COMPLETENESS OF HOLOMORPHS

BY

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1. *Introduction.* A complete group is a group without centre and without outer automorphisms. It is well-known that a group G is complete if and only if G is a direct factor of every group containing G as a normal subgroup (cf. [6], p. 80 and [2]). The question arises whether it is sufficient for a group to be complete, that it is a direct factor in its holomorph. RÉDEI [9] has given the following necessary condition for a group to be a direct factor in its holomorph: it is complete or a direct product of a complete group and a group of order 2. In section 2 I establish the following necessary and sufficient condition: it is complete or a direct product of a group of order 2 and a complete group without subgroups of index 2. Obviously a group of order 2 is a trivial example of a non-complete group which is a direct factor of its holomorph (trivial, because the group coincides with its holomorph). For non-trivial examples we need non-trivial complete groups without subgroups of index 2. We quote some examples of complete groups from the literature.

The unrestricted symmetric group S_n of n objects, where n is an arbitrary finite or infinite cardinal number is complete, except if $n=2$ or $n=6$ ([5], p. 92 and [11]). If n is infinite, S_n does not contain subgroups of index 2 ([10] and [1]; in these papers general statements about normal subgroups of S_n are proved. A direct proof of the non-existence of subgroups of index 2 is easier).

The group of automorphisms of a non-abelian elementary group is complete (a group is called elementary, if it has no proper characteristic subgroups); cf. [3], p. 96 corollary, where the theorem is proved for finite groups. The generalization to infinite groups offers no difficulties.

The holomorph of a finite abelian group of odd order is complete (cf. [7]; in [4] some special cases of this theorem are rediscovered). In section 3 we direct our attention to a generalization of this theorem. The condition of being abelian of odd order is replaced by the condition that the mapping $x \to x^2$ is an automorphism. This condition means that the group is abelian without elements of order 2, and every element is a square. Obviously a finite group satisfies this condition if and only if it is abelian of odd order. Unfortunately the statement, that the holomorph of a group, in which $x \to x^2$ is an automorphism, is complete, is false. A counterexample is the direct product of a quasicyclic group of type p^∞ and a group of order p, where p is an odd prime (for details see section 3). In theorem 3.1 we give necessary and sufficient conditions in order that the holomorph of a group, for which $x \to x^2$ is an automorphism, be complete. These conditions, however, are complicated, but it seems that the conditions such a group has to satisfy in order that its holomorph is not complete, are rather strong. In theorem 3.2 some classes of groups are given, for which the holomorph is complete.

I have not investigated the question, whether an abelian group, in which $x \to x^2$ is not an automorphism, may have a complete holomorph. It is well-known, that a non-abelian group never has a complete holomorph (cf. [8]; see also section 2, iv).

2. The elements of a group are denoted by Latin characters; the identity element is denoted by e. Automorphisms are denoted by Greek characters and are written as left multipliers; accordingly $\alpha\beta$ denotes the automorphism, arising by first applying β and then α. The identity automorphism is denoted by 1. The inner automorphism $x \to axa^{-1}$ is denoted by $\tau(a)$. The group of automorphisms of a group G is denoted by $A(G)$.

The holomorph $K(G)$ of a group G may be defined as follows. The elements of $K(G)$ are the pairs (a, α) with $a \in G$, $\alpha \in A(G)$; multiplication of elements of $K(G)$ is given by the rule

$$(a, \alpha)\,(b, \beta) = (a(\alpha b),\, \alpha\beta).$$

The unit element of $K(G)$ is $(e, 1)$, the inverse of (a, α) is $(\alpha^{-1}a^{-1}, \alpha^{-1})$.

The following facts about $K(G)$ are well-known.

i. $K(G)$ contains the normal subgroup $G' = \{(x, 1) | x \in G\}$, isomorphic to G and $K(G)/G' \cong A(G)$.
ii. $K(G)$ contains the subgroup $A(G)'' = \{(e, \xi) | \xi \in A(G)\}$, isomorphic to $A(G)$ and $K(G) = G'A(G)''$.
iii. Every automorphism of G' is induced by an inner automorphism of $K(G)$.
iv. The mapping $(x, \xi) \to (x^{-1}, \tau(x)\xi)$ is an automorphism of $K(G)$, which is outer if G is not abelian, because it maps the normal subgroup G' onto a group different from G'.

If H is a subgroup of G we denote by H' the group $\{(x, 1) | x \in H\}$; if B is a subgroup of $A(G)$ we denote by B'' the group $\{(e, \xi) | \xi \in B\}$.

Theorem 2.1. If G is a group, G' is a direct factor of $K(G)$ if and only if either G is complete or G is direct product of a group of order 2 and a complete group without subgroups of index 2.

