $\mu \in \widehat{W}_q \cdot \lambda$. Since $\lambda + L$ is $\widehat{W}_q$-stable, $\mu \in \lambda + L$. Moreover, since $\widehat{W}_q$ acts on the set of alcoves, μ belongs to some alcove, which means that for all $\alpha \in \Delta$, we have $(\mu,\alpha) \notin q\mathbb{Z}$.

Conversely, assume that $\mu \in \lambda + L$ and, for all $\alpha \in \Delta$, $(\mu,\alpha) \notin q\mathbb{Z}$. Then μ belongs to some alcove. Since $\widehat{W}_q$ acts transitively on the set of alcoves, and preserves $\lambda+L$, there exists $w \in \widehat{W}_q$ such that $w(\mu) \in C_q \cap (\lambda+L) = \{\lambda\}$. □

Remark 2.3. If $\lambda + L$ is W-stable and $qQ^\vee \subset L$ then $\lambda + L$ is $\widehat{W}_q$-stable.

Lemma 2.2. *We have $C_q \cap P = \{\rho\}$ if and only if*

$$\frac{(\theta,\theta)}{2}(h^\vee - 1) < q \le \frac{(\theta,\theta)}{2}(h^\vee + m - 1) \tag{2.3}$$

where $m = \min_{1\le i\le n} m_i$. In particular,

$$P \cap C_{\frac{1}{2}} = \{\rho\}.$$

Proof. Note that $(\rho,\theta) = \frac{(\theta,\theta)}{2}(h^\vee - 1)$, hence $\rho \in C_q \cap P$ if and only if $\frac{(\theta,\theta)}{2}(h^\vee - 1) < q$. Obviously $C_q \cap P = \{\rho\}$ if and only if $\rho + \omega_i \notin C_q$ for all $i = 1,\dots,n$. This implies

$$q \le (\rho + \omega_i, \theta) = \frac{(\theta,\theta)}{2}(h^\vee - 1) + \frac{(\theta,\theta)}{2}m_i = \frac{(\theta,\theta)}{2}(h^\vee + m_i - 1)$$

as desired. □

Note that $m = 1$ if Δ is not of type E_8; in this latter case $m = 2$.

3. Application to Euler products

The first application of the above results is connected with the work of Kostant on the powers of the Euler product $\prod_{m=1}^\infty (1 - x^m)$.

Let $\mathfrak{g}$ be a complex finite-dimensional semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and Δ the corresponding root system. In the notation of the previous section, we choose V to be the real span $\mathfrak{h}^*_\mathbb{R}$ of a fixed set of simple roots endowed with the invariant form induced by the Killing form of $\mathfrak{g}$. With this choice we have indeed that $(\theta,\theta) = \frac{1}{h^\vee}$ (see e.g. [10, Section 2]).

If $\lambda \in P^+$, let χ_λ denote the character of the irreducible $\mathfrak{g}$-module V_λ with highest weight λ. Recall relation (1.2). In [11, Theorem 2.4] a general criterion for determining the set

$$P_{alc} = \{\lambda \in P^+ \mid \chi_\lambda(a) \neq 0\}$$

is provided (see also [9, Exercise 10.19]). Kostant's theorem can be rephrased as follows:

Theorem 3.1. *We have*

$$\lambda \in P_{alc} \iff \lambda + \rho \in \widehat{W}_{\frac{1}{2}} \cdot \rho.$$

Moreover, if $\lambda + \rho = w(\rho),\ w \in \widehat{W}_{\frac{1}{2}}$, *then* $\chi_\lambda(a) = (-1)^{\ell_{\frac{1}{2}}(w)}$.

Corollary 3.2. *A weight* λ *belongs to* P_{alc} *if and only if it is dominant and*

$$(\lambda + \rho, \alpha) \notin \frac{1}{2}\mathbb{Z} \quad \text{for any } \alpha \in \Delta. \tag{3.1}$$

In such a case, λ *belongs to the root lattice* Q *and*

$$\chi_\lambda(a) = (-1)^{\sum_{\alpha\in\Delta^+} \lfloor 2(\lambda+\rho,\alpha)\rfloor}. \tag{3.2}$$

Proof. By Lemma 2.2 we have that $C_{\frac{1}{2}} \cap P = \{\rho\}$. Recall that $(\theta, \theta) = \frac{1}{h^\vee}$. Then $\frac{1}{2}Q^\vee \subset Q$, hence we can apply Proposition 2.1. Moreover, if $\lambda + \rho \in \widehat{W}_{\frac{1}{2}} \cdot \rho$, then $\lambda + \rho \in \rho + Q + \frac{1}{2}Q^\vee \subset \rho + Q$, hence $\lambda \in Q$. Finally (3.2) follows readily from Theorem 3.1 and (2.2). □

In the rest of this section we provide an explicit rendering of Corollary 3.2 for the classical root systems. We find combinatorial conditions that guarantee that $\lambda \in P_{alc}$ and determine $\chi_\lambda(a)$. For this last purpose it is convenient to use the following general fact rather than the formula (3.2). Denote by ℓ the length function in W.

Lemma 3.3. *If* $t_\tau w \in \widehat{W}_q,\ \tau \in qQ^\vee, w \in W$, *then* $\ell_q(t_\tau w) \equiv \ell(w) \mod 2$.

Proof. We shall use several times the following well-known fact from the theory of Coxeter groups (see e.g. [8, 5.8]): cancellations occur in pairs, so that if an element has an expression in terms of the generators of a certain parity, its length has the same parity. Since $t_\tau w$ has certainly an expression involving $\ell_q(t_\tau) + \ell(w)$ generators, it suffices to show that $\ell_q(t_\tau)$ is even. Since $qQ^\vee$ is the $\mathbb{Z}$-span of $qW \cdot \theta^\vee$ it suffices to prove that if $u \in W$, then $\ell_q(t_{qu(\theta^\vee)})$ is even. This follows from the relation $t_{qu(\theta^\vee)} = us_0 s_\theta u^{-1}$. □

In the classical cases we shall explicitly determine for each $\lambda \in P_{alc}$ the unique element $w \in \widehat{W}_{\frac{1}{2}}$ such that $\lambda + \rho = w\rho$ and compute $\tau \in \frac{1}{2}Q^\vee, u \in W$ such that $w = t_\tau u$. Applying Lemma 3.3 we obtain that $\chi_\lambda(a) = (-1)^{\ell(u)}$. In [15] essentially the same analysis was applied only to type A_n obtaining Theorem 1.2 of [1]. In the following we adopt the realization of the irreducible root systems as subsets of $\mathbb{R}^N$ given in [5]. We denote by $\langle \cdot, \cdot \rangle$ the standard inner product of $\mathbb{R}^N$ and by $\{e_i\}$ the canonical basis.

3.1. Type A_n

Recall that in [5] $\mathfrak{h}^*_\mathbb{R}$ is identified with the subspace of $\mathbb{R}^{n+1}$ orthogonal to $\lambda_0 = \sum_{i=1}^{n+1} e_i$. In this setting

$$\Delta^+ = \{e_i - e_j \mid i < j\}$$

and

$$Q = \left(\sum_{i=1}^{n+1} \mathbb{Z}e_i\right) \cap \mathfrak{h}^*_\mathbb{R}.$$

The map $\lambda \mapsto \bar{\lambda} = \lambda - \langle \lambda, e_{n+1}\rangle \lambda_0$ maps P bijectively onto $\sum_{i=1}^{n} \mathbb{Z}e_i$, P^+ onto

$$P_n = \left\{\sum_{i=1}^{n} \lambda_i e_i \mid \lambda_i \in \mathbb{Z}, \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0\right\}.$$

We finally recall that $\rho = \sum_{i=1}^{n+1} \frac{n-2i+2}{2} e_i$, $\theta = e_1 - e_{n+1}$, hence $h^\vee = n+1$. Since $\langle \theta, \theta \rangle = 2$ and $(\theta, \theta) = \frac{1}{h^\vee}$, we have

$$(\cdot, \cdot) = \frac{1}{2h^\vee} \langle \cdot, \cdot \rangle. \tag{3.3}$$

This implies in particular that $\frac{1}{2} Q^\vee = (n+1) Q$.

If $\lambda \in \mathfrak{h}_\mathbb{R}^*$ set $\lambda_i = \langle \bar{\lambda}, e_i \rangle$. Since $\langle \lambda_0, \alpha \rangle = 0$ for all $\alpha \in \mathfrak{h}_\mathbb{R}^*$ we see that $\langle \bar{\lambda}, \alpha \rangle = \langle \lambda, \alpha \rangle$ for all $\alpha \in \Delta$. Also recall that $\bar{\rho} = \sum_{i=1}^{n} (n-i+1) e_i$. Applying Corollary 3.2 we deduce the following result, which is the first statement of Theorem 1.2 from [1].

