Bounds on eigenvalues of the Hadamard product and the Fan product of matrices

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Abstract

We prove an upper bound for the spectral radius of the Hadamard product of nonnegative matrices and a lower bound for the minimum eigenvalue of the Fan product of $M$-matrices. These improve two existing results.

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1. Introduction

For a positive integer $n$, $N$ denotes the set $\{1, 2, \ldots, n\}$ throughout.

We write $A \succeq B$ if $a_{ij} \geq b_{ij}$ for all $i, j \in N$. We write $A \succeq 0$ if all $a_{ij} \geq 0$. If $A \succeq 0$, we say $A$ is a nonnegative matrix, and if $A > 0$, we say that $A$ is a positive matrix. The spectral radius of $A$ is denoted by $\rho(A)$. If $A$ is a nonnegative matrix, the Perron–Frobenius theorem guarantees that $\rho(A) \in \sigma(A)$ where $\sigma(A)$ is the set of all eigenvalues of $A$.

An $n \times n$ matrix $A$ is said to be reducible if there exists a permutation matrix $P$ such that

$$P^T A P = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix},$$
where \( B, D \) are square matrices of order at least one. If \( A \) is not reducible, then it is called irreducible. Note that any \( 1 \times 1 \) matrix is irreducible.

Let \( A \) be an irreducible nonnegative matrix. It is well known that there exist positive vectors \( u \) and \( v \) such that \( Au = \rho u \), and \( v^T A = \rho v^T \), \( u \) and \( v \) being called right and left Perron eigenvectors of \( A \), respectively.

Denote by \( R^{n\times n} \) the set of all \( n \times n \) real matrices. For two real matrices \( A, B \in R^{n\times n} \), the Hadamard product of \( A = [a_{ij}] \) and \( B = [b_{ij}] \) is the matrix \( A \circ B \equiv [a_{ij}b_{ij}] \).

In [2, p. 358], there is an inequality: if \( A, B \in R^{n\times n} \), \( A \geq 0 \) and \( B \geq 0 \), then \( \rho(A \circ B) \leq \rho(A)\rho(B) \). But this inequality can be very weak in some cases. Consider the example, \( A = I \), \( B = J \), the matrix of all ones. Then \( \rho(A \circ B) = \rho(A) = 1 \ll \rho(A)\rho(B) = n \) when \( n \) is very large.

For two nonnegative matrices \( A, B \in R^{n\times n} \), we exhibit an upper bound for \( \rho(A \circ B) \) in Section 2. The bound is sharper than the bound \( \rho(A)\rho(B) \) in [2, p. 358].

We denote by \( Z_n \) the class of all \( n \times n \) real matrices all of whose off-diagonal entries are nonpositive. An \( n \times n \) matrix \( A \) is called an \( M \)-matrix if there exists an \( n \times n \) nonnegative matrix \( B \) and some nonnegative real number \( \lambda \) such that \( A = \lambda I - B \) and \( \lambda \geq \rho(B) \), \( I \) being the identity matrix; if \( \lambda > \rho(B) \), we call \( A \) a nonsingular \( M \)-matrix; if \( \lambda = \rho(B) \), we call \( A \) a singular \( M \)-matrix. Denote by \( \mathbb{M}_n \) the set of nonsingular \( M \)-matrices.

Let \( A \in Z_n \) and denote \( \tau(A) = \min\{\Re(\lambda) : \lambda \in \sigma(A)\} \). Basic for our purpose are the following simple facts (see Problems 16, 19 and 28 in Section 2.5 of [2]):

(i) \( \tau(A) \in \sigma(A) \); \( \tau(A) \) is called the minimum eigenvalue of \( A \).
(ii) If \( A, B \in \mathbb{M}_n \), and \( A \geq B \), then \( \tau(A) \geq \tau(B) \).
(iii) If \( A \in \mathbb{M}_n \), then \( \rho(A^{-1}) \) is the Perron eigenvalue of the nonnegative matrix \( A^{-1} \), and \( \tau(A) = \frac{1}{\rho(A^{-1})} \) is a positive real eigenvalue of \( A \).