Proof. G' is a direct factor of $K(G)$ if and only if every coset of G' contains an element, which is permutable with every element of G', such that these elements constitute a group. If this element of the coset

$\{(x, \alpha)|x \in G\}$ is denoted by $(f(\alpha)^{-1}, \alpha)$, a short calculation gives the following two conditions:

(2.1) $$\alpha x = f(\alpha)x\, f(\alpha)^{-1},$$

(2.2) $$f(\alpha\beta) = f(\alpha)\, f(\beta).$$

From (2.1) it follows that all automorphisms of G are inner; from (2.2) and (2.1) it follows that f is an isomorphic mapping of $A(G)$ into G. If all automorphisms of G are inner, (2.1) and (2.2) are equivalent with the possibility of choosing one element from every coset of the centre C of G, such that these elements constitute a group. This is possible if and only if C is a direct factor of G. So we have proved the following statement.

G' is a direct factor of $K(G)$ if and only if all automorphisms of G are inner and the centre of G is a direct factor of G.

Let $G = C \times D$; C the centre of G. Then D has no centre. If all automorphisms of G are inner, the same holds for C and D, so D is complete. As C is abelian, it has order 1 or 2. If C has order 1, then G (and therefore G') is complete, and by a theorem, mentioned in the introduction, G' is direct factor in every group containing G' as a normal subgroup, thus G' is a direct factor of $K(G)$.

Now let $G = C \times D$; C of order 2 and D complete. An automorphism α of G induces on C the identical automorphism, because C is the centre of G and has order 2. It maps D onto a subgroup D^* of G of index 2. If α is outer, $D^* \neq D$, because D is complete and α is identical on C. Then $D^* \cap D$ is a subgroup of D of index 2 in D. Conversely if H is a subgroup of D of index 2 in D, we determine a mapping of G into G in the following way (c denotes the generating element of C):

$$\begin{aligned} h &\to h \quad \text{for } h \in H, \\ d &\to cd \quad \text{for } d \in D,\ d \notin H, \\ ch &\to ch \quad \text{for } h \in H, \\ cd &\to d \quad \text{for } d \in D,\ d \notin H. \end{aligned}$$

Obviously this mapping is an outer automorphism of G. So theorem 2.1 is proved.

3. In this section we consider abelian groups G; we adopt an additive notation for the group operation of G. If $x \to 2x$ is an automorphism of G, we denote it by 2. If G_1 and G_2 are abelian groups, the collection of all homomorphisms of G_1 into G_2 may be made in the obvious way into an additive group, which we denote by Hom (G_1, G_2).

Theorem 3.1. The holomorph $K(G)$ of an abelian group G, in which $x \to 2x$ is an automorphism, is not complete, if and only if G is a direct sum of groups B and C, satisfying the following requirements:

i. $B \neq 0$.
ii. $\mathrm{Hom}(C, B) = 0$.

iii. There exists an isomorphic mapping $x \to \delta(x)$ of B onto $\mathrm{Hom}(B, C)$.
iv. There exists a function $f(x)(x \in B,\ f(x) \in B)$, mapping B onto B and satisfying $\delta(y)f(x) = \delta(x)y$ for all $x, y \in B$.

Remark. It is possible to show that, by iii, the function $f(x)$ in iv is determined uniquely by $\delta(x)$ and that the fulfilment of iv does not depend on the choice of $\delta(x)$ in iii in this sense, that if $\delta(x)$ is replaced by another isomorphic mapping $x \to \delta_1(x)$ of B onto $\mathrm{Hom}(B, C)$, then $f(x)$ can be replaced by $f_1(x)$, satisfying iv (with indices 1 at the appropriate places). As we do not need these facts in the sequel, we omit the proof.

Proof. We first prove that for any abelian group G, in which $x \to 2x$ is an automorphism, $K(G)$ has no centre. An element (a, α) of the centre of $K(G)$ has to satisfy

$$(a, \alpha)(x, \xi) = (x, \xi)(a, \alpha),$$

for all $x \in G$, $\xi \in A(G)$. So we must have

$$a + \alpha x = x + \xi a.$$

Taking $\xi = 1$, we get $\alpha x = x$, so $\alpha = 1$. Now $a = \xi a$. Taking $\xi = 2$, we get $a = 0$, which completes the proof.