Proposition 3.4. *For $\overline{\lambda} = \sum_{i=1}^{n} \lambda_i e_i \in P_n$ we have*

$$\lambda \in P_{alc} \iff \lambda_i + n - i + 1 \not\equiv \lambda_j + n - j + 1 \bmod (n+1).$$

$(1 \le i \ne j \le n+1)$.

Note that, since $\lambda \in Q$, we have

$$\sum_{i=1}^{n+1} \lambda_i = \sum_{i=1}^{n+1} \langle \overline{\lambda}, e_i \rangle = \left(\sum_{i=1}^{n+1} \langle \lambda, e_i \rangle \right) - (n+1) \langle \lambda, e_{n+1} \rangle = -(n+1) \langle \lambda, e_{n+1} \rangle.$$

Hence $n+1$ divides $\sum_{i=1}^{n+1} \lambda_i$, so we can write

$$\lambda_i + (n-i+1) - \frac{1}{n+1} \sum_{j=1}^{n+1} \lambda_j = (n - r_i + 1) + (n+1) q_i \tag{3.4}$$

with $r_i \in \{1, 2, \ldots, n+1\}$.

Set $\tau = (n+1) \sum_{i=1}^{n+1} q_i e_i$. By Proposition 3.4 the r_i are pairwise distinct, so, by (3.4)

$$(n+1) \sum_{i=1}^{n+1} q_i = \sum_{i=1}^{n+1} (n-i+1) - \sum_{i=1}^{n+1} (n - r_i + 1) = 0,$$

hence $\tau \in \frac{1}{2} Q^\vee$. We can write

$$\begin{aligned} \lambda + \rho &= \sum_{i=1}^{n+1} \left(\lambda_i + (n-i+1) - \frac{1}{n+1} \sum_{j=1}^{n+1} \lambda_j - \frac{n}{2} \right) e_i \\ &= \sum_{i=1}^{n+1} \left(\frac{n - 2r_i + 2}{2} \right) e_i + (n+1) \sum_{i=1}^{n+1} q_i e_i. \end{aligned}$$

The action of W on V is described explicitly in [5]. In particular it is known that, if $v \in W$, then there is an element σ_v of S_n such that $v(e_i) = e_{\sigma_v(i)}$. This fact establishes the well-known isomorphism between W and S_n. Thus if we set σ to be the element of S_n such that $\sigma(i) = r_i$, and let v be the element of W such that $\sigma_v = \sigma^{-1}$, then $v(\rho) = \sum_{i=1}^{n+1} (\frac{n-2i+2}{2}) v(e_i) = \sum_{i=1}^{n+1} (\frac{n-2\sigma(i)+2}{2}) e_i = \sum_{i=1}^{n+1} (\frac{n-2r_i+2}{2}) e_i$ hence $\lambda + \rho = t_\tau v(\rho)$ and $\chi_\lambda(a) = (-1)^{\ell(v)}$.

Remark 3.1. It is well-known (and easy to prove) that $(-1)^{\ell(v)} = \mathrm{sign}(\sigma_v)$ thus $\chi_\lambda(a)$ is the sign of the permutation $i \mapsto r_i$.

3.2. Type C_n

We have $\Delta^+ = \{e_i \pm e_j \mid 1 \le i < j \le n\} \cup \{2e_i \mid 1 \le i \le n\}$, $\rho = \sum_{i=1}^{n} (n-i+1) e_i$, $\theta = 2e_1$ so that $h^\vee = n+1$. Moreover

$$P = \sum_{i=1}^{n} \mathbb{Z} e_i, \qquad Q = \left\{ \sum_{i=1}^{n} \lambda_i e_i \,\middle|\, \sum_{i=1}^{n} \lambda_i \in 2\mathbb{Z} \right\},$$

$$P^+ = \left\{ \sum_{i=1}^{n} \lambda_i e_i \in P \mid \lambda_1 \geq \lambda_2 \geq \cdots \lambda_n \geq 0 \right\}.$$

This time $\langle\theta,\theta\rangle = 4$, so that $(\cdot,\cdot) = \frac{1}{4h^\vee}\langle\cdot,\cdot\rangle$ and $\frac{1}{2}Q^\vee = 2h^\vee\mathbb{Z}^n$. By Corollary 3.2 we have

Proposition 3.5. *For $\lambda = \sum_{i=1}^{n}\lambda_i e_i \in P^+$ we have*

$$\lambda \in P_{alc} \iff \begin{array}{l} \lambda_i + n - i + 1 \not\equiv \pm(\lambda_j + n - j + 1) \bmod 2(n+1) \ (i \neq j) \\ \lambda_i + n - i + 1 \notin (n+1)\mathbb{Z}. \end{array}$$

It is well-known that the finite Weyl group W acts faithfully on $\{\pm e_1, \ldots, \pm e_n\}$ by signed permutations. It follows that $W \cdot \rho$ is the set of elements of type $\sum_{i=1}^{n} a_i e_i$ with $\{\pm a_1, \ldots, \pm a_n\} = \{\pm 1, \ldots, \pm n\}$. Now assume that $\lambda \in P_{alc}$ and $\mu = \lambda + \rho$, $\mu = \sum_{i=1}^{n}\mu_i e_i$. Denote by $\overline{\mu}_i$ the unique element in $\{\pm 1, \ldots, \pm n\}$ such that $\mu_i \equiv \overline{\mu}_i \bmod 2(n+1)$ and set $\overline{\mu} = \sum_{i=1}^{n}\overline{\mu}_i e_i$. Notice that by Proposition 3.5 the $\overline{\mu}_i$ are distinct and different from $0, n+1$. Then there exists $v \in W$ such that $\overline{\mu} = v(\rho)$. Moreover from the description of $\frac{1}{2}Q^\vee$ it follows that $\mu - \overline{\mu} \in \frac{1}{2}Q^\vee$. Set $\tau = \mu - \overline{\mu}$. It follows that $\lambda + \rho = t_\tau v(\rho)$ and hence, by Lemma 3.3, we have $\chi_\lambda(a) = (-1)^{\ell(v)}$.

Remark 3.2. If $v \in W$ define $(\pm i)^{\sigma_v} = \pm\langle v(\rho), e_{n-i+1}\rangle$ for $i = 1, \ldots, n$. Since, as observed above, W acts as signed permutations on $\{\pm e_1, \ldots, \pm e_n\}$ we have that the map $v \mapsto \sigma_v$ defines a homomorphism from W to the set of signed permutations of $\{\pm 1, \ldots, \pm n\}$. If σ is such a signed permutation then set $|\sigma|$ to be the element of S_n defined by $i^{|\sigma|} = |i^\sigma|$ and set $n_\sigma = |\{i \mid i^\sigma < 0,\ i = 1, \ldots, n\}|$. It is well-known that $\chi(\sigma) = \operatorname{sign}(|\sigma|)(-1)^{n_\sigma}$ is a character of the group of signed permutations. Since $\chi(\sigma_{s_i}) = -1$ it follows at once that $(-1)^{\ell(v)} = \chi(\sigma_v)$. This shows that $\chi_\lambda(a) = \operatorname{sign}(|\sigma_v|)(-1)^{n_{\sigma_v}}$. Observe that $|\sigma_v|$ is the permutation of $\{1, 2, \ldots, n\}$ defined by setting $i^{|\sigma_v|} = |\overline{\mu}_{n-i+1}|$ and $n_{\sigma_v} = |\{i \mid \overline{\mu}_i < 0\}|$.

3.3. Type B_n

We have $\Delta^+ = \{e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{e_i \mid 1 \leq i \leq n\}$, $\rho = \sum_{i=1}^{n}\frac{2n-2i+1}{2}e_i$, $\theta = e_1 + e_2$, hence $h^\vee = 2n-1$. Moreover

$$P = \left\{ \sum_{i=1}^{n} \frac{x_i}{2} e_i \mid x_i \text{ all even or all odd} \right\}, \qquad Q = \sum_{i=1}^{n} \mathbb{Z}e_i,$$

$$P^+ = \left\{ \sum_{i=1}^{n} \lambda_i e_i \in P \mid \lambda_1 \geq \lambda_2 \geq \cdots \lambda_n \geq 0 \right\}.$$

Since $(\theta,\theta) = 2$ we have $(\cdot,\cdot) = \frac{1}{2h^\vee}\langle\cdot,\cdot\rangle$.

