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Existence of solutions for a Dirichlet problem with a p-Laplacian ☆

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ABSTRACT

Some theorems of existence of weak solutions are obtained for p-Laplacian equations with Dirichlet boundary conditions by using a critical point theorem.

1. Introduction and the main results

Consider the existence of weak solutions for the following Dirichlet problem:

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset R^N$ $(N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $p > 1$ and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian. The nonlinearity $f \in C(\bar{\Omega} \times R, R)$ satisfies the subcritical growth condition: there are $c_1 > 0$ and $q \in (p, p^*)$ such that

$$|f(x,t)| \leq c_1(1 + |t|^{q-1}), \quad \forall (x,t) \in \bar{\Omega} \times R, \tag{2}$$

where $p^* := Np/(N-p)$ is the critical Sobolev exponent of p if $1 < p < N$, and $p^* = \infty$ if $p \geq N$. Let $W_0^{1,p}(\Omega)$ be the usual Sobolev space equipped with the norm

$$\|u\| = \left(\int_\Omega |\nabla u|^p dx \right)^{1/p}, \quad u \in W_0^{1,p}(\Omega).$$

By the Sobolev embedding theorem, for any $1 \leq \theta \leq p^*$, the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^\theta(\Omega)$ is continuous and there is a positive constant C such that

$$\|u\|_p \leq C\|u\|, \qquad \|u\|_q \leq C\|u\|, \qquad \|u\|_1 \leq C\|u\|, \qquad \|u\|_{\frac{\mu}{\mu+1-q}} \leq C\|u\| \tag{3}$$

for all $u \in W_0^{1,p}(\Omega)$, where $\|\cdot\|_\theta$ denotes the norm of $L^\theta(\Omega)$. If $1 \leq \theta < p^*$, the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^\theta(\Omega)$ is compact. In the following, C always denotes a positive constant related to the above four inequalities.

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Define the functional $J : W_0^{1,p}(\Omega) \to R$:

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \int_\Omega F(x, u)dx, \tag{4}$$

where $F(x,t) = \int_0^t f(x,s)ds$. Since $f(x,t)$ satisfies the subcritical growth condition (2), by a standard argument, $J \in C^1(W_0^{1,p}(\Omega), R)$ and from the variational point of view, it is well known that finding the weak solutions of problem (1) corresponds to looking for the critical points of the functional J in $W_0^{1,p}(\Omega)$.

We recall the eigenvalues of the following nonlinear eigenvalue problem:

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

Define the manifold

$$S = \left\{ u \in W_0^{1,p}(\Omega) : \int_\Omega |\nabla u|^p dx = 1 \right\};$$

then S is a smooth, symmetric nonempty submanifold in $W_0^{1,p}(\Omega)$. Denote by $\mathcal{A}$ the class of compact symmetric subsets of S, and let

$$\Sigma_k = \{A \in \mathcal{A} : i(A) \ge k\},$$

where i is the cohomological index of Fadell and Rabinowitz (the details can be seen in [1] or [2]); a sequence of eigenvalues of problem (5) can be variationally characterized as follows (the details can be seen in [2]):

$$\lambda_k = \inf_{A \in \Sigma_k} \sup_{u \in A} \frac{1}{\int_\Omega |u|^p dx}.$$

It is not known whether the sequence $\{\lambda_k\}_{k\in N}$ contains all the eigenvalues of problem (5), but we will refer to $\{\lambda_k\}_{k\in N}$ as the variational eigenvalues of problem (5).

The goal of this paper is to obtain the existences of weak solutions for problem (1) by using the variational method. We first state some relevant conditions on the nonlinearity F:

$$\lim_{|t|\to\infty} (f(x,t)t - pF(x,t)) = +\infty \quad \text{uniformly for } x \in \Omega, \tag{6}$$

$$\lim_{|t|\to\infty} (f(x,t)t - pF(x,t)) = -\infty \quad \text{uniformly for } x \in \Omega, \tag{7}$$

$$\lambda_{k-1} < \liminf_{|t|\to\infty} \frac{pF(x,t)}{|t|^p} \le \limsup_{|t|\to\infty} \frac{pF(x,t)}{|t|^p} \le \lambda_k \quad \text{uniformly for } x \in \Omega, \tag{8}$$

$$\lambda_{k-1} \le \liminf_{|t|\to\infty} \frac{pF(x,t)}{|t|^p} \le \limsup_{|t|\to\infty} \frac{pF(x,t)}{|t|^p} < \lambda_k \quad \text{uniformly for } x \in \Omega. \tag{9}$$

Now we are ready to introduce the main results.