Let χ be an automorphism of $K(G)$. G' is mapped by χ onto an abelian normal subgroup S of $K(G)$. If $(a, \alpha) \in S$, $(0, 2)(a, \alpha)(0, 2)^{-1}(a, \alpha)^{-1} = (a, 1) \in S$ and $(a, 1)^{-1}(a, \alpha) = (0, \alpha) \in S$. So there exist subgroups C of G and B_1 of $A(G)$, such that $S = \{(x, \xi) | x \in C,\ \xi \in B_1\}$. Obviously S is the direct product of C' and B_1''. For $c \in C$ and $\alpha \in B_1$ we have $(c, \alpha) = (c, 1)(0, \alpha) = (0, \alpha)(c, 1) = (\alpha c, \alpha)$, so $\alpha c = c$. Moreover if $b \in G$ and $\alpha \in B_1$, we have $(b, 1)(0, \alpha)(b, 1)^{-1} = (b - \alpha b, \alpha) \in S$, so $b - \alpha b \in C$. Finally $C' = S \cap G'$ and C is a characteristic subgroup of G, because C' is a normal subgroup of $K(G)$, contained in G'. We may interpret $K(G)$ also as the holomorph of S and repeat our preceding argument interchanging the rôles of S and G'. So we find that G is the direct sum of C and a subgroup B isomorphic to B_1.

As we are only interested in the question, whether χ is outer or inner, we may replace χ by another automorphism χ_1 of $K(G)$ belonging to the same automorphism class. As every automorphism of G' is induced by an inner automorphism of $K(G)$, we may choose χ_1 in such a way, that it induces on G' any prescribed isomorphic mapping of G' onto S. We choose this mapping in such a way that it induces the identical mapping on C' and maps B' onto B_1''. We need the following two lemma's.

Lemma 3.1. If the abelian group G is the direct sum of its subgroups B and C, C is a characteristic subgroup of G if and only if $\mathrm{Hom}(C, B) = 0$.

Proof. If $\zeta \in \mathrm{Hom}(C, B)$, $\zeta \neq 0$, the mapping $b \to b$ for $b \in B$ and $c \to \zeta c + c$ for $c \in C$ determines an automorphism of G; there exists a $c_1 \in C$ with $\zeta c_1 \neq 0$, so C is not a characteristic subgroup of G. We now assume $\mathrm{Hom}(C, B)=0$ and denote the projection of G onto B, corresponding to the given direct decomposition, by ϑ. If α is an automorphism of G, the restriction to C of $\vartheta\alpha$ is an element of $\mathrm{Hom}(C, B)$ and therefore zero. This means that α maps C into itself, so C is a characteristic subgroup of G.

Lemma 3.2. If the abelian group G is the direct sum of its subgroups B and C, where C is a characteristic subgroup of G, there exists a one-to-one mapping of $A(G)$ onto the set of all triples (α, β, γ), where $\alpha \in A(B)$, $\beta \in A(C)$, $\gamma \in \mathrm{Hom}(B, C)$; the mapping $\lambda \to (\alpha, \beta, \gamma)$ is determined by

$$\lambda(b+c)=\alpha b+\beta c+\gamma b \tag{3.1}$$

for $b \in B$, $c \in C$.

Proof. Let ϑ_1 and ϑ_2 denote the projections of G onto B and C respectively, corresponding to the given direct decomposition. Let $\lambda \in A(G)$. Because C is a characteristic subgroup of G, the restriction β of λ to C is an element of $A(C)$. Let γ denote the restriction of $\vartheta_2\lambda$ to B; then $\gamma \in \mathrm{Hom}(B, C)$. Let α denote the restriction of $\vartheta_1\lambda$ to B. Then (3.1) holds. It remains to prove that $\alpha \in A(B)$. Let $\alpha b=0$, then $\vartheta_1\lambda b=0$, $\lambda b \in C$, $b \in C$, $b=0$. If $b \in B$, there exists a $g \in G$, satisfying $\lambda g=b$. Now

$$b=\vartheta_1\lambda g=\vartheta_1\lambda(\vartheta_1 g+\vartheta_2 g)=\vartheta_1\lambda\vartheta_1 g=\alpha(\vartheta_1 g).$$

So $\alpha \in A(B)$.

Conversely, let $\alpha \in A(B)$, $\beta \in A(C)$, $\gamma \in \mathrm{Hom}(B, C)$. Determine λ by (3.1); obviously λ is a homomorphism. If $\lambda(b+c)=0$, then $\alpha b=0$ and $\beta c+\gamma b=0$, so $b=c=0$. Finally

$$\lambda(\alpha^{-1}b+\beta^{-1}c-\beta^{-1}\gamma\alpha^{-1}b)=b+c.$$

This completes the proof.