Proposition 3.6. *For $\lambda = \sum_{i=1}^{n}\lambda_i e_i \in P^+$ we have*

$$\lambda \in P_{alc} \iff \begin{array}{l} \lambda_i \in \mathbb{Z} \quad \text{for } i = 1, \ldots, n, \\ 2(\lambda_i + n - i) + 1 \not\equiv \pm 2(\lambda_j + n - j) + 1 \bmod 2(2n-1) \\ (i \neq j). \end{array}$$

Proof. By Corollary 3.2 we have that $\lambda \in Q = \sum_{i=1}^{n}\mathbb{Z}e_i$. The second condition follows directly from (3.1) and the observation that $\langle\lambda + \rho, e_i\rangle \notin \mathbb{Z}$ for $i = 1, \ldots, n$. $\square$

Observe that

$$\begin{aligned} \frac{1}{2}Q^\vee &= \frac{1}{2}\left\{\tau \in \mathfrak{h}^*_{\mathbb{R}} \mid (\tau, x) \in \mathbb{Z}\,\forall x \in P^+\right\} \\ &= \frac{1}{2}\left\{\tau \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle\tau, x\rangle \in 2h^\vee\mathbb{Z}\,\forall x \in P^+\right\} \\ &= h^\vee\left\{\tau \in \mathfrak{h}^*_{\mathbb{R}} \mid \langle\tau, x\rangle \in \mathbb{Z}\,\forall x \in P^+\right\} \end{aligned}$$

$$= h^\vee \left\{ \tau = \sum_{i=1}^{n} \tau_i e_i \in \mathfrak{h}_{\mathbb{R}}^* \mid \tau_i \in \mathbb{Z},\ \sum_{i=1}^{n} \tau_i \text{ even} \right\}.$$

Assume that $\lambda \in P_{alc}$ and set $\mu = \lambda + \rho$, so that $\mu = \sum_{i=1}^n \frac{\mu_i}{2} e_i$ with $\mu_i \in 2\mathbb{Z} + 1$ for $i = 1, \ldots, n$. Denote by $\overline{\mu}_i$ the unique element in $\{\pm 1, \pm 3, \ldots, \pm(2n-3)\} \cup \{2n-1\}$ such that $\mu \equiv \overline{\mu}_i \mod 2(2n-1)$ and set $\tilde{\mu} = \sum_{i=1}^n \frac{\overline{\mu}_i}{2} e_i$. Consider $\mu - \tilde{\mu}$: if $\mu - \tilde{\mu} \in \frac{1}{2} Q^\vee$ we set $\overline{\mu} = \tilde{\mu}$. Otherwise let i^* be the unique index such that $\mu_{i^*} = 2n-1$. and set $\overline{\mu} = \tilde{\mu} - \frac{2n-1}{2} e_{i^*}$. This is equivalent to changing $2n-1$ into $-(2n-1)$ in the sequence of remainders. Then we obtain that $\mu - \overline{\mu} \in \frac{1}{2} Q^\vee$. Now we observe that in any case $\overline{\mu} \in W \cdot \rho$, say $\overline{\mu} = v(\rho)$. Hence if we set $\tau = \mu - \overline{\mu}$, we obtain that $\mu = \lambda + \rho = t_\tau v(\rho)$ and $\chi_\lambda(a) = (-1)^{\ell(v)}$.

Remark 3.3. If $v \in W$, we define $(\pm i)^{\sigma_v} = \pm 2\left\langle v(\rho), e_{n-(i-1)/2} \right\rangle$ for $i = 1, 3, \ldots, 2n-1$. Since also in type B the Weyl group acts as signed permutations on $\{\pm e_1, \ldots, \pm e_n\}$ we have that the map $v \mapsto \sigma_v$ defines a homomorphism from W to the set of signed permutations of $\{\pm 1, \pm 3, \ldots, \pm(2n-1)\}$. Arguing as in type C we find that $\chi_\lambda(a) = \operatorname{sign}(|\sigma_v|)(-1)^{n_{\sigma_v}}$ where $|\sigma_v|$ is the permutation of $\{1, 3, \ldots, 2n-1\}$ defined by setting $i^{|\sigma_v|} = |\overline{\mu}_{n-(i-1)/2}|$ and $n_{\sigma_v} = |\{i \mid \overline{\mu}_i < 0\}|$.

3.4. Type D_n

We have $\Delta^+ = \{e_i \pm e_j \mid 1 \le i < j \le n\}$, $\rho = \sum_{i=1}^n (n-i) e_i$, $\theta = e_1 + e_2$, hence $h^\vee = 2n-2$. Moreover

$$P = \left\{ \sum_{i=1}^{n} \frac{\lambda_i}{2} e_i \mid \lambda_i \text{ all even or all odd} \right\},$$

$$Q = \left\{ \sum_{i=1}^{n} \lambda_i e_i \mid \sum_{i=1}^{n} \lambda_i \text{ even} \right\},$$

$$P^+ = \left\{ \sum_{i=1}^{n} \lambda_i e_i \in P \mid \lambda_1 \ge \lambda_2 \ge \cdots \ge |\lambda_n| \right\}.$$

Since $(\theta, \theta) = 2$ we have $(\cdot, \cdot) = \frac{1}{2h^\vee} \langle \cdot, \cdot \rangle$. As in type B_n, Corollary 3.2 implies the following result.

Proposition 3.7. *For $\lambda = \sum_{i=1}^n \lambda_i e_i \in P^+$ we have*

$$\lambda \in P_{alc} \iff \begin{array}{l} \lambda_i \in \mathbb{Z} \quad \text{for } i = 1, \ldots, n, \ \sum_{i=1}^{n} \lambda_i \text{ even}, \\ \lambda_i + n - i \not\equiv \pm(\lambda_j + n - j) \bmod (2n-2) \ (i \neq j). \end{array}$$

Observe that in this case $\frac{1}{2} Q^\vee = h^\vee Q$. Assume that $\lambda \in P_{alc}$ and set $\mu = \lambda + \rho$, so that $\mu = \sum_{i=1}^n \mu_i e_i$ with $\mu_i \in \mathbb{Z}$ for $i = 1, \ldots, n$. Denote by $\overline{\mu}_i$ the unique element in $\{\pm 1, \pm 2, \ldots, \pm(n-2)\} \cup \{0, n-1\}$ such that $\mu \equiv \overline{\mu}_i \mod (2n-2)$ and set $\tilde{\mu} = \sum_{i=1}^n \overline{\mu}_i e_i$. Consider $\mu - \tilde{\mu}$: if $\mu - \tilde{\mu} \in \frac{1}{2} Q^\vee$ we define $\overline{\mu} = \tilde{\mu}$. Otherwise let i^* be the unique index such that $\mu_{i^*} = n-1$ and set $\overline{\mu} = \tilde{\mu} - 2(n-1) e_{i^*}$. This is equivalent to changing $n-1$ into $-(n-1)$ in the sequence of remainders. Then we obtain that $\mu - \overline{\mu} \in \frac{1}{2} Q^\vee$. As in type B_n we have $\overline{\mu} = v(\rho),\ v \in W$ and $\mu = \lambda + \rho = t_\tau v(\rho)$ with $\tau = \mu - \overline{\mu}$. As before, $\chi_\lambda(a) = (-1)^{\ell(v)}$.

Remark 3.4. This time the action of W on ρ defines a homomorphism $v \mapsto |\sigma_v|$ onto the set of permutations of $\{0, 1, 2, \ldots, n-1\}$. The permutation $|\sigma_v|$ is defined by setting $i^{|\sigma_v|} = |\langle v(\rho), e_{n-i} \rangle|$. Since $|\sigma_{s_i}|$ is a simple transposition, it follows as before that $(-1)^{\ell(v)} = \operatorname{sign}(|\sigma_v|)$, hence $\chi_\lambda(a)$ is the sign of the permutation of $\{0, 1, \ldots, n-1\}$ defined by setting $i \mapsto |\overline{\mu}_{n-i}|$.

3.5. Type G_2

It is amusing to work out our Euclidean algorithm for type G_2 also. Following [5] we realize the root system of type G_2 in

$$V = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\right\}.$$

As above $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{R}^3$ and $\{e_1, e_2, e_3\}$ is the canonical basis. We have

$$\Delta = \{\pm(e_i - e_j) \mid 1 \le i, j \le 3\} \cup \left\{\pm(2e_i - e_j - e_k) \mid \{i, j, k\} = \{1, 2, 3\}\right\},$$

$\Pi = \{e_1 - e_2, -2e_1 + e_2 + e_3\}$, so that $\rho = -e_1 - 2e_2 + 3e_3$, $\theta = -e_1 - e_2 + 2e_3$, hence $h^\vee = 4$. Moreover

$$P = Q = V \cap \left(\sum_{i=1}^{3} \mathbb{Z}e_i\right), \qquad P^+ = \left\{\sum_{i=1}^{3} \lambda_i e_i \in P \mid 0 \ge \lambda_1 \ge \lambda_2\right\}.$$

Since $(\theta, \theta) = 6$ we have $(\cdot,\cdot) = \frac{1}{6h^\vee}\langle\cdot,\cdot\rangle$. Set $\varepsilon_i = -1$ for $i = 1, 2$ and $\varepsilon_3 = 1$. Corollary 3.2 implies the following result.

