Available online at www.sciencedirect.com

ScienceDirect

Journal of Algebra 309 (2007) 367–385

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

FI-injective and *FI*-flat modules

Lixin Mao [a,b], Nanqing Ding [a,*]

[a] *Department of Mathematics, Nanjing University, Nanjing 210093, China*
[b] *Institute of Mathematics, Nanjing Institute of Technology, Nanjing 211167, China*

Received 6 May 2006

Communicated by Luchezar L. Avramov

Abstract

Let R be a ring. A left R-module M (respectively right R-module N) is called *FI*-injective (respectively *FI*-flat) if $\mathrm{Ext}^1(G, M) = 0$ (respectively $\mathrm{Tor}_1(N, G) = 0$) for any *FP*-injective left R-module G. Suppose R is a left coherent ring. It is shown that a left R-module M is *FI*-injective if and only if M is a direct sum of an injective left R-module and a reduced *FI*-injective left R-module; a finitely presented right R-module M is *FI*-flat if and only if M is a cokernel of a flat preenvelope of a right R-module. These modules together with the left derived functors of Hom are used to study the *FP*-injective dimensions of modules and rings.

Keywords: *FP*-injective module; *FP*-injective dimension; *FI*-injective module; *FI*-flat module; (Pre)cover; (Pre)envelope

1. Introduction

We first recall some known notions and facts needed in the sequel.

Let R be a ring. A left R-module M is called *FP-injective* (or *absolutely pure*) [15,19] if $\mathrm{Ext}^1(N, M) = 0$ for all finitely presented left R-modules N. The *FP-injective dimension of* M, denoted by *FP-id*(M), is defined to be the smallest nonnegative integer n such that $\mathrm{Ext}^{n+1}(F, M) = 0$ for every finitely presented left R-module F (if no such n exists, set *FP-id*$(M) = \infty$), and $l.FP\text{-}\mathrm{dim}(R)$ is defined as $\sup\{FP\text{-}id(M)\colon\ M \text{ is a left } R\text{-module}\}$.

* Corresponding author.
E-mail addresses: maolx2@hotmail.com (L. Mao), nqding@nju.edu.cn (N. Ding).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2006.10.019

Let $\mathcal{C}$ be a class of R-modules and M an R-module. Following [7], we say that a homomorphism $\phi : M \to C$ is a *$\mathcal{C}$-preenvelope* if $C \in \mathcal{C}$ and the abelian group homomorphism $\mathrm{Hom}(\phi, C') : \mathrm{Hom}(C, C') \to \mathrm{Hom}(M, C')$ is surjective for each $C' \in \mathcal{C}$. A $\mathcal{C}$-preenvelope $\phi : M \to C$ is said to be a *$\mathcal{C}$-envelope* if every endomorphism $g : C \to C$ such that $g\phi = \phi$ is an isomorphism. Dually we have the definitions of a *$\mathcal{C}$-precover* and a *$\mathcal{C}$-cover*. $\mathcal{C}$-envelopes ($\mathcal{C}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism.

In what follows, we write ${}_R\mathcal{M}$ and $\mathcal{FI}$ for the categories of all left R-modules and all *FP*-injective left R-modules, respectively. Recall that a ring R is called *left coherent* if every finitely generated left ideal is finitely presented. It has been recently proven that every left R-module has an *FP*-injective cover over a left coherent ring R (see [16]), so every left R-module M has a *left $\mathcal{FI}$-resolution*, that is, there is a $\mathrm{Hom}(\mathcal{FI}, -)$ exact complex $\cdots \to F_1 \to F_0 \to M \to 0$ (not necessarily exact) with each F_i *FP*-injective. Write

$$K_0 = M, \qquad K_1 = \ker(F_0 \to M), \qquad K_i = \ker(F_{i-1} \to F_{i-2}) \quad \text{for } i \geqslant 2.$$

The nth kernel K_n ($n \geqslant 0$) is called the *nth $\mathcal{FI}$-syzygy of M*.

On the other hand, every left R-module M has an *FP*-injective preenvelope over any ring R (see [11]). So M has a *right $\mathcal{FI}$-resolution*, that is, there is a $\mathrm{Hom}(-, \mathcal{FI})$ exact complex $0 \to M \to F^0 \to F^1 \to \cdots$ with each F^i *FP*-injective. Obviously, the complex is exact. Let

$$L^0 = M, \qquad L^1 = \operatorname{coker}\bigl(M \to F^0\bigr), \qquad L^i = \operatorname{coker}\bigl(F^{i-2} \to F^{i-1}\bigr) \quad \text{for } i \geqslant 2.$$

The nth cokernel L^n ($n \geqslant 0$) is called the *nth $\mathcal{FI}$-cosyzygy of M*.

Note that $\mathrm{Hom}(-,-)$ is left balanced on ${}_R\mathcal{M} \times {}_R\mathcal{M}$ by $\mathcal{FI} \times \mathcal{FI}$ for a left coherent ring R (see [11, Definition 8.2.13]). Thus the *nth left derived functor* of $\mathrm{Hom}(-,-)$, which is denoted by $\mathrm{Ext}_n(-,-)$, can be computed using a right $\mathcal{FI}$-resolution of the first variable or a left $\mathcal{FI}$-resolution of the second variable. Following [11, Definition 8.4.1], the *left $\mathcal{FI}$-dimension of a left R-module M*, denoted by left $\mathcal{FI}$-dim M, is defined as $\inf\{n$: there is a left $\mathcal{FI}$-resolution of the form $0 \to F_n \to \cdots \to F_0 \to M \to 0$ of $M\}$. If there is no such n, set left $\mathcal{FI}$-dim $M = \infty$. The *global left $\mathcal{FI}$-dimension of ${}_R\mathcal{M}$*, denoted by *gl* left $\mathcal{FI}$-$\dim_R \mathcal{M}$, is defined to be sup{left $\mathcal{FI}$-dim M: $M \in {}_R\mathcal{M}$} and is infinite otherwise. The right versions can be defined similarly.

Recall that a left R-module M is called *reduced* [11] if M has no nonzero injective submodules.