Theorem 1. *Assume that $\lambda_{k-1} < \lambda_k$ $(k \ge 2)$ are two consecutive eigenvalues of (5) and (2). If the nonlinearity F satisfies (6) and (8) (or (7) and (9)), problem (1) has at least a weak solution.*

Theorem 2. *Assume that $\lambda_{k-1} < \lambda_k$ $(k \ge 2)$ are two consecutive eigenvalues of (5) and the nonlinearity F satisfies (2) and the crossing conditions*

$$F(x,t) \ge \frac{1}{p}\lambda_{k-1}|t|^p, \quad \forall (x,t) \in \bar{\Omega} \times R, \tag{10}$$

$$\limsup_{|t|\to 0} \frac{pF(x,t)}{|t|^p} \le \alpha < \lambda_k < \beta \le \liminf_{|t|\to\infty} \frac{pF(x,t)}{|t|^p} \quad \textit{uniformly for } x \in \Omega, \tag{11}$$

where α, β are two constants. If one of the following nonquadraticity conditions holds:

$$\liminf_{|t|\to\infty} \frac{f(x,t)t - pF(x,t)}{|t|^\mu} \ge a > 0 \quad \textit{uniformly for } x \in \Omega, \textit{ or} \tag{12}$$

$$\limsup_{|t|\to\infty} \frac{f(x,t)t - pF(x,t)}{|t|^\mu} \le -a < 0 \quad \textit{uniformly for } x \in \Omega, \tag{13}$$

where a is a positive constant and $\mu > (q-1)\max\{\frac{1}{p-1}, \frac{p^*}{p^*-1}\}$ if $1 < p < N$ or $\mu > (q-1)\max\{1, \frac{1}{p-1}\}$ if $p \ge N$, then problem (1) has at least a nontrivial weak solution in $W_0^{1,p}(\Omega)$.

Remark. In recent years, many people have devoted time to studying the existence and multiplicity of nontrivial solutions for problem (1) by using directly variational methods or the mini-max methods. For the linear case ($p = 2$), there have been a great many results on the existence and multiplicity of nontrivial solutions for problem (1) obtained by using the variational method (see [3–5] and the references therein). For the case $1 < p \neq 2$, in the absence of any direct decomposition of $W_0^{1,p}(\Omega)$ related to the eigenvalues of $-\Delta_p$, the variational method used in the linear case ($p = 2$) cannot be extended to the nonlinear case ($1 < p \neq 2$), and the situation becomes far more complicated, but many people are devoting time to considering the existence and multiplicity of nontrivial solutions for problem (1) (see [6–15,2] and references therein).

Eqs. (6), (7), (12) and (13) correspond to the nonquadraticity conditions with $p = 2$, (8) and (9) are the asymptotic noncrossing conditions, and (10) and (11) are the crossing conditions. They were introduced by Costa and Magalhaes in [16] and were widely used by many people to consider the existence and multiplicity of solutions for elliptic equations and elliptic systems (see [6,9,10,13] and references therein).

In [9], El Amrouss and Moussaoui obtained the existence of a weak solution for problem (1) by using an abstract critical point theory under (7) and (9) with $k = 2$, that is, the following condition holds:

$$\lambda_1 \le \liminf_{|t|\to\infty} \frac{pF(x,t)}{|t|^p} \le \limsup_{|t|\to\infty} \frac{pF(x,t)}{|t|^p} < \lambda_2 \quad \text{uniformly in } x \in \Omega.$$

Let $\{\nu_k\}$ be the another sequence of eigenvalues of problem (5) defined by the cogenus index $\gamma^+(A)$, where $A \in \mathcal{A}$:

$$\gamma^+(A) = \sup\{m \in N : \text{ there is a continuous odd surjection } \psi : R^m \setminus \{0\} \to A\}.$$

In [13], assuming that (6) and the following condition hold:

$$\nu_{k-1} < \liminf_{|t|\to\infty} \frac{pF(x,t)}{|t|^p} \le \limsup_{|t|\to\infty} \frac{pF(x,t)}{|t|^p} \le \nu_k \quad \text{uniformly for } x \in \Omega,$$

Ou and Li proved the existence of a weak solution for problem (1) by using the G-linking theorem, but they cannot prove a similar result in the other case. Hence, Theorem 1 is complementary to the relevant results of [9,13].