Proposition 3.8. *For $\lambda = \sum_{i=1}^{3} \lambda_i e_i \in P^+$ we have that $\lambda \in P_{alc}$ if and only if*

$$\lambda_i + \varepsilon_i i \not\equiv \lambda_j + \varepsilon_j j \mod (12)\ (i \ne j) \tag{3.5}$$

$$2(\lambda_i + \varepsilon_i i) \not\equiv \lambda_j + \varepsilon_j j + \lambda_k + \varepsilon_k k \mod (12)\ (\{i, j, k\} = \{1, 2, 3\}) \tag{3.6}$$

An easy calculation shows that in this case

$$\frac{1}{2}Q^\vee = 4\left\{\sum_{i=1}^{3} x_i e_i \in Q \mid x_1 \equiv x_2 \equiv x_3 \bmod (3)\right\}. \tag{3.7}$$

Assume that $\lambda \in P_{alc}$ and set $\mu = \lambda + \rho$, so that $\mu = \sum_{i=1}^{3} \mu_i e_i$ with $\mu_i = \lambda_i + \varepsilon_i i \in \mathbb{Z}$ and $\mu_1 + \mu_2 + \mu_3 = 0$. Denote by $[\mu_i]_n = \mu_i + n\mathbb{Z} \in \mathbb{Z}/n\mathbb{Z}$. By the Chinese remainder theorem the map $[\mu_i]_{12} \mapsto ([\mu_i]_3, [\mu_i]_4)$ is an isomorphism.

Since $\sum_{i=1}^{3} \mu_i = 0$, we have obviously that $\sum_{i=1}^{3} [\mu_i]_n = 0$. Relation (3.5) implies that $([\mu_i]_3, [\mu_i]_4) \ne ([\mu_j]_3, [\mu_j]_4)$ if $i \ne j$. Moreover we have the following further conditions:

$$\begin{aligned} &[\mu_j]_4 \text{ cannot be all equal}, \\ &[\mu_j]_4 \ne 0 && j = 1, 2, 3, \\ &[\mu_i]_4 + [\mu_j]_4 \ne 0 && \text{if } i \ne j. \end{aligned} \tag{3.8}$$

Let us check the first condition: if $[\mu_1]_4 = [\mu_2]_4 = [\mu_3]_4 = x$ then

$$-2([\mu_1]_3, x) + ([\mu_2]_3, x) + ([\mu_3]_3, x) = ([\mu_1]_3 + [\mu_2]_3 + [\mu_3]_3, 0) = (0, 0)$$

and this contradicts (3.6). For the second condition suppose $[\mu_i]_4 = 0$. Let j, k be such that $\{i, j, k\} = \{1, 2, 3\}$. Since $[\mu_i]_4 + [\mu_j]_4 + [\mu_k]_4 = 0$ we have that $-2[\mu_i]_4 + [\mu_j]_4 + [\mu_k]_4 = -3[\mu_i]_4 = 0$ hence

$$-2([\mu_i]_3, [\mu_i]_4) + ([\mu_j]_3, [\mu_j]_4) + ([\mu_k]_3, [\mu_k]_4) = ([\mu_i]_3 + [\mu_j]_3 + [\mu_k]_3, 0) = (0, 0).$$

The third condition is obtained in the same way.

Set $S = \{([\mu_i]_3, [\mu_i]_4) \mid i = 1, 2, 3\}$. The conditions in (3.8) imply that there are two possibilities for S: either $S = \{(a, [1]_4), (b, [1]_4), (c, [2]_4)\}$ or $S = \{(a, [3]_4), (b, [3]_4), (c, [2]_4)\}$. Relation (3.5) forces $a \ne b$, so that $a - b = \pm[1]_3$. Define the ordered sets

$$S_1 = ((a, [1]_4), (b, [1]_4), (c, [2]_4)),$$
$$S_2 = ((a, [3]_4), (b, [3]_4), (c, [2]_4)).$$

The algorithm works as follows. Let i^*, j^*, k^* be such that $([\mu_{i^*}]_{12}, [\mu_{j^*}]_{12}, [\mu_{k^*}]_{12}) = S_x$, $x = 1, 2$, and write $\mu_y = 4\tilde{q}_y + \tilde{r}_y$, $y \in \{i^*, j^*, k^*\}$, where the sequence of remainders $\tilde{r}_y$ is (1, 1, 2) if $x = 1$ and (3, 3, 2) if $x = 2$; this of course determines the $\tilde{q}_y$. Now change the sequence of quotiens $\tilde{q}_y$ into a new sequence q_y in such a way to obtain the following new remainders r_y

		$(r_{i^*}, r_{j^*}, r_{k^*})$
$x = 1$	$a - b = [1]_3$	$(1, -3, 2)$
$x = 1$	$a - b = -[1]_3$	$(-3, 1, 2)$
$x = 2$	$a - b = [1]_3$	$(3, -1, -2)$
$x = 2$	$a - b = -[1]_3$	$(-1, 3, -2)$.

This choice implies $q_{i^*} \equiv q_{j^*} \equiv q_{k^*} \mod (3)$. For instance assume $x = 1, a - b = [1]_3$. Since $a = [q_{i^*} + 1]_3$, $b = [q_{j^*}]_3$, and $c = [q_{k^*} + 2]_3$, we have that $0 = a - b - [1]_3 = [q_{i^*} - q_{j^*}]_3$ and, since $\sum_{i=1}^{3} q_i = 0$ we also obtain that $[q_{i^*} - q_{k^*}]_3 = 0$. The other cases are checked similarly.

In all cases we have that, if we set $\tau = \sum_i q_i e_i$ then $\tau \in \frac{1}{2}Q^\vee$. Moreover $\{r_1, r_2, r_3\} = \pm\{1, 2, -3\}$. We now observe that $\sum_i r_i e_i$ is in $W \cdot \rho$. This is an immediate consequence of the general fact that, if $\lambda \in P$ and $(\lambda, \lambda) = (\rho, \rho)$, then $\lambda = w\rho$ for some $w \in W$. (A less attractive proof is obtained by simply listing all twelve elements of $W \cdot \rho$.) Thus $\mu = t_\tau v(\rho)$, where v is the unique element of W such that $v(\rho) = \sum_i r_i e_i$.

A more explicit description of v and the determination of $\chi_\lambda(a)$ will be performed at the end of Section 4.

4. Affine Weyl groups as permutations of $\mathbb{Z}$

In this section we will show how one can construct realizations of the classical affine Weyl groups as permutations of $\mathbb{Z}$ from the knowledge of the orbit $\widehat{W}_q \cdot \lambda$, for an appropriate choice of λ and q. Our treatment takes into account all the representations of classical affine Weyl groups known in literature. We obtain analogous results also for $\tilde{G}_2$.

We shall use the following obvious facts several times.

Fact 4.1. Let $p \in \mathbb{N}^+$ and assume that:

(1) $A = \{a_1, \dots, a_p\}$ is a set of representatives of $\mathbb{Z}/p\mathbb{Z}$;
(2) $f : A \to \mathbb{Z}$, $a_i \mapsto a_i^f$ is a map such that $\{a_1^f, \dots, a_p^f\}$ is still a set of representatives of $\mathbb{Z}/p\mathbb{Z}$.

Then $\tilde{f} : \mathbb{Z} \to \mathbb{Z}$, $a_i + kp \mapsto a_i^f + kp$ for all $k \in \mathbb{Z}$, is a permutation of $\mathbb{Z}$ which extends f.

Fact 4.2. Let $q \in \mathbb{R}^+$ and assume that $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ is q-regular. Then $w \mapsto w(\lambda)$ is a bijection from $\widehat{W}_q$ to the orbit $\widehat{W}_q \cdot \lambda$ of λ under $\widehat{W}_q$.

Types $\tilde{A}_{n-1}$, $\tilde{C}_n$, $\tilde{B}_n$, and $\tilde{D}_n$.

We shall use the following notation: for $a, b \in \mathbb{Z}$ with $a < b$, $c \in \mathbb{Z}$ with $c > 0$, $A \subseteq \mathbb{Z}$ we set

$$[a, b] = \{z \in \mathbb{Z} \mid a \le z \le b\}, \qquad [c] = [1, c]; \qquad \pm A = A \cup -A.$$

For any set N, we denote by $S(N)$ the group of permutations of N.