In Section 2 of this paper, we introduce the concepts of *FI*-injective and *FI*-flat modules. It is shown that a left R-module M is *FI*-injective if and only if M is a kernel of an *FP*-injective precover $A \to B$ with A injective. For a left coherent ring R, we prove that a left R-module M is *FI*-injective if and only if M is a direct sum of an injective left R-module and a reduced *FI*-injective left R-module; a finitely presented right R-module M is *FI*-flat if and only if M is a cokernel of a flat preenvelope of a right R-module.

In Section 3, we investigate the *FP*-injective dimensions of modules and rings in terms of *FI*-injective and *FI*-flat modules and the left derived functors $\mathrm{Ext}_n(-,-)$. Let R be a left coherent ring. We first give some characterizations of left semihereditary rings. It is proven that R is left semihereditary (i.e., $l.FP\text{-}\dim(R) \leqslant 1$) if and only if the canonical map $\sigma : \mathrm{Ext}_0(M, N) \to \mathrm{Hom}(M, N)$ is a monomorphism for all left R-modules M and N if and only if every *FI*-injective left R-module is injective if and only if every *FI*-flat right R-module is flat. Then it is shown that $l.FP\text{-}\dim(R) \leqslant n$ ($n \geqslant 2$) if and only if $\mathrm{Ext}_{n+k}(M, N) = 0$ for all left R-modules M, N and all

$k \geqslant -1$. Moreover, we get that $l.FP\text{-dim}(R) \leqslant n$ ($n \geqslant 2$) if and only if $\mathrm{Ext}_{n+k}(M, N) = 0$ for all pure-injective left R-modules M, N and all $k \geqslant -1$. Finally we prove that a ring R is left coherent and $l.FP\text{-dim}(R) \leqslant 2$ if and only if every left R-module has an FP-injective cover with the unique mapping property if and only if R is a left coherent ring and $\mathrm{Ext}_k(M, N) = 0$ for all left R-modules M, N and all $k \geqslant 1$ if and only if R is a left coherent ring and every finitely presented FI-flat right R-module has an epic flat (pre)envelope.

Throughout this paper, R is an associative ring with identity and all modules are unitary. M_R (${}_RM$) denotes a right (left) R-module. For an R-module M, $E(M)$ stands for the injective envelope of M, the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is denoted by M^+, and $id(M)$ ($fd(M)$) is the injective (flat) dimension of M. Let M and N be R-modules. $\mathrm{Hom}(M, N)$ (respectively $\mathrm{Ext}^n(M, N)$) means $\mathrm{Hom}_R(M, N)$ (respectively $\mathrm{Ext}^n_R(M, N)$), and similarly $M \otimes N$ (respectively $\mathrm{Tor}_n(M, N)$) denotes $M \otimes_R N$ (respectively $\mathrm{Tor}^R_n(M, N)$) for an integer $n \geqslant 1$. For unexplained concepts and notations, we refer the reader to [11,18,20].

2. *FI*-injective modules and *FI*-flat modules

We begin with the following

Definition 2.1. A left R-module M is called *FI-injective* if $\mathrm{Ext}^1(G, M) = 0$ for any FP-injective left R-module G.

A right R-module N is said to be *FI-flat* if $\mathrm{Tor}_1(N, G) = 0$ for any FP-injective left R-module G.

Remark 2.2. (1) We note that any FI-injective left R-module is copure injective in sense of [9] and any FI-flat right R-module is copure flat in sense of [10]. If R is a left noetherian ring, then FI-injective left R-modules and FI-flat right R-modules coincide with copure injective left R-modules and copure flat right R-modules, respectively.

(2) A right R-module M is FI-flat if and only if M^+ is FI-injective by the standard isomorphism: $\mathrm{Ext}^1(N, M^+) \cong \mathrm{Tor}_1(M, N)^+$ for any left R-module N.

Proposition 2.3. *The following hold for a left coherent ring R:*

(1) *A left R-module M is injective if and only if M is FI-injective and FP-$id(M) \leqslant 1$.*
(2) *A right R-module N is flat if and only if N is FI-flat and $fd(N) \leqslant 1$.*

Proof. (1) "Only if" part is trivial.

"If" part. Let M be an FI-injective left R-module and FP-$id(M) \leqslant 1$. Then there is an exact sequence $0 \to M \to E \to L \to 0$ with E injective. Note that L is FP-injective by [19, Lemma 3.1] since R is a left coherent ring. So the exact sequence is split, and hence M is injective.

(2) "Only if" part is trivial.

"If" part. For any FI-flat right R-module N with $fd(N) \leqslant 1$, we have N^+ is FI-injective by Remark 2.2(2). Thus N^+ is injective by (1) since FP-$id(N^+) \leqslant 1$. So N is flat. □

Proposition 2.4. *The following are equivalent for a left R-module M:*

(1) *M is FI-injective.*

(2) *For every exact sequence* $0 \to M \to E \to L \to 0$, *where* E *is FP-injective,* $E \to L$ *is an FP-injective precover of* L.
(3) M *is a kernel of an FP-injective precover* $f : A \to B$ *with* A *injective.*
(4) M *is injective with respect to every exact sequence* $0 \to A \to B \to C \to 0$, *where* C *is FP-injective.*

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (4) are clear by definitions.

(2) $\Rightarrow$ (3) is obvious since there exists a short exact sequence $0 \to M \to E(M) \to E(M)/M \to 0$.

(3) $\Rightarrow$ (1). Let M be a kernel of an *FP*-injective precover $f : A \to B$ with A injective. Then we have an exact sequence $0 \to M \to A \to A/M \to 0$. So, for any *FP*-injective left R-module N, the sequence $\mathrm{Hom}(N, A) \to \mathrm{Hom}(N, A/M) \to \mathrm{Ext}^1(N, M) \to 0$ is exact. It is easy to verify that $\mathrm{Hom}(N, A) \to \mathrm{Hom}(N, A/M) \to 0$ is exact by (3). Thus $\mathrm{Ext}^1(N, M) = 0$, and so (1) follows.