Let \( A \) be an irreducible nonsingular \( M \)-matrix. It is well known that there exist positive vectors \( u \) and \( v \) such that \( Au = \tau(A)u \), and \( v^T A = \tau(A)v^T \), \( u \) and \( v \) being called right and left Perron eigenvectors of \( A \), respectively.

For two real matrices \( A, B \in \mathbb{M}_n \), the Fan product of \( A \) and \( B \) is denoted by \( A \star B = C = [c_{ij}] \in \mathbb{M}_n \) and is defined by

\[
c_{ij} = \begin{cases} -a_{ij}b_{ij}, & \text{if } i \neq j, \\ a_{ii}b_{ii}, & \text{if } i = j. \end{cases}
\]

If \( A, B \in \mathbb{M}_n \), then \( A \star B \) is also an \( M \)-matrix. For two nonsingular \( M \)-matrices \( A, B \in \mathbb{M}_n \), we exhibit a lower bound for \( \tau(A \star B) \) in Section 3. The bound is sharper than the bound \( \tau(A)\tau(B) \) in [2, p. 359].

2. An upper bound for the spectral radius of the Hadamard product of nonnegative matrices

In this section, we will exhibit an upper bound for \( \rho(A \circ B) \).

**Lemma 1.** If \( P \) is an \( n \times n \) irreducible nonnegative matrix, and \( Pz \leq kz \) for a nonnegative nonzero vector \( z \), then \( \rho(P) \leq k \).
Proof. $P$ is an $n \times n$ irreducible nonnegative matrix, so $A$ has a positive left Perron vector $y$; that is, $y^TP = \rho(P)y^T$. We are given that $Pz \leqslant kz$, so $Pz-kz \leqslant 0$ and hence $y^T(Pz-kz) = (\rho(P)-k)y^Tz \leqslant 0$. Since $y^Tz > 0$, we have $\rho(P) \leqslant k$. □

Lemma 2 [1]. Let $A \in \mathbb{R}^{n \times n}$ be a given nonnegative matrix. Then either $A$ is irreducible or there exists a permutation $P$ such that

$$P^TAP = \begin{pmatrix} A_1 & A_{12} & \cdots & A_{1k} \\ A_2 & \cdots & A_{2k} \\ \vdots & \ddots & \vdots \\ A_k & & & \end{pmatrix}$$

in which each $A_i$ is irreducible, $i = 1, \ldots, k$.

Remark. Eq. (1) is called the irreducible normal form. Note that $\sigma(A) = \bigcup_{i=1}^{k} \sigma(A_i)$ and $\rho(A) = \max\{\rho(A_i) : i = 1, \ldots, k\}$.

Lemma 3 [1]. $A$ be an $n \times n$ nonnegative matrix. If $A_k$ is a principal submatrix of $A$, then $\rho(A_k) \leqslant \rho(A)$. If $A$ is irreducible and $A_k \neq A$, then $\rho(A_k) < \rho(A)$.

Theorem 4. If $A, B$ are two $n \times n$ nonnegative matrices, then

$$\rho(A \circ B) \leqslant \max_{1 \leq i \leq n} \{2a_{ii}b_{ii} + \rho(A)\rho(B) - a_{ii}\rho(B) - b_{ii}\rho(A)\}. \tag{2}$$

Proof. It is quite evident that (2) holds with equality for $n = 1$.

Below we assume that $n \geq 2$. Let us distinguish two cases.