We realize the classical root systems as in [5], except that we reverse the order of the canonical basis of $\mathbb{R}^n$. Thus if $\{e_i \mid i \in [n]\}$ is the canonical basis of $\mathbb{R}^n$, the simple roots and the highest root are:

for $A_{n-1} : \alpha_i = e_{i+1} - e_i$ for $i = 1, \dots, n-1$; $\theta = e_n - e_1$;
for $C_n : \alpha_1 = 2e_1$, $\alpha_i = e_i - e_{i-1}$ for $i = 2, \dots, n$; $\theta = 2e_n$;
for $B_n : \alpha_1 = e_1, \alpha_i = e_i - e_{i-1}$ for $i = 2, \dots, n$; $\theta = e_{n-1} + e_n$;
for $D_n : \alpha_1 = e_1 + e_2, \alpha_i = e_i - e_{i-1}$ for $i = 2, \dots, n$; $\theta = e_{n-1} + e_n$.

If Δ is of type A_{n-1}, then Δ is a subset of $V = \{\sum_{i=1}^n x_i e_i \mid \sum_{i=1}^n x_i = 0\}$. We extend the faithful action of W on V to $\mathbb{R}^n$ by fixing pointwise $V^\perp$. We also naturally extend the translation action of $\widehat{W}_q$ to $\mathbb{R}^n$.

Set

$$\lambda = \sum_{i \in [n]} i e_i .$$

Observe that

$$\lambda = \begin{cases} n\lambda_0 + \overline{\rho} & \text{in type } A_{n-1}, \\ \rho & \text{in type } C_n, \\ \rho + \omega_1 & \text{in type } B_n, \\ \rho + 2\omega_1 & \text{in type } D_n. \end{cases}$$

We set $Q^\vee_{\langle\cdot,\cdot\rangle} = \sum_{\alpha \in \Pi} \mathbb{Z} \frac{2\alpha}{\langle\alpha,\alpha\rangle}$, thus

$$Q^\vee_{\langle\cdot,\cdot\rangle} = \frac{1}{c} Q^\vee$$

with $c = \langle\theta, \theta\rangle h^\vee$. The element λ is $\frac{p}{c}$-regular where

$$p = \begin{cases} n & \text{in type } A_{n-1}, \\ 2n+1 & \text{in types } B_n, C_n, \text{ and } D_n. \end{cases}$$

In particular, by Fact 4.2, $w \mapsto w(\lambda)$ is a bijection from $\widehat{W}_{\frac{p}{c}}$ to $\widehat{W}_{\frac{p}{c}} \cdot \lambda$. We notice that

$$\widehat{W}_{\frac{p}{c}} = p Q^\vee_{\langle\cdot,\cdot\rangle} \rtimes W,$$

where we identify $pQ^\vee_{\langle\cdot,\cdot\rangle}$ with the group of translations of $\mathbb{R}^n$ by elements of $pQ^\vee_{\langle\cdot,\cdot\rangle}$. We also observe that for types A_n and C_n we have $\frac{p}{c} = \frac{1}{2}$.

We set

$$I = \begin{cases} [n] & \text{in type } A_{n-1}, \\ [-n, n] & \text{in types } B_n, C_n, D_n. \end{cases}$$

Thus I is a set of representatives of $\mathbb{Z}/p\mathbb{Z}$. For types B_n, C_n, and D_n, we set

$$e_0 = 0, \qquad e_{-i} = -e_i$$

for all $i \in [n]$. Thus e_i is defined for all $i \in I$. It is well-known that the finite Weyl group W permutes $\{e_i \mid i \in I\}$.

For all $w \in \widehat{W}_{\frac{p}{c}}$, and $i \in I$, we set

$$i^{w_*} = \langle w(\lambda), e_i \rangle. \tag{4.1}$$

Then, by Fact 4.2, w_* determines w. Since $\langle\cdot,\cdot\rangle$ is W-invariant and W permutes the e_i, for $w \in W$ we have that

$$e_{i^{w_*}} = w^{-1} e_i .$$

This makes clear that $w \mapsto w_*$ is an injective homomorphism of the finite Weyl group W into $S(I)$. In fact, this is the usual permutation representation of W. For A_{n-1}, $\{w_* \mid w \in W\}$ is the whole symmetric group S_n; for both C_n and B_n, $\{w_* \mid w \in W\}$ is the group of all permutations of $[-n, n]$ such that $(-i)^{w_*} = -i^{w_*}$; for D_n, $\{w_* \mid w \in W\}$ is the group of all permutations of $[-n, n]$ such that $(-i)^{w_*} = -i^{w_*}$ and $|\{i \in [n] \mid i^{w_*} < 0\}|$ is even.

We recall that for type A_{n-1} the lattice $Q^\vee_{\langle\cdot,\cdot\rangle}$ is the subgroup of $\sum_{i\in[n]} \mathbb{Z}e_i$ with zero coordinate sum. For type C_n, $Q^\vee_{\langle\cdot,\cdot\rangle} = \sum_{i\in[n]} \mathbb{Z}e_i$, while for both B_n and D_n, $Q^\vee$ is the subgroup of $\sum_{i\in[n]} \mathbb{Z}e_i$ of all elements with even coordinate sum. In particular, since $\widehat{W}_{\frac{p}{c}} = pQ^\vee_{\langle\cdot,\cdot\rangle} \rtimes W$, we obtain in any case that for all $w \in \widehat{W}_{\frac{p}{c}}$ and $i \in I$

$$i^{w_*} \in \mathbb{Z} \quad \text{and} \quad \{i^{w_*} \bmod p \mid i \in I\} = \{i \bmod p \mid i \in I\}.$$

Thus, since I is a set of representatives of $\mathbb{Z}/p\mathbb{Z}$, the map w_* satisfies conditions (1), (2) of Fact 4.1. It follows that w_* extends to a bijection of $\mathbb{Z}$ onto itself, which we still denote by w_*, defined by

$$(i + kp)^{w_*} = i^{w_*} + kp \tag{4.2}$$

for all $i \in I$. We notice that in types C_n, B_n, and D_n, since $0^{w_*} = 0$, we have that $z^{w_*} = z$ for all $z \in p\mathbb{Z} = (2n+1)\mathbb{Z}$ and $w \in \widehat{W}_{\frac{p}{c}}$.

We shall verify that $w \mapsto w_*$ is an injective homomorphism of $\widehat{W}_{\frac{p}{c}}$ into the group of all permutations of $\mathbb{Z}$. It is obvious that w_* is uniquely determined by I^{w_*}, and hence by $w(\lambda)$, so injectivity follows immediately from Fact 4.2. Assume $\widehat{w}, \widehat{u} \in \widehat{W}$, $\widehat{w} = t_\eta w$, $\widehat{u} = t_\tau u$, with $w, u \in W$ and $\tau, \eta \in pQ^\vee_{\langle\cdot,\cdot\rangle}$. Then for $i \in I$

$$i^{\widehat{w}_*} = \langle \widehat{w}(\lambda), e_i\rangle = \langle \eta, e_i\rangle + \langle w(\lambda), e_i\rangle = \langle \eta, e_i\rangle + i^{w_*},$$

and since $\langle \eta, e_i\rangle \in p\mathbb{Z}$ and $e_{i^{w_*}} = w^{-1}(e_i)$, we obtain

$$(i^{\widehat{w}_*})^{\widehat{u}_*} = \langle \eta, e_i\rangle + i^{w_*\widehat{u}_*} = \langle \eta, e_i\rangle + \langle \tau, e_{i^{w_*}}\rangle + i^{w_*u_*} = \langle \eta + w(\tau), e_i\rangle + i^{w_*u_*}.$$

On the other hand, $\widehat{w}\widehat{u} = t_{\eta+w(\tau)}wu$, hence

$$i^{(\widehat{w}\widehat{u})_*} = \langle \eta + w(\tau), e_i\rangle + i^{(wu)_*},$$

and since $i^{(wu)_*} = i^{w_*u_*}$, we obtain that $i^{(\widehat{w}\widehat{u})_*} = i^{\widehat{w}_*\widehat{u}_*}$.

Remark 4.3. Suppose that we are given a homomorphism $w \mapsto w'$ from $\widehat{W}_{\frac{p}{c}}$ to $S(\mathbb{Z})$ such that (4.1) holds. Then, for all $w \in \widehat{W}_{\frac{p}{c}}$, $w'_{|I} = w_{*|I}$. If $w \in \widehat{W}_{\frac{p}{c}}$ and $u = t_\eta$, $\eta \in pQ^\vee_{\langle\cdot,\cdot\rangle}$, then, for $i \in I$,

$$\begin{aligned} i^{(uw)'} &= \langle \eta + w(\lambda), e_i\rangle = \langle \eta, e_i\rangle + i^{w'} \\ &= i^{u'w'} = \langle \eta + \lambda, e_i\rangle^{w'} = (\langle \eta, e_i\rangle + i)^{w'}. \end{aligned}$$

From the explicit description of $Q^\vee_{\langle\cdot,\cdot\rangle}$, it is clear that for all $i \in I$ and $k \in \mathbb{Z}$ there exists $\eta \in Q^\vee_{\langle\cdot,\cdot\rangle}$ such that $\langle \eta, e_i\rangle = k$. It follows that relation (4.2) holds with w' in place of w_*, and therefore $w' = w_*$. Thus the w_* are the only permutations of $\mathbb{Z}$ such that (4.1) holds and $w \mapsto w_*$ is a homomorphism of $\widehat{W}_{\frac{p}{c}}$ into $S(\mathbb{Z})$.