(4) $\Rightarrow$ (1). For each *FP*-injective left R-module N, there exists a short exact sequence $0 \to K \to P \to N \to 0$ with P projective, which induces an exact sequence $\mathrm{Hom}(P, M) \to \mathrm{Hom}(K, M) \to \mathrm{Ext}^1(N, M) \to 0$. Note that $\mathrm{Hom}(P, M) \to \mathrm{Hom}(K, M) \to 0$ is exact by (4). Hence $\mathrm{Ext}^1(N, M) = 0$, as desired. □

Proposition 2.5. *Let* R *be a left coherent ring. Then the following are equivalent for a left* R*-module* M:

(1) M *is a reduced FI-injective left* R*-module.*
(2) M *is a kernel of an FP-injective cover* $f : A \to B$ *with* A *injective.*

Proof. (1) $\Rightarrow$ (2). By Proposition 2.4, the natural map $\pi : E(M) \to E(M)/M$ is an *FP*-injective precover. Note that $E(M)/M$ has an *FP*-injective cover, and $E(M)$ has no nonzero direct summand K contained in M since M is reduced. It follows that $\pi : E(M) \to E(M)/M$ is an *FP*-injective cover by [20, Corollary 1.2.8], and hence (2) follows.

(2) $\Rightarrow$ (1). Let M be a kernel of an *FP*-injective cover $\alpha : A \to B$ with A injective. By Proposition 2.4, M is *FI*-injective. Now let K be an injective submodule of M. Suppose $A = K \oplus L$, $p : A \to L$ is the projection and $i : L \to A$ is the inclusion. It is easy to see that $\alpha(ip) = \alpha$ since $\alpha(K) = 0$. Therefore ip is an isomorphism since α is a cover. Thus i is epic, and hence $A = L$, $K = 0$. So M is reduced. □

Theorem 2.6. *Let* R *be a left coherent ring. Then a left* R*-module* M *is FI-injective if and only if* M *is a direct sum of an injective left* R*-module and a reduced FI-injective left* R*-module.*

Proof. "If" part is clear.

"Only if" part. Let M be an *FI*-injective left R-module. Consider the exact sequence $0 \to M \to E(M) \to E(M)/M \to 0$. Note that $E(M) \to E(M)/M$ is an *FP*-injective precover of $E(M)/M$ by Proposition 2.4. But $E(M)/M$ has an *FP*-injective cover $L \to E(M)/M$, so we have the commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{f} & L & \longrightarrow & E(M)/M & \longrightarrow & 0 \\
 & & \phi\big\downarrow & & \gamma\big\downarrow & & \big\| & & \\
0 & \longrightarrow & M & \xrightarrow[\alpha]{} & E(M) & \longrightarrow & E(M)/M & \longrightarrow & 0 \\
 & & \sigma\big\downarrow & & \beta\big\downarrow & & \big\| & & \\
0 & \longrightarrow & K & \xrightarrow{f} & L & \longrightarrow & E(M)/M & \longrightarrow & 0.
\end{array}$$

Note that $\beta\gamma$ is an isomorphism, and so $E(M) = \ker(\beta) \oplus \mathrm{im}(\gamma)$. Thus L and $\ker(\beta)$ are injective (for $\mathrm{im}(\gamma) \cong L$). Therefore K is a reduced *FI*-injective module by Proposition 2.5. Since $\sigma\phi$ is an isomorphism by the Five Lemma, we have $M = \ker(\sigma) \oplus \mathrm{im}(\phi)$, where $\mathrm{im}(\phi) \cong K$. In addition, we get the commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \ker(\sigma) & \longrightarrow & \ker(\beta) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & M & \xrightarrow[\alpha]{} & E(M) & \longrightarrow & E(M)/M & \longrightarrow & 0 \\
 & & \sigma\big\downarrow & & \beta\big\downarrow & & \big\| & & \\
0 & \longrightarrow & K & \xrightarrow{f} & L & \longrightarrow & E(M)/M & \longrightarrow & 0. \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Hence $\ker(\sigma) \cong \ker(\beta)$ by the 3×3 Lemma [18, Exercise 6.16, p.175]. This completes the proof. □

It is well known that R is a left coherent ring if and only if every right R-module has a flat preenvelope (see [7]). Here we have

Proposition 2.7. *Let R be a left coherent ring.*

(1) *If L is a cokernel of a flat preenvelope $f : K \to F$ of a right R-module K, then L is FI-flat.*
(2) *If M is a finitely presented FI-flat right R-module, then M is a cokernel of a flat preenvelope.*

Proof. (1) There is an exact sequence $0 \to \mathrm{im}(f) \xrightarrow{i} F \to L \to 0$. It is clear that $i : \mathrm{im}(f) \to F$ is a flat preenvelope. For any *FP*-injective left R-module N, N^+ is flat by [12, Theorem 2.2]. Thus we obtain an exact sequence $\mathrm{Hom}(F, N^+) \to \mathrm{Hom}(\mathrm{im}(f), N^+) \to 0$, which yields the exactness of $(F \otimes N)^+ \to (\mathrm{im}(f) \otimes N)^+ \to 0$. So the sequence $0 \to \mathrm{im}(f) \otimes N \to F \otimes N$ is exact. But the flatness of F implies the exactness of $0 \to \mathrm{Tor}_1(L, N) \to \mathrm{im}(f) \otimes N \to F \otimes N$, and hence $\mathrm{Tor}_1(L, N) = 0$.

(2) Let M be a finitely presented FI-flat right R-module. There is an exact sequence $0 \to K \to P \to M \to 0$ with P projective and both P and K finitely generated. We claim that $K \to P$ is a flat preenvelope. In fact, for any flat right R-module F, we have $\mathrm{Tor}_1(M, F^+) = 0$, and so we get the following commutative diagram with the first row exact:

$$\begin{array}{ccccc} 0 & \longrightarrow & K \otimes F^+ & \xrightarrow{\alpha} & P \otimes F^+ \\ & & \Big\downarrow \tau_{K,F} & & \Big\downarrow \tau_{P,F} \\ & & \mathrm{Hom}(K, F)^+ & \xrightarrow{\theta} & \mathrm{Hom}(P, F)^+. \end{array}$$

Note that $\tau_{K,F}$ is an epimorphism and $\tau_{P,F}$ is an isomorphism by [4, Lemma 2]. Thus θ is a monomorphism, and hence $\mathrm{Hom}(P, F) \to \mathrm{Hom}(K, F)$ is epic, as required. □

Recall that R is said to be a *QF ring* if R is left noetherian and ${}_R R$ is injective.