We remark that if $\lambda_1 \to (\alpha_1, \beta_1, \gamma_1)$ and $\lambda_2 \to (\alpha_2, \beta_2, \gamma_2)$, then $\lambda_1\lambda_2 \to$ $\to (\alpha_1\alpha_2, \beta_1\beta_2, \gamma_1\alpha_2+\beta_1\gamma_2)$.

Let $\lambda \in B_1$ and $\lambda \to (\alpha, \beta, \gamma)$ according to lemma 3.2. If $c \in C$, then $c=\lambda c=\beta c$, so $\beta=1$. If $b \in B$, then $b-\lambda b=b-\alpha b-\gamma b \in C$, so $b-\alpha b=0$, $\alpha=1$. Moreover $(1, 1, \gamma_1)(1, 1, \gamma_2)=(1, 1, \gamma_1+\gamma_2)$. So an isomorphic mapping of B onto B_1 induces an isomorphic mapping of B into $\mathrm{Hom}(B, C)$; we denote the mapping corresponding to χ_1 by $b \to \delta(b)$; the mapping of B onto B_1 then reads $b \to (1, 1, \delta(b))$.

According to lemma 3.2 we now write the elements of $K(G)$ in the form

$$(b+c, \alpha, \beta, \gamma),$$

with $b \in B$, $c \in C$, $\alpha \in A(B)$, $\beta \in A(C)$, $\gamma \in \mathrm{Hom}(B, C)$.

We return to the automorphism χ_1. We have

$$\chi_1(b+c, 1, 1, 0)=(c, 1, 1, \delta(b)). \tag{3.2}$$

We put

$$\chi_1(0, \alpha, \beta, \gamma)=(B(\lambda)+C(\lambda), \boldsymbol{\Xi}(\lambda), \mathbf{H}(\lambda), \mathbf{Z}(\lambda)), \tag{3.3}$$

where $B(\lambda) \in B$, $C(\lambda) \in C$, $\Xi(\lambda) \in A(B)$, $\mathbf{H}(\lambda) \in A(C)$, $\mathbf{Z}(\lambda) \in \mathrm{Hom}(B, C)$, $\lambda = (\alpha, \beta, \gamma)$.

Because χ_1 is a homomorphism, we have

$$(B(\lambda_1\lambda_2)+C(\lambda_1\lambda_2),\ \Xi(\lambda_1\lambda_2),\ \mathbf{H}(\lambda_1\lambda_2),\ \mathbf{Z}(\lambda_1\lambda_2)) = (B(\lambda_1)+C(\lambda_1)+\Xi(\lambda_1)B(\lambda_2)+ \\ +\mathbf{H}(\lambda_1)C(\lambda_2)+\mathbf{Z}(\lambda_1)B(\lambda_2),\ \Xi(\lambda_1)\Xi(\lambda_2),\ \mathbf{H}(\lambda_1)\mathbf{H}(\lambda_2),\ \mathbf{Z}(\lambda_1)\Xi(\lambda_2)+\mathbf{H}(\lambda_1)\mathbf{Z}(\lambda_2)).$$

So we get

$$B(\lambda_1\lambda_2) = B(\lambda_1) + \Xi(\lambda_1)B(\lambda_2). \tag{3.4}$$

$$C(\lambda_1\lambda_2) = C(\lambda_1) + \mathbf{H}(\lambda_1)C(\lambda_2) + \mathbf{Z}(\lambda_1)B(\lambda_2). \tag{3.5}$$

$$\mathbf{Z}(\lambda_1\lambda_2) = \mathbf{Z}(\lambda_1)\Xi(\lambda_2) + \mathbf{H}(\lambda_1)\mathbf{Z}(\lambda_2). \tag{3.6}$$

Moreover we have

$$\chi_1(0, \alpha, \beta, \gamma)\chi_1(b+c, 1, 1, 0) = \chi_1(\alpha b+\beta c+\gamma b, 1, 1, 0)\chi_1(0, \alpha, \beta, \gamma),$$

$$(B(\lambda)+C(\lambda), \Xi(\lambda), \mathbf{H}(\lambda), \mathbf{Z}(\lambda))(c, 1, 1, \delta(b)) = (\beta c+\gamma b, 1, 1, \delta(\alpha b))(B(\lambda)+C(\lambda).$$

$$\Xi(\lambda),\ \mathbf{H}(\lambda),\ \mathbf{Z}(\lambda)),\ (B(\lambda)+C(\lambda)+\mathbf{H}(\lambda)c,\ \Xi(\lambda),\ \mathbf{H}(\lambda),\ \mathbf{Z}(\lambda)+\mathbf{H}(\lambda)\delta(b)) = \\ = (\beta c+\gamma b+B(\lambda)+C(\lambda)+\delta(\alpha b)B(\lambda),\ \Xi(\lambda),\ \mathbf{H}(\lambda),\ \delta(\alpha b)\Xi(\lambda)+\mathbf{Z}(\lambda)),$$