Combining the previous discussion with the results of Section 2 we obtain Lusztig's description of the affine group of type $\tilde{A}_{n-1}$ [12, Section 3.6]. Recall that, in this case, $p = n = h^\vee$, and $\frac{p}{c} = \frac{1}{2}$.

Theorem 4.1. *If Δ is of type A_{n-1}, the map $w \mapsto w_*$ is a permutation representation of $\widehat{W}_{\frac{1}{2}}$ in $S(\mathbb{Z})$. Its image $\{w_* \mid w \in \widehat{W}_{\frac{1}{2}}\}$ is the group of all $f \in S(\mathbb{Z})$ such that*

(1) *$(z+n)^f = z^f + n$ for all $z \in \mathbb{Z}$;*
(2) *$\sum_{i=1}^n i^f = \sum_{i=1}^n i$.*

Proof. The first statement has already been proved. It is clear from definitions that $(z+n)^{w_*} = z^{w_*} + n$ for all $w \in \widehat{W}_{\frac{1}{2}}$. It is also clear that condition (2) holds for all $v \in W$. If $w \in \widehat{W}_{\frac{p}{c}}$, $w = t_\eta v$, $\eta \in nQ^\vee_{\langle\cdot,\cdot\rangle}$, $v \in W$, then

$$\sum_{i=1}^n i^{w_*} = \sum_{i=1}^n \langle \eta, e_i\rangle + \sum_{i=1}^n i^{v_*}.$$

But it is obvious, by the explicit description of $Q^\vee_{\langle\cdot,\cdot\rangle}$, that $\sum_{i=1}^n \langle \eta, e_i\rangle = 0$, hence (2) holds for w.

It remains to prove that if $f \in S(\mathbb{Z})$ satisfies (1), (2), then there exists $w \in \widehat{W}_{\frac{1}{2}}$ such that $f = w_*$. Let f be such that (1), (2) hold and set $a_i = i^f$, $i = 1, \ldots, n$. Then $a_i \neq a_j \mod n$ if $i \neq j$ (otherwise f is not a bijection). It follows from Proposition 2.1 that

$$\sum_{i=1}^n \left(a_i - \frac{1}{n}\sum_{j=1}^n a_j \right) e_i = w(\rho)$$

for some $w \in \widehat{W}_{\frac{1}{2}}$. Observe that $\frac{1}{n}\sum_{j=1}^n a_j = \frac{n+1}{2}$, hence $\sum_{i=1}^n a_i e_i = \frac{n+1}{2}\lambda_0 + w(\rho) = w(\lambda)$. This implies that $\langle w(\lambda), e_i\rangle = a_i$, hence $f = w_*$. □

Remark 4.4. The affine reflection s_0 is equal to $t_{\frac{\theta^\vee}{2}} s_\theta$. Since $\theta = e_n - e_1$ and $\frac{\theta^\vee}{2} = n(e_n - e_1)$, we obtain that

$$\begin{aligned} j^{s_{0*}} &= \langle t_{n(e_n-e_1)} s_\theta(\lambda), e_j\rangle \\ &= \left\langle n(e_n - e_1) + ne_1 + \sum_{i=2}^{n-1} ie_i + e_n, e_j \right\rangle = \begin{cases} 0 & \text{for } j = 1, \\ j & \text{for } 2 \le j \le n-1, \\ n+1 & \text{for } j = n. \end{cases} \end{aligned}$$

Clearly, for $i \in [n-1]$, s_{i*} acts on $[n]$ as the transposition $(i, i+1)$.

Remark 4.5. We may apply formula (2.1) with $\mu = \lambda$. Since positive roots in A_{n-1} are of the form $\alpha_{ij} = e_j - e_i$, $i < j$, we deduce, using (3.3), the following relation

$$k(w, \alpha_{ij}) = \left\lfloor \frac{(w(\lambda), \alpha_{ij})}{\frac{1}{2}} \right\rfloor = \left\lfloor \frac{\langle w(\lambda), e_j - e_i\rangle}{h^\vee} \right\rfloor = \left\lfloor \frac{j^{w_*} - i^{w_*}}{n} \right\rfloor.$$

This is one statement of Theorem 4.1 from [14] (taking into account the different notational conventions). We also have, by (2.2)

$$\ell_{\frac{1}{2}}(w) = \sum_{1 \le i < j \le n} \left| \left\lfloor \frac{j^{w_*} - i^{w_*}}{n} \right\rfloor \right|,$$

a formula which appears, with different derivations, in [2,7,13,14].

Theorem 4.2. *If Δ is of type C_n, then $w \mapsto w_*$ is an injective homomorphism of $\widehat{W}_{\frac{1}{2}}$ into $S(\mathbb{Z})$. Its image $\{w_* \mid w \in \widehat{W}_{\frac{1}{2}}\}$ is the subgroup of all permutations f of $\mathbb{Z}$ such that*

(1) $(-z)^f = -z^f$ *for all* $z \in \mathbb{Z}$;
(2) $(z + k(2n+1))^f = z^f + k(2n+1)$ *for all* $z, k \in \mathbb{Z}$.

Proof. Recall that in this case $p = 2n + 1$ and $\frac{p}{c} = \frac{1}{2}$. It follows directly from definitions that, for all $w \in \widehat{W}_{\frac{p}{c}}$, w_* satisfies conditions (1), (2). It remains to prove that all permutations of $\mathbb{Z}$ which satisfy conditions (1), (2) lie in $\{w_* \mid w \in \widehat{W}_{\frac{p}{c}}\}$.

The anti-symmetry condition (1) implies in particular that $0^f = 0$, hence any odd $f \in S(\mathbb{Z})$ satisfies (2) if and only if it permutes the non zero cosets in $\mathbb{Z}/p\mathbb{Z}$. This means that $\{0, \pm 1^f, \ldots, \pm n^f\}$ is a set of representative of $\mathbb{Z}/p\mathbb{Z}$ or, equivalently, that

$$i^f \not\equiv 0, \qquad i^f \pm j^f \not\equiv 0 \bmod p, \qquad \text{for } 1 \le i < j \le n \tag{4.3}$$

(notice that p being odd, $i^f \not\equiv 0 \bmod p$ if and only if $2i^f \not\equiv 0 \bmod p$).

Now we recall that $P = Q^\vee_{\langle\cdot,\cdot\rangle}$ and $\lambda = \rho$, so that, by Lemma 2.2, $\lambda + Q^\vee_{\langle\cdot,\cdot\rangle} \cap C_{\frac{p}{c}} = \{\lambda\}$. Since $\widehat{W}_{\frac{p}{c}} = pQ^\vee_{\langle\cdot,\cdot\rangle} \rtimes W$, it is clear that $\widehat{W}_{\frac{p}{c}}$ acts on $\lambda + P$. By Proposition 2.1 we obtain that $\widehat{W}_{\frac{p}{c}} \cdot \lambda$ is the set of all $\mu \in \lambda + P$ such that $(\mu, \alpha) \notin \frac{p}{c}\mathbb{Z}$ or, equivalently, $\langle \mu, \alpha\rangle \notin pQ^\vee_{\langle\cdot,\cdot\rangle}$ for each root α. By the explicit description of the root system, this means that, if $\mu = \sum_{i=1}^n \mu_i e_i$, then

$$2\mu_i, \ \mu_i \pm \mu_j \notin p\mathbb{Z} \quad \text{for } 1 \le i < j \le n.$$

Comparing the above conditions with (4.3), we deduce that for each $f \in S(\mathbb{Z})$ such that (1), (2) hold, there exists $w \in \widehat{W}_{\frac{p}{c}}$ such that $\sum_{i=1}^n i^f e_i = w(\lambda)$, and therefore such that $f = w_*$. □

Remark 4.6. In our setting, the affine reflection s_0 is equal to $t_{\frac{2n+1}{c}\theta^\vee} s_\theta$. Since $\theta = 2e_n$ and $\frac{1}{c}\theta^\vee = \frac{1}{2}\theta = e_n$, we obtain that

$$j^{s_{0*}} = \langle t_{(2n+1)e_n} s_\theta(\lambda), e_j\rangle = \langle e_n + \lambda, e_j\rangle = \begin{cases} j & \text{for } 1 \le j < n, \\ n+1 & \text{for } 1 \le j < n. \end{cases}$$

Clearly, for $i \in [n-1]$, s_{i*} acts on $[n]$ as the transposition $(i, i-1)$, while s_{n*} acts on $[-n, n]$ as the transposition $(-n, n)$.