Case 1. If $A \circ B$ is irreducible, then $A$ and $B$ are irreducible. Lemma 3 yields that

$$\rho(A) - a_{ii} > 0 \quad \forall i \in N$$

and

$$\rho(B) - b_{ii} > 0 \quad \forall i \in N.$$

Let $u = (u_i)$, $v = (v_i)$ be the right Perron eigenvectors of $A$ and $B^T$. Then we have

$$u_ia_{ii} + \sum_{j \neq i} u_j a_{ij} = \rho(A)u_i \quad \forall i \in N.$$

Also we have

$$v_jb_{jj} + \sum_{i \neq j} v_i b_{ij} = \rho(B)v_j \quad \forall j \in N.$$

Thus

$$b_{ij} \leqslant \frac{[\rho(B) - b_{jj}]v_j}{v_i} \quad \text{for all } i \neq j.$$

Let $z$ be the vector $(z_i)$, where

$$z_i = \frac{u_i}{[\rho(B) - b_{ii}]v_i} > 0 \quad \forall i \in N.$$
We define $C = A \circ B$. For any $i \in N$,
\[
(Cz)_i = a_{ii} b_{ii} z_i + \sum_{j \neq i} a_{ij} b_{ij} z_j \\
\leq a_{ii} b_{ii} z_i + \sum_{j \neq i} a_{ij} \left[ \rho(B) - b_{jj} \right] v_j u_j / v_i \left[ \rho(B) - b_{jj} \right] v_j \\
= a_{ii} b_{ii} z_i + \frac{1}{v_i} \sum_{j \neq i} a_{ij} u_j \\
= a_{ii} b_{ii} z_i + \frac{1}{v_i} (\rho(A) - a_{ii}) u_i \\
= a_{ii} b_{ii} z_i + (\rho(A) - a_{ii})(\rho(B) - b_{ii}) z_i \\
= (2a_{ii} b_{ii} + \rho(A) \rho(B) - a_{ii} \rho(B) - b_{ii} \rho(A)) z_i.
\]

By Lemma 1, this shows that
\[
\rho(A \circ B) \leq \max_{1 \leq i \leq n} \{ 2a_{ii} b_{ii} + \rho(A) \rho(B) - a_{ii} \rho(B) - b_{ii} \rho(A) \}.
\]

**Case 2.** Let $A \circ B$ be reducible. If we denote by $T$ the $n \times n$ permutation matrix $(t_{ij})$ with
\[
t_{12} = t_{23} = \cdots = t_{n-1,n} = t_{n1} = 1,
\]
the remaining $t_{ij}$ zero, then both $A + \varepsilon T$ and $B + \varepsilon T$ are irreducible nonnegative matrices for any chosen positive real number $\varepsilon$. Now we substitute $A + \varepsilon T$ and $B + \varepsilon T$ for $A$ and $B$, respectively in the previous case, and then letting $\varepsilon \to 0$, the result follows by continuity. □

In fact when $A \circ B$ is reducible and $n \geq 2$, we can get another result. We define $D = A \circ B$. From Lemma 2, we get that there exists a permutation $P$ such that
\[
P^T DP = \begin{pmatrix}
D_1 & D_{12} & \cdots & D_{1k} \\
D_{2} & \cdots & D_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
D_k & & & D_k
\end{pmatrix}
\]
in which each $D_i = A_i \circ B_i$ is irreducible, $i = 1, \ldots, k$. Then
\[
\rho(A \circ B) = \max \{ \rho(D_i) = \rho(A_i \circ B_i) : i = 1, \ldots, k \}.
\]
If
\[
\rho(A \circ B) = \rho(D_{i_0}) = \rho(A_{i_0} \circ B_{i_0}).
\]
If $D_{i_0}$ is a $1 \times 1$ principal submatrix of $D$, then
\[
\rho(A \circ B) = \rho(D_{i_0}) = a_{ii} b_{ii}
\]
for some $i \in N$.
If $D_{i_0}$ is a $t \times t$ principal submatrix of $D$ and $2 \leq t \leq n - 1$, then
\[
\rho(A \circ B) \leq \max_{i \in \alpha} \{ 2a_{ii} b_{ii} + \rho(A[\alpha]) \rho(B[\alpha]) - a_{ii} \rho(B[\alpha]) - b_{ii} \rho(A[\alpha]) \},
\]
where $A[\alpha]$ and $B[\alpha]$ are two principal submatrices of $A$ and $B$ with the same index set $\alpha \subset N$ and $|\alpha| = t$ (|\alpha| denotes the cardinality of the set \alpha).
Remark 1. We have
\[
\rho(A)\rho(B) - (2a_{ii}b_{ii} + \rho(A)\rho(B) - a_{ii}\rho(B) - b_{ii}\rho(A)) \\
= (\rho(A) - a_{ii})b_{ii} + (\rho(B) - b_{ii})a_{ii}.
\]
If \(A\) is irreducible, then
\[
\rho(A) - a_{ii} > 0 \quad \forall i \in N.
\]
If \(A\) is reducible, then
\[
\rho(A) - a_{ii} \geq 0 \quad \forall i \in N.
\]
Thus
\[
(\rho(A) - a_{ii})b_{ii} + (\rho(B) - b_{ii})a_{ii} \geq 0.
\]
So we can get
\[
\max_{1 \leq i \leq n} [2a_{ii}b_{ii} + \rho(A)\rho(B) - a_{ii}\rho(B) - b_{ii}\rho(A)] \leq \rho(A)\rho(B).
\]
Therefore, the bound in (2) is sharper than the known one \(\rho(A)\rho(B)\) in [2, p. 358].