So we get

$$\mathbf{H}(\lambda)c = \beta c + \gamma b + \delta(\alpha b)B(\lambda). \tag{3.7}$$

$$\delta(\alpha b)\Xi(\lambda) = \mathbf{H}(\lambda)\delta(b). \tag{3.8}$$

Putting $b = 0$ into (3.7) we get $\mathbf{H}(\lambda)c = \beta c$, and so

$$\mathbf{H}(\lambda) = \beta. \tag{3.9}$$

From (3.7) and (3.9) we infer

$$\delta(\alpha b)B(\lambda) + \gamma b = 0. \tag{3.10}$$

We now use the fact that 2 is an automorphism of $K(G)$; obviously 2 is permutable with all other automorphisms of $K(G)$. Moreover 2 corresponds to the triple (2, 2, 0).

So we infer from (3.4)

$$B(\lambda) + \Xi(\lambda)B(2) = B(2) + \Xi(2)B(\lambda), \tag{3.11}$$

from (3.5), using (3.9),

$$C(\lambda) + \beta C(2) + \mathbf{Z}(\lambda)B(2) = C(2) + 2C(\lambda) + \mathbf{Z}(2)B(\lambda), \tag{3.12}$$

and from (3.6), using (3.9),

$$\mathbf{Z}(\lambda)\Xi(2) + \beta\mathbf{Z}(2) = \mathbf{Z}(2)\Xi(\lambda) + 2\mathbf{Z}(\lambda). \tag{3.13}$$

From (3.12) we infer

$$C(\lambda) = \beta C(2) + \mathbf{Z}(\lambda)B(2) - C(2) - \mathbf{Z}(2)B(\lambda). \tag{3.14}$$

We are going to use now the fact that χ_1 is a mapping of $K(G)$ onto $K(G)$. Using (3.2), (3.3) and (3.9) we get

$$\chi_1(b+c, \alpha, \beta, \gamma) = (B(\lambda)+c+C(\lambda)+\delta(b)B(\lambda),\ \Xi(\lambda), \beta, \delta(b)\Xi(\lambda)+\mathbf{Z}(\lambda)). \tag{3.15}$$

To every collection consisting of $b \in B$, $c \in C$ and a triple $\lambda = (\alpha, \beta, \gamma)$ there exist $x \in B$, $y \in C$ and a triple $\omega = (\xi, \eta, \zeta)$ such that

$$B(\omega) = b, \tag{3.16}$$

$$\boldsymbol{\Xi}(\omega) = \alpha, \tag{3.17}$$

$$\eta = \beta, \tag{3.18}$$

$$\delta(x)\alpha + \mathbf{Z}(\omega) = \gamma, \tag{3.19}$$

$$y + C(\omega) + \delta(x)b = c. \tag{3.20}$$

Substituting ω for λ in (3.11) and using (3.16) and (3.17) we get

$$b + \alpha B(2) = B(2) + \boldsymbol{\Xi}(2)b.$$

Taking $\alpha = 1$, we get $b = \boldsymbol{\Xi}(2)b$ for every $b \in B$, so

$$\boldsymbol{\Xi}(2) = 1. \tag{3.21}$$

Taking $\alpha = 2$ and using (3.21) we get

$$B(2) = 0. \tag{3.22}$$

Using (3.21) we infer from (3.13)

$$\mathbf{Z}(\lambda) = \beta\mathbf{Z}(2) - \mathbf{Z}(2)\boldsymbol{\Xi}(\lambda). \tag{3.23}$$

Using (3.17), (3.18) and (3.23) we infer from (3.19)

$$\delta(x)\alpha + \beta\mathbf{Z}(2) - \mathbf{Z}(2)\alpha = \gamma.$$

Taking $\alpha = \beta = 1$ we get $\delta(x) = \gamma$ and this means, that $x \to \delta(x)$ is a mapping of B *onto* $\mathrm{Hom}(B, C)$.