Remark 4.7. The representation of the Weyl group of type $\tilde{C}_n$ as a subgroup of $S(\mathbb{Z})$ obtained in Theorem 4.2 coincides with the one presented by Bedard [4]. A different representation appears in literature (see [14,13]). We can also get this representation in our framework. Indeed, we note that, with the notation of Lemma 2.2, there are two possible values of q verifying Eq. (2.3): $2n+1$ and $2n+2$. Hence we can define an injective homomorphism $w \mapsto w_{**}$ of $\widehat{W}_{\frac{2n+2}{c}}$ into $S(\mathbb{Z})$ setting

$$i^{w_{**}} = \langle w(\lambda), e_i \rangle \qquad \text{for } i \in [-n, n], \ \pm (n+1)^{w_{**}} = \pm(n+1),$$
$$(i + k(2n+2))^{w_{**}} = i^{w_{**}} + k(2n+2).$$

Then s_{i**} and s_{i*} have the same action on $[-n, n]$, for $i \in [n]$. The action of s_{0**} is defined by $j^{s_{0**}} = j$ for $1 \le j < n$, $n^{s_{0*}} = n+2$, and by the condition of compatibility with translation by $2n+2$.

Theorem 4.3. *If Δ is of type B_n then $w \mapsto w_*$ is an injective homomorphism of $\widehat{W}_{\frac{p}{c}}$ into $S(\mathbb{Z})$. Its image $\{w_* \mid w \in \widehat{W}_{\frac{p}{c}}\}$ is the subgroup of all permutations f of $\mathbb{Z}$ such that*

(1) $(-z)^f = -z^f$ *for all* $z \in \mathbb{Z}$;
(2) $(z + k(2n+1))^f = z^f + k(2n+1)$ *for all* $z, k \in \mathbb{Z}$;
(3) $\sum_{i=1}^n i^f \equiv \binom{n+1}{2} \mod 2$.

Proof. It remains to prove that w_* satisfies (3) for all $w \in \widehat{W}_{\frac{p}{c}}$ and that each $f \in S(\mathbb{Z})$ such that (1), (2), (3) hold is equal to some w_*, $w \in \widehat{W}_{\frac{p}{c}}$.

If $w \in W$, then $\{1^{w_*}, \dots, n^{w_*}\}$ differs from $[n]$ at most in the sign of elements, hence it is clear that

$$\binom{n+1}{2} = \sum_{i=1}^n i \equiv \sum_{i=1}^n i^{w_*} \mod 2.$$

Since $Q^\vee_{\langle\cdot,\cdot\rangle}$ is the the set of all elements in $\sum_{i=1}^n \mathbb{Z}e_i$ with even coordinate sum and $\widehat{W}_{\frac{p}{c}} = pQ^\vee_{\langle\cdot,\cdot\rangle} \rtimes W$, it is clear that w_* satisfies (3) for all $w \in \widehat{W}_{\frac{p}{c}}$.

The above argument also shows that $\lambda + Q^\vee_{\langle\cdot,\cdot\rangle}$ is $\widehat{W}_{\frac{p}{c}}$-stable. Moreover, it is easily seen that $\lambda + Q^\vee_{\langle\cdot,\cdot\rangle} \cap C_{\frac{p}{c}} = \{\lambda\}$. Thus we may apply Proposition 2.1, with $L = Q^\vee_{\langle\cdot,\cdot\rangle}$, so as to obtain that $\widehat{W}_{\frac{p}{c}} \cdot \lambda$ is the set of all $\mu \in \lambda + Q^\vee_{\langle\cdot,\cdot\rangle}$ such that $\langle \mu, \alpha \rangle \notin p\mathbb{Z}$, for each root α. From the explicit description of $Q^\vee_{\langle\cdot,\cdot\rangle}$ and of the root system, we obtain that, if $\mu = \sum_{i=1}^n \mu_i e_i \in \sum_{i=1}^n \mathbb{Z}e_i$, then $\mu \in \widehat{W}_{\frac{p}{c}} \cdot \lambda$ if and only if

$$\sum_{i=1}^n \mu_i \equiv \binom{n+1}{2} \mod 2, \quad \text{and} \quad \mu_i,\ \mu_i \pm \mu_j \notin p\mathbb{Z} \quad \text{for } 1 \le i < j \le n.$$

Now it is clear that the same argument used in the proof of Theorem 4.2 shows that if $f \in S(\mathbb{Z})$ satisfies condition (1), then condition (2) is equivalent to (4.3). We easily conclude that each $f \in S(\mathbb{Z})$ such that (1), (2), (3) hold is equal to w_* for some $w \in \widehat{W}_{\frac{p}{c}}$. □

Remark 4.8. Condition (3) in Theorem 4.3 can be replaced by the following one:

(3′) $\sum_{i=1}^n \left(i^f - \overline{i^f}\right) \in 2(2n+1)\mathbb{Z}$.
or, equivalently,
(3″) $|\{i \le n \mid i^f > n\}|$ is even.

In fact, if we set $i^f = k_i(2n+1) + \overline{i^f}$, then we have $\sum_{i=1}^n \overline{i^f} \equiv \binom{n+1}{2}$ hence $\sum_{i=1}^n i^f \equiv \binom{n+1}{2}$ if and only if $\sum_{i=1}^n k_i$ is even, which is equivalent to condition (3′). Moreover, $\{j \le n \mid j^f > n\} = \sum_{i=1}^n |k_i| \equiv \sum_{i=1}^n k_i$, and since $k_i(2n+1) = i^f - \overline{i^f}$, we obtain that (3″) is equivalent to (3′) and hence to (3).

We finally deal with type D_n. In this case, we identify $\widehat{W}_{\frac{p}{c}}$ with a subgroup of its $\tilde{B}_n$-analog. Namely, if W_{B_n} is the finite Weyl group for type B_n, we may identify the finite Weyl group of D_n with the subgroup of W_{B_n}

$$W' = \{w \in W_{B_n} \mid i^{w_*} < 0 \text{ for an even number of } i \in [n]\}$$

and we set

$$\widehat{W}_{\frac{p}{c}} = pQ^{\vee}_{\langle\cdot,\cdot\rangle} \rtimes W'.$$

For $j \in \mathbb{Z}$ we denote by $\overline{j}$ its residue modulo p. It is clear that if $w \in \widehat{W}_{\frac{p}{c}}$, $w = t_\eta v$, with $\eta \in pQ^{\vee}_{\langle\cdot,\cdot\rangle}$ and $v \in W'$, then $i^{v_*} = \overline{i^{w_*}}$ for all $i \in [n]$, and $\eta = \sum_{i=1}^{n}(i^{\widehat{w}_*} - i^{w_*})e_i$, hence from Theorem 4.3 we directly obtain the following result.

Theorem 4.4. *If Δ is of type D_n then $w \mapsto w_*$ is an injective homomorphism of $\widehat{W}_{\frac{p}{c}}$ into $S(\mathbb{Z})$. Its image $\{w_* \mid w \in \widehat{W}_{\frac{p}{c}}\}$ is the subgroup of all permutations f of $\mathbb{Z}$ such that*

(1) $(-z)^f = -z^f$ *for all* $z \in \mathbb{Z}$*;*
(2) $(z + k(2n+1))^f = z^f + k(2n+1)$ *for all* $z, k \in \mathbb{Z}$*;*
(3) $\sum_{i=1}^{n} i^f \equiv \binom{n+1}{2} \bmod 2$*, and* $|\{i \in [n] \mid \overline{i^f} < 0\}|$ *is even.*

Remark 4.9. For both types B_n and D_n we find that $s_0 = t_{(2n+1)\theta}s_\theta$ and hence

$$s_0(\lambda) = (2n+1)\theta + \lambda - \langle\lambda, \theta\rangle\theta = \lambda + 2\theta = \sum_{i=1}^{n-2} ie_i + (n+1)e_{n-1} + (n+2)e_n.$$

It follows that

$$i^{s_{0*}} = i \quad \text{for } i \in [n-2], \qquad (n-1)^{s_{0*}} = n+1, \qquad n^{s_{0*}} = n+2.$$

Since $n+1 = -n+(2n+1)$, and $n+2 = -(n-1)+(2n+1)$, we have that $(n+1)^{s_{0*}} = n-1$, and $(n+2)^{s_{0*}} = n$. Thus s_{0*} acts on $\{-n+2, \dots, n+2\}$ as the product of transpositions $(n-1, n+1)(n, n+2)$. For $i \in [n]$, the action of s_{i*} on $[-n, n]$ is the usual one, hence, for $2 \le i \le n$, s_{i*} is the product of transpositions $(i-1, i)(-(i-1), -i)$; s_{1*} is the transposition $(1, -1)$ for B_n, while is the product of transpositions $(1, -2)(2, -1)$ for D_n.