Costa and Magalhaes in [6] obtained the existence of a nontrivial solution for problem (1) under the following condition:

$$\limsup_{|t|\to 0} \frac{pF(x,t)}{|t|^p} \le \alpha < \lambda_1 < \beta \le \liminf_{|t|\to\infty} \frac{pF(x,t)}{|t|^p} \quad \text{uniformly for } x \in \Omega,$$

which implies that F crosses the first eigenvalue λ_1 of problem (5). Hence, Theorem 2 is a generalization of Theorem 1.1 in [9].

2. Preliminaries and proofs of theorems

In this section, we will introduce two abstract critical point theorems, which are based on the deformation lemma and on a linking structure. The deformation lemma may be ensured by adding a compactness condition: the (PS) condition or the (Ce) condition. Let W be a real Banach space; the functional I satisfies the $(PS)_c$ condition at the level $c \in R$ if any sequence $\{u_n\} \subset W$ such that $I(u_n) \to c$, $I'(u_n) \to 0$ as $n \to \infty$, has a convergent subsequence. The functional I satisfies the (PS) condition if I satisfies the $(PS)_c$ condition at any $c \in R$. The functional I satisfies the $(Ce)_c$ condition at the level $c \in R$ if any sequence $\{u_n\} \subset W$ such that $I(u_n) \to c$, $(1 + \|u_n\|)\|I'(u_n)\|_{W^*} \to 0$ as $n \to \infty$ has a convergent subsequence. The functional I satisfies the (Ce) condition if I satisfies the $(Ce)_c$ condition at any $c \in R$. The (Ce) condition was introduced by Cerami [3], and it is a weaker version of the (PS) condition. Next, we will introduce the notions of the homotopical linking and relative homotopical linking.

Definition 1 (*See [15]*)**.** Let A be a closed proper subset of a topological space X and W be a Banach space, $g \in C(A, W)$ such that $g(A)$ is closed and bounded, B a nonempty closed subset of W such that $\operatorname{dist}(g(A), B) > 0$, and

$$\Gamma = \{\gamma \in C(X, W) : \gamma(X) \text{ is closed}, \gamma|_A = g\}.$$

We say that (A, g) homotopically links B with respect to X if

$$\gamma(X) \cap B \neq \emptyset, \quad \forall \gamma \in \Gamma.$$

When $A \subset W$ and $g : A \to W$ is the inclusion and $X = IA = \{tx : x \in A, t \in [0, 1]\}$, we will simply say that A homotopically links B.

Here, we will introduce an example of a homotopical linking set. Let $\mathcal{M}$ be a bounded symmetric subset radially homeomorphic to the unit sphere S, i.e., the restriction of the radial projection π_S to $\mathcal{M}$ is a homeomorphism. Then the radial projection from $W \setminus \{0\}$ to $\mathcal{M}$ is given by

$$\pi_{\mathcal{M}} = (\pi_S|_{\mathcal{M}})^{-1} \circ \pi_S.$$

Let $A_0 \neq \emptyset$ and B_0 be disjoint closed symmetric subsets of $\mathcal{M}$ such that

$$i(A_0) = i(\mathcal{M} \setminus B_0) < \infty,$$

where i is the cohomological index; we have the following property.

Proposition 1 (*See [15]*). *$A = RA_0$ homotopically links $B = \pi_{\mathcal{M}}^{-1}(B_0) \cup \{0\}$ for any $R > 0$.*

Definition 2 (*Relative Homotopical Linking; See [4]*). Let W be a metric space and A, B, P, S be four subsets of W with $S \subset P$ and $B \subset A$. We say that (P, S) links (A, B) if $S \cap A = B \cap P = \emptyset$ and for every deformation $\eta : P \times [0, 1] \to W \setminus B$ with $\eta(S \times [0, 1]) \cap A = \emptyset$, we have $\eta(P \times \{1\}) \cap A \neq \emptyset$.

Definition 3 (*Relative Cohomological Linking; See [8]*). Let W be a metric space and A, B, P, S be four subsets of W with $S \subset P$ and $B \subset A$; let m be a nonnegative integer and $\mathcal{K}$ be a field. We say that (P, S) links (A, B) cohomologically in dimension m over $\mathcal{K}$ if $S \cap A = B \cap P = \emptyset$ and the restriction homomorphism $H^m(W \setminus B, W \setminus A, \mathcal{K}) \to H^m(P, S, \mathcal{K})$ is not identically zero, where H^* denotes the Alexander–Spanier cohomology (see [17]).