Now we consider again the numerical example in the introduction. Let \(A = I, B = J\), the matrix of all ones. It is easy to observe that
\[
\rho(A) = 1, \quad \rho(B) = n, \\
\rho(A \circ B) = 1,
\]
and
\[
\max_{1 \leq i \leq n} [2a_{ii}b_{ii} + \rho(A)\rho(B) - a_{ii}\rho(B) - b_{ii}\rho(A)] = 2 + n - n - 1 = 1.
\]
It is surprise to see that our bound is the real spectral radius of \(\rho(A \circ B)\).

Remark 2. If we only consider the case when \(A \circ B\) is reducible and \(n \geq 2\), we can get a similar result. Since
\[
a_{ii}b_{ii} \leq \rho(A)\rho(B) \quad \forall i \in N
\]
and by Remark 1 and Lemma 3 we can get
\[
\max_{i \in \alpha} [2a_{ii}b_{ii} + \rho(A[\alpha])\rho(B[\alpha]) - a_{ii}\rho(B[\alpha]) - b_{ii}\rho(A[\alpha])] \\
\leq \rho(A[\alpha])\rho(B[\alpha]) \leq \rho(A)\rho(B).
\]
From Theorem 4 we can get the following corollary:

Corollary 5. Let \(A, B\) be two \(n \times n\) nonnegative matrices. We have the following inequalities:
\[
|\det(A \circ B)| \leq [\rho(A \circ B)]^n \leq \max_{1 \leq i \leq n} [2a_{ii}b_{ii} + \rho(A)\rho(B) - a_{ii}\rho(B) - b_{ii}\rho(A)]^n \\
\leq (\rho(A)\rho(B))^n.
\]
3. A lower bound for the minimum eigenvalue of the Fan product of $M$-matrices

In this section, we will derive a lower bound for the minimum eigenvalue of the Fan product of $M$-matrices.

**Lemma 6** [1]. If $Q$ is irreducible, and $Q \in M_n$, $Qz \geq k z$ for a nonnegative nonzero vector $z$, then $k \leq \tau(Q)$.

**Lemma 7** [1]. Let $A \in M_n$ be given. Then either $A$ is irreducible or there exists a permutation $P$ such that

$$
P^TAP = 
\begin{pmatrix}
A_1 & A_{12} & \cdots & A_{1k} \\
A_2 & \cdots & A_{2k} \\
\vdots & & \ddots & \vdots \\
A_k & & & A_k
\end{pmatrix}
$$

in which each $A_i$ is irreducible, $i = 1, \ldots, k$.

**Remark.** Eq. (6) is called the irreducible normal form. Note that $\sigma(A) = \bigcup_{i=1}^{k} \sigma(A_i)$ and $\tau(A) = \min\{\tau(A_i) : i = 1, \ldots, k\}$.