Proposition 2.8. *R is a QF ring if and only if every left R-module is FI-injective.*

Proof. It follows from the fact that R is a *QF* ring if and only if every (*FP*-)injective left R-module is projective. □

Recall that R is called a *left IF ring* [4] if every injective left R-module is flat.

Proposition 2.9. *The following are equivalent for a ring R:*

(1) *R is a left IF ring.*
(2) *Every pure-injective left R-module is FI-injective.*
(3) *Every right R-module is FI-flat.*
(4) *Every finitely presented right R-module is FI-flat.*

Proof. (1) ⇒ (2). Let M be any pure-injective left R-module. For any *FP*-injective left R-module N, there is a pure exact sequence $0 \to N \to E \to L \to 0$ with E injective. So N is flat since E is flat. On the other hand, there is an exact sequence $0 \to K \to P \to N \to 0$ with P projective. Note that the sequence is also pure since N is flat. Thus the sequence $\mathrm{Hom}(P, M) \to \mathrm{Hom}(K, M) \to 0$ is exact, and so $\mathrm{Ext}^1(N, M) = 0$. Therefore, M is *FI*-injective.

(2) ⇒ (3). Let M be a right R-module. Then M^+ is pure-injective, and so it is *FI*-injective by (2). Thus M is *FI*-flat by Remark 2.2(2).

(3) ⇒ (4) is trivial.

(4) ⇒ (1). Let E be an injective left R-module. Then $\mathrm{Tor}_1(M, E) = 0$ for any finitely presented right R-module M by (4). So E is flat. □

We shall say that a right R-module M is *strongly FI-flat* if $\mathrm{Tor}_i(M, G) = 0$ for all *FP*-injective left R-modules G and all $i \geqslant 1$. Similarly, a left R-module N will be called *strongly FI-injective* if $\mathrm{Ext}^i(G, N) = 0$ for all *FP*-injective left R-modules G and all $i \geqslant 1$.

Theorem 2.10. *Let R be a left and right coherent ring. Consider the following conditions:*

(1) *FP-id*$(R_R) \leqslant 1$.
(2) *Every submodule of an FI-flat right R-module is FI-flat.*
(3) *Every FI-flat right R-module is strongly FI-flat.*
(4) *Every FI-injective left R-module is strongly FI-injective.*
(5) *Every quotient of an FI-injective left R-module is FI-injective.*

Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftarrow$ (4) $\Leftarrow$ (5). (1) $\Rightarrow$ (5) *holds in case R is a left perfect ring.*

Proof. (1) $\Rightarrow$ (2). Let A be a submodule of an *FI*-flat right R-module B and M an *FP*-injective left R-module. Then one gets an exact sequence $\mathrm{Tor}_2(B/A, M) \to \mathrm{Tor}_1(A, M) \to \mathrm{Tor}_1(B, M) = 0$. On the other hand, there is a pure exact sequence $0 \to M \to \Pi(R_R)^+$ since $(R_R)^+$ is a cogenerator in ${}_R\mathcal{M}$. Thus we get a split exact sequence $(\Pi(R_R)^+)^+ \to M^+ \to 0$. Note that $fd((R_R)^+) = FP\text{-}id(R_R) \leqslant 1$ by [12, Theorem 2.2], and so $fd(\Pi(R_R)^+) \leqslant 1$ since R is right coherent. It follows that $FP\text{-}id((\Pi(R_R)^+)^+) = fd(\Pi(R_R)^+) \leqslant 1$ by [12, Theorem 2.1]. Hence $fd(M) = FP\text{-}id(M^+) \leqslant 1$. Thus $\mathrm{Tor}_2(B/A, M) = 0$, and so $\mathrm{Tor}_1(A, M) = 0$. Therefore, A is *FI*-flat.

(2) $\Rightarrow$ (3). Let M be an *FI*-flat right R-module. Then there is an exact sequence $0 \to K \to P \to M \to 0$ with P projective. So K is *FI*-flat by (2). Thus M is strongly *FI*-flat by induction.

(3) $\Rightarrow$ (1). Let M be a right R-module. Then there is an exact sequence $0 \to K \to P \to M \to 0$ with P projective. Note that K has a flat preenvelope $f : K \to F$ since R is left coherent. So f is a monomorphism, and we get an exact sequence $0 \to K \to F \to L \to 0$, where L is *FI*-flat by Proposition 2.7. Thus L is strongly *FI*-flat by (3), and so K is *FI*-flat. There is an induced exact sequence $0 = \mathrm{Tor}_2(P, (R_R)^+) \to \mathrm{Tor}_2(M, (R_R)^+) \to \mathrm{Tor}_1(K, (R_R)^+) = 0$. Thus $\mathrm{Tor}_2(M, (R_R)^+) = 0$ and hence $fd((R_R)^+) \leqslant 1$. So $FP\text{-}id(R_R) \leqslant 1$ by [12, Theorem 2.2].

(5) $\Rightarrow$ (4). Let M be an *FI*-injective left R-module. Then there is an exact sequence $0 \to M \to E \to L \to 0$ with E injective. So L is *FI*-injective by (5). It is easy to check that M is strongly *FI*-injective by induction.

(4) $\Rightarrow$ (3) holds by Remark 2.2(2) and the standard isomorphism: $\mathrm{Ext}^n(N, M^+) \cong \mathrm{Tor}_n(M, N)^+$ for any right R-module M, any left R-module N and any $n \geqslant 1$ (see [18, p. 360]).

(1) $\Rightarrow$ (5). Suppose that R is a left perfect ring. Then the projective (flat) dimension of any *FP*-injective left R-module is at most 1 by the proof of (1) $\Rightarrow$ (2). So (5) holds. □

3. *FP*-injective dimensions and the left derived functors of Hom

As is mentioned in the introduction, if R is a left coherent ring, then $\mathrm{Hom}(-,-)$ is left balanced on ${}_R\mathcal{M} \times {}_R\mathcal{M}$ by $\mathcal{FI} \times \mathcal{FI}$. Let $\mathrm{Ext}_n(-,-)$ denote the nth left derived functor of $\mathrm{Hom}(-,-)$ with respect to the pair $\mathcal{FI} \times \mathcal{FI}$. Then, for two left R-modules M and N, $\mathrm{Ext}_n(M, N)$ can be computed using a right $\mathcal{FI}$-resolution of M or a left $\mathcal{FI}$-resolution of N.