According to this fact there exists a $b_1 \in B$ satisfying $\delta(b_1) = \mathbf{Z}(2)$. Let ψ be the inner automorphism $\tau((b_1 + C(2), 1, 1, 0))$ of $K(G)$ and $\chi_2 = \chi_1\psi$. We get

$$\psi(0, 2, 2, 0) = (-b_1 - C(2), 2, 2, 0).$$

Using (3.15), (3.21) and (3.22) we get

$$\chi_2(0, 2, 2, 0) = (0, 1, 2, 0). \tag{3.24}$$

Now χ_2 belongs to the automorphism class of χ, and it induces on G' the same mapping as χ_1, i.e. (3.2) holds with χ_1 replaced by χ_2. We replace χ_1 by χ_2 and take the functions $B(\lambda)$, $C(\lambda)$, $\boldsymbol{\Xi}(\lambda)$, $\mathbf{H}(\lambda)$ and $\mathbf{Z}(\lambda)$ corresponding to χ_2. All formulas deduced for these functions remain valid and moreover from (3.24) we infer

$$C(2) = 0, \tag{3.25}$$

$$\mathbf{Z}(2) = 0. \tag{3.26}$$

Using (3.22), (3.25) and (3.26), (3.14) turns into

$$C(\lambda) = 0. \tag{3.27}$$

Using (3.26), (3.23) turns into

$$\mathbf{Z}(\lambda)=0. \tag{3.28}$$

We return to the fact, that χ_2 is a mapping onto $K(G)$. Substituting ω for λ and t for b in (3.8) and using (3.9), (3.17) and (3.18) we get

$$\delta(\xi t)=\beta\delta(t)\alpha^{-1}. \tag{3.29}$$

Substituting ω for λ and t for b in (3.10) and using (3.16) we get

$$\delta(\xi t)b+\zeta t=0,$$

and using (3.29), this turns into

$$\zeta t= -\beta\delta(t)\alpha^{-1}b. \tag{3.30}$$

Obviously (3.29) determines ξ uniquely as a function of α and β and (3.30) determines ζ uniquely as a function of α, β and b. Moreover the mapping ξ determined by (3.29) is an element of $A(B)$, and the mapping ζ determined by (3.30) is an element of $\mathrm{Hom}(B, C)$.

By (3.28), (3.19) turns into

$$\delta(x)=\gamma\alpha^{-1}, \tag{3.31}$$

and by (3.27) and (3.31), (3.20) turns into

$$y=c-\gamma\alpha^{-1}b. \tag{3.32}$$

Using (3.18), (3.29), (3.30), (3.31) and (3.32) we find that χ_2^{-1} is the mapping

$$(b+c,\ \alpha,\ \beta,\ \gamma) \rightarrow (X_1(\gamma\alpha^{-1})+c-\gamma\alpha^{-1}b,\ \Xi_1(\alpha,\ \beta),\ \beta,\ \beta\mathbf{Z}_1(\alpha^{-1}b)), \tag{3.33}$$

where the functions $X_1(\gamma)$, $\Xi_1(\alpha, \beta)$ and $\mathbf{Z}_1(b)$ are determined by

$$\delta(X_1(\gamma))=\gamma, \tag{3.34}$$

$$\delta(\Xi_1(\alpha,\ \beta)t)=\beta\delta(t)\alpha^{-1}, \tag{3.35}$$

$$\mathbf{Z}_1(b)t= -\delta(t)b. \tag{3.36}$$

Resuming the facts proved thus far, we get the following statement.

If χ is an automorphism of $K(G)$, which maps G' onto S, G is the direct sum of B and C, where C is a characteristic subgroup of G and $C'=S\cap G'$. Moreover there exists an isomorphic mapping $x \rightarrow \delta(x)$ of B onto $\mathrm{Hom}(B,C)$ such that the mapping determined by (3.33), (3.34), (3.35) and (3.36) is the inverse of an automorphism χ_2 of $K(G)$, belonging to the automorphism class of χ.

Obviously $S=G'$, if and only if $B=0$. If $B=0$, (3.33) is the identical mapping. So $K(G)$ contains an outer automorphism if and only if G is the direct sum of subgroups B and C, satisfying i, ii and iii and the mapping (3.33) is an automorphism of $K(G)$.

It is a matter of straightforward verification that (3.33) is always a homomorphism. This homomorphism is an isomorphism if and only if

(3.37) $$X_1(\gamma\alpha^{-1})=0,$$

(3.38) $$c-\gamma\alpha^{-1}b=0,$$

(3.39) $$\boldsymbol{\Xi}_1(\alpha,\ 1)=1,$$

(3.40) $$\mathbf{Z}_1(\alpha^{-1}b)=0$$

together imply $b=c=\gamma=0$ and $\alpha=1$. Now, by (3.34), (3.37) implies $\gamma=0$, and then (3.38) implies $c=0$. By (3.35), (3.39) is equivalent to

(3.41) $$\delta(t)\alpha=\delta(t) \text{ for all } t\in B,$$

and by (3.36) and (3.41), (3.40) and (3.41) together are equivalent to (3.41) and