From [8], if (P, S) links (A, B) cohomologically, then (P, S) links (A, B).

Definition 4. Let W be a real Banach space. A subset E of W is said to be symmetric if $-u \in E$ for any $u \in E$. A subset E of W is said to be a star set if $tu \in E$ for any $u \in E$ and $t > 0$.

Proposition 2 (*See [8]*). *Let W be a real normed space and C_-, C_+ be two star sets in W such that C_+ is closed in W, $C_- \cap C_+ = \{0\}$ and such that $(W, C_- \setminus \{0\})$ links C_+ cohomologically in dimension m over Z_2. Let $r_-, r_+ > 0$ and $e \in W$ with $-e \notin C_-$; define*

$$D_- = \{u \in C_- : \|u\| \le r_-\}, \qquad S_+ = \{u \in C_+ : \|u\| = r_+\},$$
$$Q = \{u + te : u \in C_-, t \ge 0, \|u + te\| \le r_-\},$$
$$H = \{u + te : u \in C_-, t > 0, \|u + te\| = r_-\}.$$

If $r_- > r_+ > 0$, $(Q, D_- \cup H)$ links S_+ cohomologically in dimension $m + 1$ over Z_2. In fact, $D_- \cup H$ is the relative boundary of Q.

Finally, we will recall two abstract critical point theorems, and then prove Theorem 1 by using Theorem A and Theorem 2 by using Theorem B.

Theorem A (*See [15]*). *Let A be a closed subset of a topological space X and W be a Banach space, $g \in C(A, W)$ such that (A, g) homotopically links B with respect to X, and $f \in C^1(W, R)$. If*

$$c = \inf_{\gamma \in \Gamma} \sup_{u \in \gamma(X)} f(u)$$

is finite, where $\Gamma = \{\gamma \in C(X, W) : \gamma(X)$ is closed, $\gamma|_A = g\}$, $\sup f(g(A)) \le \inf f(B)$ and f satisfies the $(Ce)_c$ condition, then $c \ge \inf f(B)$ and is a critical value of f. If $c = \inf f(B)$, then J has a critical point with critical value c on B.

Theorem B (*See [4]*). *Let W be a Banach space and $f \in C^1(W, R)$. Let A, B, P, S be four subsets of W with $S \subset P$ and $B \subset A$ such that (P, S) links (A, B) and*

$$\sup_S f \le \inf_A f, \qquad \sup_P f \le \inf_B f$$

(we agree that $\sup \emptyset = -\infty$, $\inf \emptyset = +\infty$). Define

$$c = \inf_{\eta \in \mathcal{N}} \sup f(\eta(P \times \{1\})),$$

where $\mathcal{N}$ is the set of deformations $\eta : P \times [0, 1] \to W \setminus B$ with $\eta(S \times [0, 1]) \cap A = \emptyset$. Then we have

$$\inf_A f \le c \le \sup_P f.$$

Moreover, if f satisfies the $(PS)_c$ condition, c is a critical value of f.

Proof of Theorem 1. Here, we assume that (7) and (9) hold, since the proof is similar to that under conditions (6) and (8). We will divide the proof into two steps.

Step 1. We will prove that the functional J satisfies the (Ce) condition. Let $\{u_n\}$ be a (Ce) sequence for the functional J, that is,

$$J(u_n) \to c \in R \quad \text{and} \tag{14}$$

$$\|J'(u_n)\|(1 + \|u_n\|) \to 0 \quad \text{as } n \to \infty. \tag{15}$$

We first claim that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, and then prove that $\{u_n\}$ has a convergent subsequence. Suppose, by contradiction, that $\|u_n\| \to \infty$ as $n \to \infty$. From (14) and (15), there exists a positive constant M_0 such that

$$\begin{aligned} M_0 &\le \limsup_{n\to\infty}(pJ(u_n) - \langle J'(u_n), u_n\rangle) \\ &= \limsup_{n\to\infty} \int_\Omega f(x,u_n)u_n - pF(x,u_n)dx. \end{aligned} \tag{16}$$