Let $0 \to M \xrightarrow{g} F^0 \xrightarrow{f} F^1 \to \cdots$ be a right $\mathcal{FI}$-resolution of M. Applying $\mathrm{Hom}(-, N)$, we obtain the deleted complex $\cdots \to \mathrm{Hom}(F^1, N) \xrightarrow{f^*} \mathrm{Hom}(F^0, N) \to 0$. Then $\mathrm{Ext}_n(M, N)$ is exactly the nth homology of the complex above. There is a canonical map

$$\sigma : \mathrm{Ext}_0(M, N) = \mathrm{Hom}\big(F^0, N\big)/\mathrm{im}\big(f^*\big) \to \mathrm{Hom}(M, N)$$

defined by $\sigma(\alpha + \mathrm{im}(f^*)) = \alpha g$ for $\alpha \in \mathrm{Hom}(F^0, N)$.

Proposition 3.1. *Let R be a left coherent ring. The following are equivalent for a left R-module M:*

(1) *M is FP-injective.*
(2) *The canonical map $\sigma:\mathrm{Ext}_0(M,N)\to\mathrm{Hom}(M,N)$ is an epimorphism for any left R-module N.*
(3) *The canonical map $\sigma:\mathrm{Ext}_0(M,M)\to\mathrm{Hom}(M,M)$ is an epimorphism.*

Proof. (1) $\Rightarrow$ (2) is obvious by letting $F^0=M$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). By (3), there exists $\alpha\in\mathrm{Hom}(F^0,M)$ such that $\sigma(\alpha+\mathrm{im}(f^*))=\alpha g=1_M$. Thus M is isomorphic to a direct summand of F^0, and hence it is *FP*-injective. □

Corollary 3.2. *The following are equivalent for a left coherent ring R:*

(1) *${}_RR$ is FP-injective.*
(2) *The canonical map $\sigma:\mathrm{Ext}_0({}_RR,N)\to\mathrm{Hom}({}_RR,N)$ is an epimorphism for any left R-module N.*
(3) *The canonical map $\sigma:\mathrm{Ext}_0({}_RR,{}_RR)\to\mathrm{Hom}({}_RR,{}_RR)$ is an epimorphism.*
(4) *Every (finitely presented) left R-module has an epic FP-injective cover.*
(5) *Every (finitely presented) right R-module has a monic flat preenvelope.*
(6) *Every (finitely presented) right R-module is a submodule of a flat right R-module.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow from Proposition 3.1.

(1) $\Rightarrow$ (4). Let M be a left R-module, then M has an *FP*-injective cover g. On the other hand, there is an exact sequence $F\to M\to 0$ with F free. Since F is *FP*-injective by (1), g is an epimorphism.

(4) $\Rightarrow$ (1). Let $f:N\to{}_RR$ be an epic *FP*-injective cover. Then ${}_RR$ is isomorphic to a direct summand of N, and so ${}_RR$ is *FP*-injective.

(1) $\Rightarrow$ (5). Note that R is a right *IF* ring by [4, Theorem 1], and so (5) follows.

(5) $\Rightarrow$ (1) is clear by [14, Theorem 2.3] since every finitely presented right R-module is torsionless.

(5) $\Rightarrow$ (6) is obvious.

(6) $\Rightarrow$ (5) follows since R is a left coherent ring. □

Proposition 3.3. *Let R be a left coherent ring. Then the following are equivalent for a left R-module M:*

(1) *right $\mathcal{FI}$-dim $M\leqslant 1$.*
(2) *The canonical map $\sigma:\mathrm{Ext}_0(M,N)\to\mathrm{Hom}(M,N)$ is a monomorphism for any left R-module N.*

Proof. (1) $\Rightarrow$ (2). By (1), M has a right $\mathcal{FI}$-resolution $0\to M\to F^0\to F^1\to 0$. Thus we get an exact sequence $0\to\mathrm{Hom}(F^1,N)\to\mathrm{Hom}(F^0,N)\to\mathrm{Hom}(M,N)$ for any left R-module N. Hence σ is a monomorphism.

(2) ⇒ (1). Consider the exact sequence $0 \to M \to F^0 \to L^1 \to 0$, where $M \to F^0$ is an *FP*-injective preenvelope. We only need to show that L^1 is *FP*-injective. By [11, Theorem 8.2.3], we have the commutative diagram with exact rows:

$$\begin{array}{ccccccc} & & \mathrm{Ext}_0(L^1, L^1) & \longrightarrow & \mathrm{Ext}_0(F^0, L^1) & \longrightarrow & \mathrm{Ext}_0(M, L^1) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\sigma_1} & & \downarrow{\scriptstyle\sigma_2} & & \downarrow{\scriptstyle\sigma_3} & & \\ 0 & \longrightarrow & \mathrm{Hom}(L^1, L^1) & \longrightarrow & \mathrm{Hom}(F^0, L^1) & \longrightarrow & \mathrm{Hom}(M, L^1). & & \end{array}$$

Note that σ_2 is an epimorphism by Proposition 3.1 and σ_3 is a monomorphism by (2). Hence σ_1 is an epimorphism by the Snake Lemma [18, Theorem 6.5]. Thus L^1 is *FP*-injective by Proposition 3.1, and so (1) follows. □

Let $wD(R)$ denote the weak global dimension of a ring R. We have the following lemma which will be needed frequently.

Lemma 3.4. *Let R be a left coherent ring. Then*

(1) *right $\mathcal{FI}$-dim $M = FP$-$id(M)$ for any left R-module M;*
(2) *$wD(R) = l.FP$-dim$(R) =$ gl right $\mathcal{FI}$-dim$_R\,\mathcal{M}$.*

Proof. (1) It is clear that $FP\text{-}id(M) \leqslant$ right $\mathcal{FI}$-dim M. Conversely, we may assume that $FP\text{-}id(M) = n < \infty$. Let $0 \to M \to F^0 \to F^1 \to \cdots \to F^{n-1}$ be a partial right $\mathcal{FI}$-resolution of M. Then we get an exact sequence $0 \to M \to F^0 \to F^1 \to \cdots \to F^{n-1} \to L \to 0$. Therefore, L is *FP*-injective by [19, Lemma 3.1], and so right $\mathcal{FI}$-dim $M \leqslant n$, as desired.