Type G_2.

In this case we shall define an injective homorphism of $\widehat{W}$ $(=\widehat{W}_1)$ into $S(\mathbb{Z})$. We omit everywhere the subscript 1, so T is the subgroup of translations of $\widehat{W}$ and C is the fundamental alcove. The rest of the notation is the same as Section 3.5. The map $w \mapsto w_*$, $\widehat{W} \to S(\mathbb{Z})$, we are going to define is determined by $w(\rho)$. Injectivity will be an immediate consequence of the fact that $\rho \in C$.

We set $e_{-i} = -e_i$ for $i \in [3]$, $\varepsilon_i = -1$ for $i = \pm 1, \pm 2$, $\varepsilon_3 = \varepsilon_{-3} = 1$. Then we define, for all $w \in \widehat{W}$,

$$0^{w_*} = 0, \qquad i^{w_*} = \varepsilon_i \langle w(\rho), e_i\rangle \quad \text{for } i \in \pm[3].$$

If $v \in W$, and $i \in \pm[3]$, then there exist unique $j \in \pm[3]$ and $v_i \in V^\perp$ such that $v(e_i) = e_j + v_i$. Then for $w = v^{-1}$ we have $\langle w(\rho), e_i\rangle = \langle \rho, v(e_i)\rangle = \varepsilon_j j$, hence

$$w^{-1}(e_i) = \varepsilon_i \varepsilon_{i^{w_*}} e_{i^{w_*}} + v_i,$$

with $v_i \in V^\perp$. It follows directly that for all $w, w' \in W$, $(ww')_* = w_* w'_*$, hence $w \mapsto w^*$ is an injective homomorphism of W into the set of all permutations of $[-3, 3]$.

It is easily seen that the image W_* of W under this homomorphism is the set (group) of all permutations f of $[-3, 3]$ such that $(-i)^f = -i^f$ and $\sum_{i \in [3]} \varepsilon_i i^f = 0$. Notice that this last condition is equivalent to $\{-1^f, -2^f, 3^f\}$ being equal to either $\{-1, -2, 3\}$ or $\{1, 2, -3\}$. By restricting maps to $\pm[3]$ we obtain that the map $w \mapsto w_*$ defines an isomorphism between W and the group of functions $f : \pm[3] \to \pm[3]$ such that $(-i)^f = -i^f$ and $\{-1^f, -2^f, 3^f\} = \pm\{1, 2, -3\}$.

We recall that

$$Q^\vee = 8\left\{\sum_{i=1}^{3} x_i e_i \in Q \mid x_1 \equiv x_2 \equiv x_3 \bmod 3\right\},$$

in particular, for each $t \in T$ and $i \in \pm[3]$, $i^{t_*} \equiv i \bmod 8$. For all $w \in \widehat{W}$, we define $4^{w_*} = 4$. Then it is clear that w_* maps the set of representatives $[-3, 4]$ of $\mathbb{Z}/8\mathbb{Z}$ into some set of representatives of $\mathbb{Z}/8\mathbb{Z}$, hence Fact 4.1 applies and

w_* can be extended to a bijection w_* of $\mathbb{Z}$ onto itself by setting $(i+8k)^{w_*} = i^{w_*} + 8k$ for all $k \in \mathbb{Z}$. Notice that w_* fixes pointwise $4\mathbb{Z}$.

We next verify that $w \mapsto w_*$ is an injective homomorphism of the whole $\widehat{W}$ into the group of all permutations of $\mathbb{Z}$. It is obvious that w_* is determined by $[-3,3]^{w_*}$, hence by $w(\rho)$, so, as remarked above, injectivity is immediate. Assume $\widehat{w} \in \widehat{W}$, $\widehat{w} = t_\eta w$ with $w \in W$ and $\eta \in Q^\vee$.Then for $i \in \pm[3]$

$$i^{\widehat{w}_*} = \langle \widehat{w}(\rho), e_i\rangle = \varepsilon_i\langle \eta, e_i\rangle + \varepsilon_i\langle w(\rho), e_i\rangle = \varepsilon_i(\eta, e_i) + i^{w_*}.$$

Let also $\widehat{u} \in \widehat{W}$, $\widehat{u} = t_\tau u$ with $u \in W$ and $\tau \in Q^\vee$.Then

$$\begin{aligned}(i^{\widehat{w}_*})^{\widehat{u}_*} &= \varepsilon_i\langle \eta, e_i\rangle + i^{w_*\widehat{u}_*} = \varepsilon_i\langle \eta, e_i\rangle + \varepsilon_{i^{w_*}}\langle \tau, e_{i^{w_*}}\rangle + i^{w_*u_*}\\ &= \varepsilon_i\langle \eta, e_i\rangle + \varepsilon_i\langle \tau, w^{-1}e_i\rangle + i^{w_*u_*} = \varepsilon_i\langle \eta + w(\tau), e_i\rangle + i^{w_*u_*}.\end{aligned}$$

On the other hand we have $\widehat{w}\widehat{u} = t_{\eta+w(\tau)}wu$, hence

$$i^{(\widehat{w}\widehat{u})_*} = \varepsilon_i\langle \eta + w(\tau), e_i\rangle + i^{(wu)_*},$$

and since $i^{(wu)_*} = i^{w_*u_*}$, we finally obtain that $i^{(\widehat{w}\widehat{u})_*} = i^{\widehat{w}_*\widehat{u}_*}$. Thus we have that $\widehat{W}$ is isomorphic to the subgroup $\widehat{W}_* = \{w_* \mid w \in \widehat{W}\}$ of permutations of $\mathbb{Z}$.

For $a \in \mathbb{Z}$ let $\overline{a}$ be the representative of a mod 8 in $[-3,4]$. Then using the explicit description of $Q^\vee$ given above, we obtain the following permutation representation of $\widehat{W}$.

Theorem 4.5. *If Δ is of type G_2, then $\widehat{W}$ is isomorphic to the group of all permutations f of $\mathbb{Z}$ such that*

(1) $(-z)^f = -z^f$ *for all* $z \in \mathbb{Z}$;
(2) $(z+8k)^f = z^f + 8k$ *and* $(4k)^f = 4k$ *for all* $z, k \in \mathbb{Z}$;
(3) $-1^f - 2^f + 3^f = 0$, $\{\overline{-1^f}, \overline{-2^f}, \overline{3^f}\} = \{-1,-2,3\}$ *or* $\{\overline{-1^f}, \overline{-2^f}, \overline{3^f}\} = \{1,2,-3\}$, *and* $-(1^f - \overline{1^f}) \equiv -(2^f - \overline{2^f}) \equiv (3^f - \overline{3^f})$ mod 3.

Proof. The statement follows directly from the above discussion. □

Remark 4.10. From the explicit description of α_1, it is clear that s_{1*} acts on $[-3,4]$ as $(1,2)(-1,-2)$. For s_2 we have $s_2(\rho) = \rho - \alpha_2 = e_1 - 3e_2 + 2e_3$, hence s_{2*} acts on $[-3,4]$ as $(1,-1)(2,3)(-2,-3)$.

For $w \in W$, let $|w_*|$ be the permutation of [3] defined by $i^{|w_*|} = |i^{w_*}|$, for $i = 1,2,3$. Then from the explicit description of s_{1*} and s_{2*} it is clear that, for $w \in W$, the parity of $\ell(w_*)$, and hence of $\ell(w)$, is exactly the sign of $|w_*|$. This observation, combined with Lemma 3.3 and the discussion developed in Section 3.5, solves the problem of determining explicitly $\chi_\lambda(a)$, $\lambda \in P_{alc}$. With this identification $(-1)^{\ell(v)}$ is the sign of the permutation $|v_*|$ hence, if $\lambda \in P_{alc}$ and we write $\lambda + \rho = \mu = \tau + \sum_i r_i e_i$ as described in Section 3.5, then $\chi_\lambda(a)$ is the sign of the permutation $i \mapsto |r_i|$, $i = 1,2,3$.

Finally, we have $s_0(\rho) = \theta^\vee + s_\theta(\rho) = 8\theta + \rho - 3\theta = \rho + 5\theta = -6e_1 - 7e_2 + 13e_3$, hence s_{0*} is the unique permutation f of $\mathbb{Z}$ which has properties (1) and (2) of Theorem 4.5 and such that $1^f = 6$, $2^f = 7$, $3^f = 13$.

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