(3.42) $$\delta(t)b=0 \text{ for all } t\in B.$$

We now consider the conditions (3.33) has to satisfy in order that it is a mapping onto $K(G)$. To every collection consisting of $b\in B$, $c\in C$ and a triple (α, β, γ), there have to exist $x\in B$, $y\in C$ and a triple (ξ, β, ζ), satisfying

(3.43) $$X_1(\zeta\,\xi^{-1})=b,$$

(3.44) $$y-\zeta\,\xi^{-1}x=c,$$

(3.45) $$\boldsymbol{\Xi}_1(\xi,\beta)=\alpha,$$

(3.46) $$\beta\mathbf{Z}_1(\xi^{-1}x)=\gamma.$$

By (3.34), (3.43) may be replaced by

(3.47) $$\zeta=\delta(b)\xi.$$

By (3.47), (3.44) may be replaced by

(3.48) $$y=c+\delta(b)x.$$

By (3.35), (3.45) may be replaced by

(3.49) $$\delta(\alpha t)\xi=\beta\delta(t) \text{ for all } t\in B.$$

By (3.36) and (3.49), (3.46) may be replaced by

(3.50) $$\delta(t)x=-\gamma\alpha^{-1}t \text{ for all } t\in B.$$

If x and ξ are found, y and ζ follow from (3.47) and (3.48).

Suppose that i, ii, iii are satisfied and the mapping (3.33) is an automorphism of $K(G)$. Take an element u of B and put $\gamma=-\delta(u)$, $\alpha=1$. The solvability of (3.50) implies the existence of a function $f(u)(u\in B$, $f(u)\in B)$, satisfying $\delta(t)f(u)=\delta(u)t$ for all u, $t\in B$. In order to prove that $f(u)$ maps B onto B we take an element b of B. Clearly the mapping $t\to\delta(t)b$ $(t\in B)$ is an element of $\mathrm{Hom}(B, C)$, so, by iii, there exists a $u\in B$, satisfying $\delta(t)b=\delta(u)t=\delta(t)f(u)$ and therefore $\delta(t)(f(u)-b)=0$. Because (3.42) implies $b=0$, we find $f(u)=b$, so iv is satisfied.

Suppose conversely that i, ii, iii, iv are satisfied. We first prove that if $b \in B$ and $\delta(x)b=0$ for all $x \in B$, then $b=0$. By iv, $\delta(x)b=0$ implies $\delta(b)f(x)=0$; because $f(x)$ is a mapping onto B, this implies $\delta(b)=0$ and, by iii, $b=0$. We now prove that the mapping $x \to f(x)$ is an automorphism of B. Clearly

$$\delta(y)(f(x_1+x_2)-f(x_1)-f(x_2))=0,$$

so $f(x_1+x_2)=f(x_1)+f(x_2)$. If $f(x)=0$, $\delta(x)y=0$ for all $y \in B$, $\delta(x)=0$ and, by iii, $x=0$. So $x \to f(x)$ is an automorphism; we denote it by σ and we have $\delta(y)\sigma x=\delta(x)y$ for all $x, y \in B$. We have proved already that (3.42) implies $b=0$. If (3.41) is satisfied, we have $\delta(t)(\alpha u-u)=0$ for all $u, t \in B$, so $\alpha u=u$ for all $u \in B$, so $\alpha=1$. In order to solve (3.50) we take a $u \in B$ satisfying $\delta(u)=-\gamma\alpha^{-1}$, which is possible by iii. Now $x=f(u)$ solves (3.50). In order to solve (3.49) we remark that, for all $u \in B$, $\beta\delta(u)\sigma\alpha^{-1}\sigma^{-1} \in$ $\in \mathrm{Hom}(B, C)$. So, by iii, there exists a function $g(u)(u \in B,\ g(u) \in B)$, satisfying $\delta(g(u))=\beta\delta(u)\sigma\alpha^{-1}\sigma^{-1}$. It is easy to show that the mapping $u \to g(u)$ is an automorphism of B; we denote it by ξ. We now have for all $u, t \in B$:

$$\delta(\xi u)\sigma\alpha t=\beta\delta(u)\sigma t,$$

$$\delta(\alpha t)\xi u=\beta\delta(t)u,$$

so ξ indeed solves (3.49). This completes the proof of theorem 3.1.

From the proof of theorem 3.1 it follows that an automorphism of $K(G)$ which maps G onto itself is inner. Moreover the mapping (3.33) maps G onto S, and this means that in the group of automorphism classes of $K(G)$ the square of every element equals 1. This gives the following corollary.

Corollary 3.1. If G is an abelian group, in which $x \to 2x$ is an automorphism, every automorphism of $K(G)$ which maps G onto itself is inner and the group of automorphism classes of $K(G)$ is a direct product of groups of order 2 or a group of order 1.