(2) follows from [19, Theorem 3.3] and (1). □

Proposition 3.5. *The following are equivalent for a left coherent ring R:*

(1) *FP-$id(_RR) \leqslant 1$.*
(2) *The canonical map $\sigma : \mathrm{Ext}_0(_RR, N) \to \mathrm{Hom}(_RR, N)$ is a monomorphism for any left R-module N.*
(3) *Every finitely presented FI-flat right R-module has a monic flat preenvelope.*

Proof. (1) ⇔ (2) holds by Proposition 3.3 and Lemma 3.4.

(1) ⇒ (3). Let M be a finitely presented *FI*-flat right R-module. Then M is cokernel of a flat preenvelope $K \to F^0$ of a right R-module K by Proposition 2.7(2). Thus we have a right $\mathcal{F}lat$-resolution

$$0 \to K \to F^0 \to F^1 \to \cdots,$$

where $M = \mathrm{coker}(K \to F^0)$ and $\mathcal{F}lat$ is the class of all flat right R-modules. This resolution is exact at F^0 by (1) and [11, Theorem 8.4.31], and hence M has a monic flat preenvelope.

(3) ⇒ (1). Let $0 \to M \xrightarrow{f} F^0 \to F^1 \to \cdots$ be a right $\mathcal{P}roj_{fg}$-resolution of a finitely presented right R-module M, where $\mathcal{P}roj_{fg}$ is the class of all finitely generated projective right R-modules. Then coker(f) is a finitely presented *FI*-flat right R-module by Proposition 2.7(1), and hence it

has a monic flat preenvelope by (3). It follows that the above complex is exact at F^k for $k \geqslant 0$. So (1) holds by [11, Theorem 8.4.31]. □

Lemma 3.6. *Let $\mathcal{C}$ be a class of R-modules and M an R-module.*

(1) *If $F \to M$ and $G \to M$ are $\mathcal{C}$-precovers with kernels K and L, respectively, then $K \oplus G \cong L \oplus F$.*
(2) *If $M \to F$ and $M \to G$ are $\mathcal{C}$-preenvelopes with cokernels K and L, respectively, then $K \oplus G \cong L \oplus F$.*

Proof. (1) follows from [11, Lemma 8.6.3]. The proof of (2) is dual to that of [11, Lemma 8.6.3]. □

Theorem 3.7. *The following are equivalent for a left coherent ring R:*

(1) *R is a left semihereditary ring (i.e. $l.FP\text{-}\mathrm{dim}(R) \leqslant 1$).*
(2) *$\sigma : \mathrm{Ext}_0(M, N) \to \mathrm{Hom}(M, N)$ is monic for all left R-modules M and N.*
(3) *Every left R-module has a monic FP-injective cover.*
(4) *Every FI-injective left R-module is injective.*
(5) *Every FI-injective left R-module is FP-injective.*
(6) *Every (finitely presented) FI-flat right R-module is flat.*
(7) *Every right R-module has an epic flat (pre)envelope.*
(8) *Every finitely presented right R-module has an epic flat (pre)envelope.*
(9) *The kernel of any FP-injective (pre)cover of a left R-module is FP-injective.*
(10) *The cokernel of any FP-injective preenvelope of a left R-module is FP-injective.*
(11) *The cokernel of any flat preenvelope of a right R-module is flat.*
(12) *The kernel of any flat (pre)cover of a right R-module is flat.*

Proof. (1) ⇔ (2) holds by Proposition 3.3 and Lemma 3.4.

(1) ⇒ (4) follows from Proposition 2.3 and Lemma 3.4.

(4) ⇒ (5) is trivial.

(5) ⇒ (6). Let M be an FI-flat right R-module. Then M^+ is FI-injective by Remark 2.2(2), and hence M^+ is FP-injective by (5). So M is flat by [12, Theorem 2.1].

(6) ⇒ (8). Let M be a finitely presented right R-module. Then M has a flat preenvelope $f : M \to F$ with F finitely generated and projective. It is easy to see that the inclusion $i : \mathrm{im}(f) \to F$ is a flat preenvelope. Thus $F/\mathrm{im}(f)$ is finitely presented and FI-flat by Proposition 2.7(1), and hence it is flat by (6). It follows that $\mathrm{im}(f)$ is flat, and $M \to \mathrm{im}(f)$ is an epic flat (pre)envelope.

(8) ⇒ (7). Let M be any right R-module. Then $M = \varinjlim M_i$ with M_i finitely presented for each i. By (8), each M_i has an epic flat (pre)envelope $M_i \to F_i$. It is easy to see that $\{F_i\}$ is a direct system and $M \to \varinjlim F_i$ is an epic flat (pre)envelope.

(1) ⇒ (3). Let M be a left R-module. Then M has an FP-injective cover $f : N \to M$. Note that $\mathrm{im}(f)$ is FP-injective by (1) and [15, Theorem 2]. So the inclusion $\mathrm{im}(f) \to M$ is a monic FP-injective cover.

(3) ⇒ (9). Let $f : F \to M$ be an FP-injective precover of a left R-module M and $K = \ker(f)$. Since there exists a monic FP-injective cover $g : G \to M$ by (3), we have $K \oplus G \cong F$ by Lemma 3.6(1). So K is FP-injective.

(9) $\Rightarrow$ (1). It is enough to show that any quotient of an *FP*-injective left R-module is *FP*-injective. Let M be a quotient of an *FP*-injective left R-module. Note that M has an *FP*-injective cover $f: F \to M$. So f is an epimorphism. Since $\ker(f)$ is *FP*-injective by (9), M is *FP*-injective by [19, Lemma 3.1] (for R is a left coherent ring).

(1) $\Leftrightarrow$ (10) follows from Lemma 3.4.

(7) $\Rightarrow$ (11). The proof is dual to that of (3) $\Rightarrow$ (9).

(11) $\Rightarrow$ (1). By a proof dual to that of (9) $\Rightarrow$ (1), we can show that any submodule of a flat right R-module is flat. Thus R is a left semihereditary ring.

(1) $\Leftrightarrow$ (12) is obvious. □

Remark 3.8. We note that the equivalences of (1), (3), (7) and (8) were known earlier (see [1,3,6,8,17]).

As an immediate consequence of the above theorem, we have the following result which was proven in a different way by Enochs and Jenda (see [9, Corollary 2.4]).