**Lemma 8.** Let $A \in M_n$ be given. If $A_k \in M_k$ is a principal submatrix of $A$, then $\tau(A_k) \geq \tau(A)$.

**Proof.** Suppose $A = \lambda I_n - P$ where $P$ is an $n \times n$ nonnegative matrix and $\lambda$ is a nonnegative real number. Then $A_k = \lambda I_k - P_k$ where $P_k$ is a principal submatrix of $P$. $\tau(A) = \lambda - \rho(P)$ and $\tau(A_k) = \lambda - \rho(P_k)$. Since $\rho(P) \geq \rho(P_k)$, then $\tau(A_k) \geq \tau(A)$.

**Theorem 9.** For two matrices $A, B \in M_n$,

$$
\tau(A \star B) \geq \min_{1 \leq i \leq n} \{a_{ii} \tau(B) + b_{ii} \tau(A) - \tau(A) \tau(B)\}.
$$

**Proof.** It is quite evident that (7) holds with equality for $n = 1$.

Below we assume that $n \geq 2$. Let us distinguish two cases.

**Case 1.** If $A \star B$ is irreducible, then $A$ and $B$ are irreducible. Since $A - \tau(A)I$ and $B - \tau(B)I$ are singular irreducible $M$-matrices, Theorem 6.4.16 of [1] yields that

$$
a_{ii} - \tau(A) > 0 \quad \forall i \in N
$$

and

$$
b_{ii} - \tau(B) > 0 \quad \forall i \in N.
$$

Let $u = (u_i)$, $v = (v_i)$ be the right Perron eigenvectors of $A$ and $B^T$. Then we have

$$
u_i a_{ii} - \sum_{j \neq i} u_j |a_{ij}| = \tau(A)u_i \quad \forall i \in N.
$$

Also we have

$$
v_j b_{jj} - \sum_{i \neq j} v_i |b_{ij}| = \tau(B)v_j \quad \forall j \in N.
$$
Thus
\[ |b_{ij}| \leq \frac{[b_{jj} - \tau(B)]v_j}{v_i} \] for all \( i \neq j \).

Let \( z \) be the vector \((z_i)\), where
\[ z_i = \frac{u_i}{[b_{ij} - \tau(B)]v_i} > 0 \quad \forall i \in N. \]

We define \( C = A \star B \). For any \( i \in N \),
\[
(Cz)_i = a_{ii}b_{ii}z_i - \sum_{j \neq i} |a_{ij}| |b_{ij}| z_j \\
\geq a_{ii}b_{ii}z_i - \sum_{j \neq i} |a_{ij}| \frac{[b_{jj} - \tau(B)]v_j}{v_i} \frac{u_j}{[b_{jj} - \tau(B)]v_j} \\
= a_{ii}b_{ii}z_i - \frac{1}{v_i} \sum_{j \neq i} |a_{ij}| u_j \\
= a_{ii}b_{ii}z_i - \frac{1}{v_i} (a_{ii} - \tau(A))u_i \\
= a_{ii}b_{ii}z_i - (a_{ii} - \tau(A))(b_{ii} - \tau(B))z_i \\
= (a_{ii} - \tau(B))z_i.
\]

By Lemma 6, this shows that
\[ \tau(A \star B) \geq \min_{1 \leq i \leq n} \{ a_{ii} - \tau(B) + b_{ii} - \tau(A) \}. \]

**Case 2.** If \( A \star B \) is reducible. It is well known that a matrix in \( Z_n \) is a nonsingular \( M \)-matrix if and only if all its leading principal minors are positive (see condition (E17) of Theorem 6.2.3 of [1]). If we denote by \( T \) the \( n \times n \) permutation matrix \((t_{ij})\) with
\[ t_{12} = t_{23} = \cdots = t_{n-1,n} = t_{n1} = 1 \]
the remaining \( t_{ij} \) zero, then both \( A - \varepsilon T \) and \( B - \varepsilon T \) are irreducible nonsingular \( M \)-matrices for any chosen positive real number \( \varepsilon \), sufficiently small such that all the leading principal minors of both \( A - \varepsilon T \) and \( B - \varepsilon T \) are positive. Now we substitute \( A - \varepsilon T \) and \( B - \varepsilon T \) for \( A \) and \( B \), respectively in the previous case, and then letting \( \varepsilon \to 0 \), the result follows by continuity. \( \square \)