From the right of (9), for fixed $\varepsilon_0 > 0$, there exists $R_0 > 0$ such that

$$|F(x,t)| \le \frac{\lambda_k + \varepsilon_0}{p}|t|^p, \quad \forall x \in \Omega,\ |t| \ge R_0,$$

and hence, from the continuity of F, there exists $M_1 = M_1(\varepsilon_0) > 0$ such that

$$|F(x,t)| \le \frac{\lambda_k + \varepsilon_0}{p}|t|^p + M_1, \quad \forall (x,t) \in \Omega \times R.$$

From (4), (14) and the above inequality, for n large enough, we have

$$\begin{aligned} \|u_n\|^p &= pJ(u_n) + p\int_\Omega F(x,u_n)dx \\ &\le p(c+1) + (\lambda_k + \varepsilon_0)\int_\Omega |u_n|^p dx + pM_1|\Omega| \\ &\le M_2 + (\lambda_k + \varepsilon_0)\|u_n\|_p^p, \end{aligned} \tag{17}$$

where $M_2 = p(c + 1 + M_1|\Omega|)$ and $|\Omega|$ denotes the Lebesgue measure of Ω. Let $\tilde{u}_n = u_n/\|u_n\|^p$; then there is $\tilde{u} \in W_0^{1,p}(\Omega)$ such that $\tilde{u}_n \rightharpoonup \tilde{u}$ weakly in $W_0^{1,p}(\Omega)$ and $\tilde{u}_n \to \tilde{u}$ strongly in $L^p(\Omega)$. Dividing (17) by $\|u_n\|^p$ and letting $n \to \infty$, we have

$$1 \le (\lambda_k + \varepsilon_0)\|\tilde{u}\|_p^p,$$

which implies that there exists $\widetilde{\Omega} \subset \Omega$ with positive measure such that $\tilde{u}(x) \ne 0$ for $x \in \widetilde{\Omega}$. Consequently, from the definition of $\tilde{u}_n$, it follows that $|u_n(x)| \to \infty$ as $n \to \infty$ for $x \in \widetilde{\Omega}$. From (7), there is a positive constant M_3 such that

$$f(x,t)t - pF(x,t) < M_3, \quad \text{for all } (x,t) \in \Omega \times R.$$

By Fatou's lemma, (7) and the above inequality, we obtain

$$\begin{aligned} \limsup_{n\to\infty}\int_\Omega (f(x,u_n)u_n - pF(x,u_n))dx &\le \int_{\widetilde{\Omega}} \limsup_{n\to\infty}\left(f(x,u_n)u_n - pF(x,u_n)\right)dx + M_3|\Omega \setminus \tilde{\Omega}| \\ &\to -\infty, \end{aligned}$$

which contradicts (16). Hence, $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, that is, there is $M_4 > 0$ such that

$$\|u_n\| \le M_4, \quad \text{for all } n \in N. \tag{18}$$

Thus, there is a subsequence of $\{u_n\}$, without any loss of generality still denoted by $\{u_n\}$, and $u \in W_0^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ and $u_n \to u$ strongly in $L^q(\Omega)$. Consequently, from (15) and (18), one has

$$\lim_{n\to\infty}\langle J'(u_n), u_n - u\rangle = 0. \tag{19}$$

From (2), Hölder's inequality, and (4) and (18), it follows that

$$\begin{aligned} \left|\int_\Omega f(x,u_n)(u_n - u)dx\right| &\le c_1\int_\Omega (1 + |u_n|^{q-1})|u_n - u|dx \\ &\le c_1\left(|\Omega|^{\frac{q-1}{q}} + \|u_n\|_q^{q-1}\right)\|u_n - u\|_q \\ &\le c_1\left(|\Omega|^{\frac{q-1}{q}} + C\|u_n\|^{q-1}\right)\|u_n - u\|_q \\ &\to 0 \end{aligned}$$

as $n \to \infty$. Combining the above inequality and (19), we get

$$\int_\Omega (|\nabla u_n|^{p-2}\nabla u_n, \nabla(u_n - u))dx \to 0 \quad \text{as } n \to \infty.$$

Similarly, we also obtain

$$\lim_{n\to\infty}\int_\Omega (|\nabla u|^{p-2}\nabla u, \nabla(u_n-u))dx = 0,$$

and hence,

$$\lim_{n\to\infty}\int_\Omega ((|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u), \nabla(u_n-u))dx = 0.$$

From Clarkson's inequality, that is, there being a $C_p > 0$ such that for all $\mu, \nu \in R^N$,

$$|\mu-\nu|^p \le \begin{cases} C_p(|\mu|^{p-2}\mu - |\nu|^{p-2}\nu)(\mu-\nu), & p \ge 2, \\ C_p((|\mu|^{p-2}\mu - |\nu|^{p-2}\nu)(\mu-\nu))^{\frac{p}{2}}(|\mu|+|\nu|)^{\frac{(2-p)p}{2}}, & 1 < p < 2, \end{cases}$$

it follows that

$$\lim_{n\to\infty}\int_\Omega |\nabla u_n - \nabla u|^p dx = 0,$$

that is, $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$.