Corollary 3.9. *Let R be a left noetherian ring. Then R is a left hereditary ring if and only if every copure injective left R-module is injective.*

Proposition 3.10. *Let R be a left coherent ring and an integer $n \geqslant 2$. The following are equivalent for a left R-module M:*

(1) *right* $\mathcal{FI}\text{-}\dim M \leqslant n$.
(2) $\mathrm{Ext}_{n+k}(M, N) = 0$ *for all left R-modules N and all $k \geqslant -1$.*
(3) $\mathrm{Ext}_{n-1}(M, N) = 0$ *for all left R-modules N.*

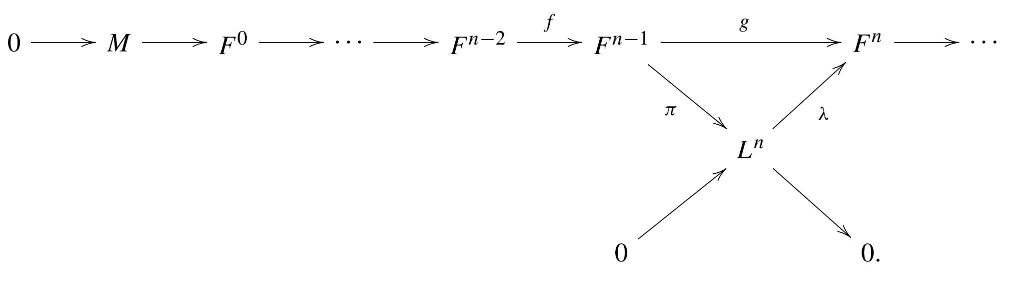
Proof. (1) $\Rightarrow$ (2). Let $0 \to M \to F^0 \to F^1 \to \cdots \to F^n \to 0$ be a right $\mathcal{FI}$-resolution of M, which induces an exact sequence

$$0 \to \mathrm{Hom}(F^n, N) \to \mathrm{Hom}(F^{n-1}, N) \to \mathrm{Hom}(F^{n-2}, N)$$

for any left R-module N. Hence $\mathrm{Ext}_n(M, N) = \mathrm{Ext}_{n-1}(M, N) = 0$. Note that it is clear that $\mathrm{Ext}_{n+k}(M, N) = 0$ for all $k \geqslant 1$. Then (2) holds.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). Let $0 \to M \to F^0 \to F^1 \to \cdots$ be a right $\mathcal{FI}$-resolution of M with $L^n = \operatorname{coker}(F^{n-2} \to F^{n-1})$. We only need to show that L^n is *FP*-injective. In fact, we have the following exact commutative diagram:



By (3), $\mathrm{Ext}_{n-1}(M, L^n) = 0$. Thus the sequence

$$\mathrm{Hom}(F^n, L^n) \xrightarrow{g^*} \mathrm{Hom}(F^{n-1}, L^n) \xrightarrow{f^*} \mathrm{Hom}(F^{n-2}, L^n)$$

is exact. Since $f^*(\pi) = \pi f = 0$, $\pi \in \ker(f^*) = \mathrm{im}(g^*)$. Thus there exists $h \in \mathrm{Hom}(F^n, L^n)$ such that $\pi = g^*(h) = hg = h\lambda\pi$, and hence $h\lambda = 1$ since π is epic. Therefore L^n is *FP*-injective. □

Corollary 3.11. *The following are equivalent for a left coherent ring R and an integer $n \geqslant 2$:*

(1) $l.FP\text{-}\mathrm{dim}(R) \leqslant n$.
(2) $\mathrm{Ext}_{n+k}(M, N) = 0$ *for all left R-modules M, N and all $k \geqslant -1$.*
(3) $\mathrm{Ext}_{n-1}(M, N) = 0$ *for all left R-modules M and N.*

Proof. It follows from Lemma 3.4 and Proposition 3.10. □

Lemma 3.12. *The following are true for any ring R:*

(1) *A left R-module N is FP-injective if and only if, for every pure-injective left R-module G, every homomorphism $f : N \to G$ factors through an injective left R-module.*
(2) *If M is a pure-injective left R-module, and $f : F \to M$ is an FP-injective cover of M, then F is injective.*

Proof. (1) "Only if" part. There is an exact sequence $0 \to N \xrightarrow{i} E \to L \to 0$ with E injective. Since the exact sequence is pure, there exists $g : E \to G$ such that $gi = f$, as required.

"If" part. It is enough to show that the exact sequence $0 \to N \xrightarrow{i} E(N) \to L \to 0$ is pure. Let H be any right R-module. Then H^+ is pure-injective. For any $f : N \to H^+$, there exist an injective left R-module Q and $g : N \to Q$ and $h : Q \to H^+$ such that $f = hg$ by hypothesis. Thus there exists $\alpha : E(N) \to Q$ such that $g = \alpha i$, and so $f = (h\alpha)i$. Therefore we get an exact sequence $\mathrm{Hom}(E(N), H^+) \to \mathrm{Hom}(N, H^+) \to 0$, which gives the exactness of the sequence $(H \otimes E(N))^+ \to (H \otimes N)^+ \to 0$. It follows that $0 \to H \otimes N \to H \otimes E(N)$ is exact. So N is *FP*-injective.

(2) By (1), there exist an injective left R-module E and $g : F \to E$ and $h : E \to M$ such that $f = hg$. So there exists $\varphi : E \to F$ such that $f\varphi = h$ since f is a cover. Therefore $f\varphi g = f$ and hence φg is an isomorphism. It follows that F is isomorphic to a direct summand of E, and so F is injective. □

Lemma 3.13. *Let R be a left coherent ring. If M is an FI-injective left R-module, then there exists an FP-injective cover $N \to M$ with N injective.*

Proof. M has an *FP*-injective cover $f : N \to M$ since R is left coherent. Consider the short exact sequence $0 \to N \xrightarrow{i} E \to L \to 0$ with E injective. Note that L is *FP*-injective by [19, Lemma 3.1] since R is left coherent. So there exists $g : E \to M$ such that $gi = f$ since M is *FI*-injective. Thus there exists $h : E \to N$ such that $fh = g$ since f is a cover. Therefore $fhi = f$, and hence hi is an isomorphism. It follows that N is injective, as desired. □