In fact when \( A \star B \) is reducible and \( n \geq 2 \), we can get another result. We define \( D = A \star B \). From Lemma 2, we get that there exists a permutation \( P \) such that
\[
P^TDP = \begin{pmatrix}
D_1 & D_{12} & \cdots & D_{1k} \\
D_{12} & D_2 & \cdots & D_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
D_{1k} & D_{2k} & \cdots & D_k
\end{pmatrix}
\] (8)
in which each $D_i = A_i \star B_i$ is irreducible, $i = 1, \ldots, k$. Then

$$\tau(A \star B) = \min\{\tau(D_i) = \tau(A_i \star B_i) : i = 1, \ldots, k\}.$$ 

If

$$\tau(A \star B) = \tau(D_{i_0}) = \tau(A_{i_0} \star B_{i_0}).$$

If $D_{i_0}$ is a $1 \times 1$ principal submatrix of $D$, then

$$\tau(A \star B) = \tau(D_{i_0}) = a_{i_0}b_{i_0}$$

for some $i_0 \in N$.

If $D_{i_0}$ is a $t \times t$ principal submatrix of $D$ and $2 \leq t \leq n - 1$, then

$$\tau(A \star B) \geq \min_{i \in \alpha} \{a_{i_0} \tau(B[\alpha]) + b_{i_0} \tau(A[\alpha]) - \tau(A[\alpha]) \tau(B[\alpha])\},$$

where $A[\alpha]$ and $B[\alpha]$ are two principal submatrices of $A$ and $B$ with the same index set $\alpha \subset N$ and $|\alpha| = t$ ($|\alpha|$ denotes the cardinality of the set $\alpha$).

**Remark 3.** We have

$$[a_{i_0} \tau(B) + b_{i_0} \tau(A) - \tau(A) \tau(B)] - \tau(A) \tau(B) = (a_{i_0} - \tau(A)) \tau(B) + (b_{i_0} - \tau(B)) \tau(A).$$

If $A \in M_n$ is irreducible, then

$$a_{i_0} - \tau(A) > 0 \quad \forall i \in N.$$ 

If $A \in M_n$ is reducible, then

$$a_{i_0} - \tau(A) \geq 0 \quad \forall i \in N.$$ 

Thus

$$(a_{i_0} - \tau(A)) \tau(B) + (b_{i_0} - \tau(B)) \tau(A) \geq 0.$$ 

So we can get

$$\min_{1 \leq i \leq n} \{a_{i_0} \tau(B) + b_{i_0} \tau(A) - \tau(A) \tau(B)\} \geq \tau(A) \tau(B).$$

So our bound is sharper than the bound $\tau(A) \tau(B)$ in [2, p. 359].

**Remark 4.** If we only consider the case when $A \star B$ is reducible and $n \geq 2$, we can get a similar result. Since

$$a_{i_0}b_{i_0} \geq \tau(A) \tau(B) \quad \forall i \in N$$

and by Remark 3 and Lemma 8 we can get

$$\min_{i \in \alpha} \{a_{i_0} \tau(B[\alpha]) + b_{i_0} \tau(A[\alpha]) - \tau(A[\alpha]) \tau(B[\alpha])\} \geq \tau(A[\alpha]) \tau(B[\alpha]) \geq \tau(A) \tau(B).$$

From Theorem 9 and [1, p. 380] we can get the following corollary:

**Corollary 10.** For two matrices $A, B \in M_n$, we have the following inequalities:

$$\det(A \star B) \geq \tau(A \star B)^n \geq \left(\min_{1 \leq i \leq n} \{a_{i_0} \tau(B) + b_{i_0} \tau(A) - \tau(A) \tau(B)\}\right)^n \geq (\tau(A) \tau(B))^n.$$
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