Step 2. We will verify that the functional J satisfies the homotopical linking given by Proposition 1.

First of all, we define the two subsets of S as follows:

$$A_0 = \left\{u \in S : \int_\Omega |\nabla u|^p dx \le \lambda_{k-1}\int_\Omega |u|^p dx\right\},$$

$$B_0 = \left\{u \in S : \int_\Omega |\nabla u|^p dx \ge \lambda_{k}\int_\Omega |u|^p dx\right\}.$$

From [15], we have

$$i(A_0) = i(S \setminus B_0) < \infty.$$

From the left of (9) and the continuity of F, for any $\varepsilon > 0$, there is a positive constant $M_5 = M_5(\varepsilon)$ such that

$$F(x,t) \le \frac{\lambda_k - \varepsilon}{p}|t|^p + M_5 \quad \text{for all } (x,t) \in \Omega \times R.$$

For any $u \in B_0$, we have

$$\begin{aligned} J(u) &\ge \frac{1}{p}(\|u\|^p - (\lambda_k-\varepsilon)\|u\|_p^p) - M_5|\Omega| \\ &\ge \frac{\varepsilon}{p\lambda_k}\|u\|^p - M_5|\Omega|. \end{aligned} \tag{20}$$

On the other hand, defining

$$G(x,s) = F(x,s) - \frac{1}{p}\lambda_{k-1}|s|^p,$$

we obtain

$$G'(x,s)s - pG(x,s) = f(x,s)s - pF(x,s).$$

From (7), for any $M > 0$, there is an $L > 0$ such that

$$G'(x,s)s - pG(x,s) \le -M$$

for any $x \in \Omega$ and $|s| > L$. Hence, for $s > 0$, from the above inequality, we have

$$\frac{d}{ds}\left(\frac{G(x,s)}{s^p}\right) = \frac{G'(x,s)s - pG(x,s)}{s^{p+1}} \le -\frac{M}{s^{p+1}}.$$

Integrating the above expression over the interval $[t,T] \subset [L,\infty)$, one has

$$\frac{G(x,t)}{t^p} \ge \frac{G(x,T)}{T^p} - \frac{M}{p}\left(\frac{1}{T^p} - \frac{1}{t^p}\right).$$

Since $\limsup_{T\to+\infty} \frac{G(x,T)}{T^p} \geq 0$, letting $T \to +\infty$, we obtain $G(x, t) \geq M/p$ for $t \geq L$ and $x \in \Omega$. Similarly, $G(x, t) \geq M/p$ for $t \leq -L$ and $x \in \Omega$. Hence, we have

$$\lim_{|t|\to\infty} G(x, t) = +\infty \quad \text{uniformly for } x \in \Omega.$$

Therefore, letting $R > 0$, for any $u \in A_0$, from the definition of A_0 and the above expression, one has

$$\begin{aligned} J(Ru) &= \frac{R^p}{p}\|u\|^p - \int_\Omega F(x, Ru)dx \\ &= \frac{R^p}{p}(\|u\|^p - \lambda_{k-1}\|u\|_p^p) - \int_\Omega G(x, Ru)dx \\ &\leq -\int_\Omega G(x, Ru)dx \\ &\to -\infty, \end{aligned}$$

as $R \to +\infty$, which combined with (20) implies that for fixed $\varepsilon > 0$, there exists $R_0 > 0$ such that

$$\max_{u\in A_0, R\geq R_0} J(Ru) < \inf_{u\in B_0} J(u).$$

Therefore, Theorem 1 is proved by Theorem A and Proposition 1, where $A = \{R_0 u : u \in A_0\}$ and $B = B_0$. $\square$

Proof of Theorem 2. We will divide the proof into two steps.