We now give an example of a group G, in which $x \to 2x$ is an automorphism and for which $K(G)$ is not complete. Let p be an odd prime, B a group of order p, C a quasicyclic group of type p^∞ and G the direct sum of B and C. Obviously $x \to 2x$ is an automorphism of G and i holds. A homomorphic image of C is zero or isomorphic to C, so ii holds. Let a be a generator of B and $d_1, d_2, \ldots$ be generators of C, satisfying $pd_1=0$, $pd_{n+1}=d_n$ $(n=1, 2, \ldots)$. An element of $\mathrm{Hom}(B, C)$ is determined by $a \to kd_1$ $(k=0, 1, \ldots, p-1)$; we denote it by k. We determine $\delta(x)$ by putting $ka \to k$; then iii is satisfied. If we take for $x \to f(x)$ in iv the identical mapping, then iv is satisfied. By theorem 3.1 we find that $K(G)$ is not complete.

In theorem 3.2 we give some sufficient conditions in order that $K(G)$ be complete.

Theorem 3.2. If G is an abelian group, in which $x \to 2x$ is an automorphism, the holomorph $K(G)$ of G is complete, if at least one of the following three conditions is satisfied:

A. G is directly indecomposable.
B. G is a direct sum of cyclic groups.
C. G is a divisible group.

Proof. We assume that $K(G)$ is not complete. Then G is a direct sum of B and C, where B and C satisfy the conditions of theorem 3.1. If $C=0$, then $\mathrm{Hom}(B, C)=0$, so, by iii, $B=0$, contradicting i. So $C \neq 0$. This proves already case A.

Let G be a direct sum of cyclic groups; the same holds for B and C ([5], p. 174). Suppose that C contains an infinite cyclic summand C_∞. By i, B contains a cyclic direct summand $\neq 0$. We may map C_∞ homomorphically $\neq 0$ onto this summand of B and the other summands of C onto 0; this gives an element $\neq 0$ of $\mathrm{Hom}(C, B)$, contradicting ii. So C is periodic. Suppose that B contains a primary direct summand of order p^n. By iii, there exists a $\gamma \in \mathrm{Hom}(B, C)$ of order p^n. The image of γ is not zero and consists of elements, whose order divides p^n. So C has a direct summand of order p^k. We may map this summand of C non-trivially into the summand of order p^n of B and the other summands of C into 0. This gives an element $\neq 0$ of $\mathrm{Hom}(C, B)$, contradicting ii. So B is torsion-free and is a direct sum of infinite cyclic groups. Because $C \neq 0$ and C is periodic, C contains an element c of order p. We map the generators of the cyclic summands of G onto c; this gives an element of order p of $\mathrm{Hom}(B, C)$, which contradicts iii. So case B is proved.

Let G be divisible; the same holds for B and C ([5], p. 163). B and C both are direct sums of groups of types p^∞ and R (additive group of the rational numbers). Suppose B contains a direct summand of type p^∞; this group contains an element of order p. By iii, $\mathrm{Hom}(B, C)$ contains an element of order p. This implies, that C contains an element of order p and therefore a direct summand of type p^∞. This clearly contradicts ii. So B is torsion-free and contains at least one direct summand of type R. Suppose that C contains a direct summand of type R; we obtain again a contradiction with ii. So C is periodic. Because $C \neq 0$, it contains a direct summand of type p^∞. It is not difficult to show, that $\mathrm{Hom}(R, p^\infty)$ is isomorphic to the additive group of the field of p-adic numbers, which has order $\aleph$ (cardinal number of the continuum). Let B be a direct sum of α times a group of type R. The order of $\mathrm{Hom}(B, C) \geqslant$ the order of $\mathrm{Hom}(B, p^\infty)=$ the order of the unrestricted direct sum of α times $\mathrm{Hom}(R, p^\infty)$. This order is $\aleph^\alpha$. If α is finite, the order of B is $\aleph_0 < \aleph^\alpha$ and if α is infinite, the order of B is $\alpha < \aleph^\alpha$. This contradicts iii. So case C is proved.

We remark that case B of theorem 3.2 implies Miller's theorem ([7]), mentioned in the introduction.

If R is the additive group of the rational numbers, $A(R)$ is isomorphic to the multiplicative group of the rational numbers $\neq 0$. By theorem 3.2, $K(R)$ is complete. So the group consisting of the pairs (a, b) with $a, b \in R$ and $b \neq 0$ and with the multiplication rule $(a,b)(c,d)=(a+bc,bd)$ is a countable complete group.

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