Step 1. We will prove that the functional J satisfies the (PS) condition. Here we assume that condition (12) holds, and similarly for (13). Let $\{u_n\}$ be a (PS) sequence of the functional J, that is,

$$J(u_n) \to c \in R \quad \text{and} \quad J'(u_n) \to 0 \quad \text{as } n \to \infty. \tag{21}$$

From Step 1 of proof of Theorem 1, we only prove that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. From (12), there is a positive constant M_6 such that

$$f(x, t)t - pF(x, t) \geq a|t|^\mu - M_6, \quad \forall (x, t) \in \Omega \times R,$$

and hence, we have

$$\begin{aligned} pJ(u_n) - \langle J'(u_n), u_n\rangle &= \int_\Omega (f(x, u_n)u_n - pF(x, u_n))dx \\ &\geq a\int_\Omega |u_n|^\mu dx - M_6|\Omega|. \end{aligned}$$

Combining (21) and the above inequality implies that

$$\frac{\int_\Omega |u_n|^\mu dx}{\|u_n\|} \to 0 \quad \text{as } n \to \infty. \tag{22}$$

From the Hölder inequality and (2) and (3), we have

$$\begin{aligned} \langle J'(u_n), u_n\rangle &= \|u_n\|^p - \int_\Omega f(x, u_n)u_n dx \\ &\geq \|u_n\|^p - \int_\Omega |f(x, u_n)| \cdot |u_n|dx \\ &\geq \|u_n\|^p - c_1\int_\Omega (|u_n|^{q-1} \cdot |u_n| + |u_n|)dx \\ &\geq \|u_n\|^p - c_1\left(\int_\Omega |u_n|^{(q-1)\cdot\frac{\mu}{q-1}}dx\right)^{\frac{q-1}{\mu}} \cdot \left(\int_\Omega |u_n|^{\frac{\mu}{\mu+1-q}}dx\right)^{\frac{\mu+1-q}{\mu}} - \|u_n\|_1 \\ &\geq \|u_n\|^p - c_1C\|u_n\|_\mu^{q-1} \cdot \|u_n\| - c_1C\|u_n\| \end{aligned}$$

for all n. Noting that $\mu > (q-1)\max\{\frac{1}{p-1}, \frac{p^*}{p^*-1}\}$ if $1 < p < N$ or $\mu > (q-1)\max\{1, \frac{1}{p-1}\}$ if $p \geq N$, by (22) and the above inequality, we have that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$.

Step 2. We will verify that the functional J satisfies relative cohomological linking in Proposition 2.

First of all, for $\lambda_{k-1} < \lambda_k$, let

$$C_- = \left\{ u \in W_0^{1,p}(\Omega) : \int_\Omega |\nabla u|^p dx \le \lambda_{k-1} \int_\Omega |u|^p dx \right\},$$
$$C_+ = \left\{ u \in W_0^{1,p}(\Omega) : \int_\Omega |\nabla u|^p dx \ge \lambda_k \int_\Omega |u|^p dx \right\},$$

where C_-, C_+ are the symmetric closed subsets in $W_0^{1,p}(\Omega)$ and $C_- \cap C_+ = \{0\}$. From Theorems 2.7 and 3.2 of [8], it follows that $(W_0^{1,p}(\Omega), C_- \setminus \{0\})$ links C_+ cohomologically in dimension $i(C_- \setminus \{0\}) = k - 1$ over Z_2. Let e be an eigenfunction corresponding to λ_k.

From (4) and (10), we have

$$J(u) \le \frac{1}{p}(\|u\|^p - \lambda_{k-1}\|u\|_p^p) \le 0, \quad \forall\, u \in C_-. \tag{23}$$

From the right of (11), fixing a constant $\hat{\beta}$ such that $\lambda_k < \hat{\beta} < \beta$, there is a positive constant M_7 such that

$$F(x,t) \ge \frac{\hat{\beta}}{p}|t|^p - M_7, \quad \forall (x,t) \in \Omega \times R.$$

Hence, from (4) and the above inequality, for all $u \in \{u + te : u \in C_-, t > 0\}$, we have

$$J(u) \le \frac{1}{p}(\|u\|^p - \hat{\beta}\|u\|_p^p) + M_7|\Omega|.$$

Hence, there is $r_- > 0$ such that

$$J(u) \le 0, \tag{24}$$

for all $u \in H = \{u + te : u \in C_-, t > 0, \|u + te\| = r_-\}$. Combining (23) with (24) yields that

$$J(u) \le 0,$$

for any $u \in H \cup \{u \in C_- : \|u\| \le r_-\}$.

On the other hand, from the left of (11), fixing a constant $\hat{\alpha}$ such that $\alpha < \hat{\alpha} < \lambda_k$, there is $\delta > 0$ such that

$$F(x,t) \le \frac{\hat{\alpha}}{p}|t|^p, \quad \forall x \in \Omega,\ |t| \le \delta.$$

Combining (2) and the above inequality, there is a positive constant M_8 such that

$$F(x,t) \le \frac{\hat{\alpha}}{p}|t|^p + M_8|t|^q, \quad \forall (x,t) \in \Omega \times R. \tag{25}$$

Hence, from (4) and (25), for all $u \in C_+$, we have

$$\begin{aligned} J(u) &\ge \frac{1}{p}(\|u\|^p - \hat{\alpha}\|u\|_p^p) - M_8\|u\|_q^q \\ &\ge \frac{\lambda_k - \hat{\alpha}}{p\lambda_k}\|u\|^p - CM_8\|u\|^q. \end{aligned}$$

From $\lambda_k > \hat{\alpha}$ and $q > p$, there are two positive constants ϖ and $r_- > r_+ > 0$ such that

$$J(u) \ge \varpi, \quad \forall u \in C_+, \qquad \|u\| = r_+.$$

Finally, from Theorem B and Proposition 2, Theorem 2 is proved. □

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References

[1] E.R. Fadell, P.H. Rabinowitz, Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems, Invent. Math. 45 (1978) 139–174.

[2] K. Perera, A. Szulkin, p-Laplacian problems where the nonlinearity crosses an eigenvalue, Discrete Contin. Dyn. Syst. 13 (2005) 743–753.

[3] P. Bartolo, V. Benci, D. Fortunato, Abstract critical point theorems and applications to some nonlinear problems with strong resonance at infinity, Nonlinear Anal. 7 (1983) 981–1012.
[4] M. Frigon, On a new notion of linking and application to elliptic problems at resonance, J. Differential Equations 153 (1) (1999) 96–120.
[5] P.H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, in: CBMS. Reg. Conf. Ser. in Math., vol. 65, Amer. Math. Soc., Providence, RI, 1986, pp. 1–100.
[6] D.G. Costa, C.A. Magalhaes, Existence results for perturbations of the p-Laplacian, Nonlinear Anal. 24 (1995) 409–418.
[7] P. Drábek, S.B. Robinson, Resonance problems for the p-Laplacian, J. Funct. Anal. 169 (1) (1999) 189–200.
[8] M. Degiovanni, S. Lancelotti, Linking over cones and nontrivial solutions for p-Laplace equations with p-superlinear nonlinearity, Ann. Inst. H. Poincaré Anal. Non Linéaire 24 (2007) 907–919.
[9] A.R. El Amrouss, M. Moussaoui, Minimax principles for critical point theory in applications to quasilinear boundary-value problems, Electron. J. Differential Equations 18 (2001) 1–9.
[10] X. Fan, Z. Li, Linking and existence results for perturbations of the p-Laplacian, Nonlinear Anal. 42 (8) (2000) 1413–1420.
[11] Q.S. Jiu, J.B. Su, Existence and multiplicity results for Dirichlet problem with p-Laplacian, J. Math. Anal. Appl. 281 (2003) 209–222.
[12] G. Li, H.S. Zhou, Multiple solutions to p-Laplacian problems with asymptotic nonlinearity as u^{p-1} at infinity, J. Lond. Math. Soc. 65 (2002) 123–138.
[13] Z.Q. Ou, C. Li, Existence of solutions for Dirichlet problem with p-Laplacian, Nonlinear Anal. 75 (2012) 4914–4919.
[14] K. Perera, Nontrivial critical groups in p-Laplacian problems via the Yang index, Topol. Methods Nonlinear Anal. 21 (2) (2003) 301–310.
[15] K. Perera, R.P. Agarwal, D. O'Regan, Morse-theoretic aspects of p-Laplacian type operators, in: Math. Surveys Monogr., vol. 161, Amer. Math. Soc., Providence, RI, 2010.
[16] D.G. Costa, C.A. Magalhaes, A unified approach to a class of strongly indefinite functionals, J. Differential Equations 125 (1995) 521–547.
[17] E.H. Spanier, Algebraic Topology, McGraw-Hill Book Co., New York, 1966.