Corollary 3.14. *Let R be a left coherent ring. If M is a pure-injective left R-module, then M has a minimal left $\mathcal{FI}$-resolution $\cdots \to F_{n-2} \to F_{n-3} \to \cdots \to F_1 \to F_0 \to M \to 0$ with each F_i injective.*

Proof. By Lemma 3.12, M has an *FP*-injective cover $f : F_0 \to M$ with F_0 injective. Note that $\ker(f)$ is *FI*-injective by Proposition 2.4. Hence $\ker(f)$ has an *FP*-injective cover $g : F_1 \to \ker(f)$ with F_1 injective by Lemma 3.13. Note that $\ker(g)$ is *FI*-injective by Proposition 2.4. So we can continue the above process to get the desired minimal left $\mathcal{FI}$-resolution of M. □

Theorem 3.15. *Let R be a left coherent ring. Consider the following conditions for a left R-module N and an integer $n \geqslant 2$:*

(1) *left* $\mathcal{FI}\text{-dim}\, N \leqslant n-2$.
(2) $\mathrm{Ext}_{n+k}(M, N) = 0$ *for all left R-modules M and all $k \geqslant -1$.*
(3) $\mathrm{Ext}_{n-1}(M, N) = 0$ *for all left R-modules M.*

Then (1) ⇒ (2) ⇒ (3). *The converses hold if N is pure-injective.*

Proof. (1) ⇒ (2). By (1), N has a left $\mathcal{FI}$-resolution

$$0 \to F_{n-2} \to \cdots \to F_1 \to F_0 \to N \to 0.$$

Then we have the following complex

$$0 \to \mathrm{Hom}(M, F_{n-2}) \to \mathrm{Hom}(M, F_{n-3}) \to \cdots \to \mathrm{Hom}(M, F_0) \to 0$$

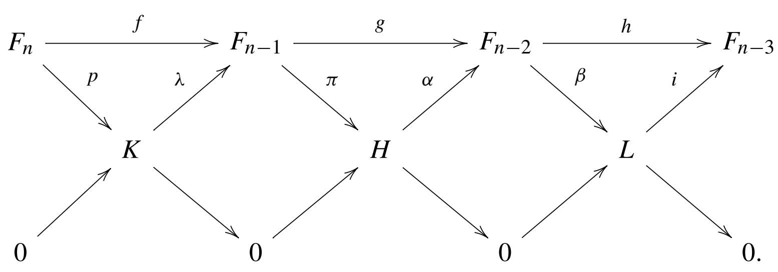
for any left R-module M. Hence $\mathrm{Ext}_{n+k}(M, N) = 0$ for all $k \geqslant -1$.

(2) ⇒ (3) is trivial.

(3) ⇒ (1). Since N is pure-injective, N has a minimal left $\mathcal{FI}$-resolution:

$$\cdots \to F_n \xrightarrow{f} F_{n-1} \xrightarrow{g} F_{n-2} \xrightarrow{h} F_{n-3} \xrightarrow{j} \cdots \to F_1 \to F_0 \to N \to 0$$

with each F_i injective by Corollary 3.14. Put $K = \ker(g)$, $H = F_{n-1}/K$. Let $\lambda : K \to F_{n-1}$ be the inclusion and $\pi : F_{n-1} \to H$ the canonical projection. Then there exists $p : F_n \to K$ such that $f = \lambda p$ and there exists a monomorphism $\alpha : H \to F_{n-2}$ such that $g = \alpha\pi$. Put $L = F_{n-2}/\operatorname{im}(\alpha)$ and let $\beta : F_{n-2} \to L$ be the canonical projection. Then there exists a homomorphism $i : L \to F_{n-3}$ such that $h = i\beta$. So we have the following commutative diagram:



By (3), $\mathrm{Ext}_{n-1}(K, N) = 0$. Thus the sequence

$$\mathrm{Hom}(K, F_n) \xrightarrow{f_*} \mathrm{Hom}(K, F_{n-1}) \xrightarrow{g_*} \mathrm{Hom}(K, F_{n-2})$$

is exact. Since $g_*(\lambda) = g\lambda = 0$, $\lambda \in \ker(g_*) = \mathrm{im}(f_*)$. So $\lambda = f_*(l) = fl$ for some $l \in \mathrm{Hom}(K, F_n)$. But $f = \lambda p$, and hence $\lambda = \lambda pl$. Thus $pl = 1$ since λ is monic, and so K is injective. It follows that H and L are injective. We claim that the complex

$$0 \to L \xrightarrow{i} F_{n-3} \to \cdots \to F_1 \to F_0 \to N \to 0$$

is a left $\mathcal{FI}$-resolution of N. In fact, it is enough to show that the complex

$$0 \longrightarrow \mathrm{Hom}(G, L) \xrightarrow{i_*} \mathrm{Hom}(G, F_{n-3}) \xrightarrow{j_*} \mathrm{Hom}(G, F_{n-4})$$

is exact for any FP-injective left R-module G. Note that we have the following exact commutative diagram:

$$\begin{array}{ccccccccc}
\mathrm{Hom}(G, F_{n-1}) & & \xrightarrow{\quad g_* \quad} & & \mathrm{Hom}(G, F_{n-2}) & & \xrightarrow{\quad h_* \quad} & & \mathrm{Hom}(G, F_{n-3}) \\
& \searrow^{\pi_*} & & \nearrow^{\alpha_*} & & \searrow^{\beta_*} & & \nearrow^{i_*} & \\
& & \mathrm{Hom}(G, H) & & & & \mathrm{Hom}(G, L) & & \\
& \nearrow & & \searrow & & & & \searrow & \\
0 & & & & 0 & & & & 0.
\end{array}$$

So $\ker(i_*\beta_*) = \ker(h_*) = \mathrm{im}(g_*) = \mathrm{im}(\alpha_*\pi_*) = \mathrm{im}(\alpha_*) = \ker(\beta_*)$. Let $\theta \in \ker(i_*)$. Since β_* is epic, $\theta = \beta_*(\gamma)$ for some $\gamma \in \mathrm{Hom}(G, F_{n-2})$. Thus $i_*\beta_*(\gamma) = 0$, and hence $\theta = \beta_*(\gamma) = 0$. It follows that i_* is monic. On the other hand, $\ker(j_*) = \mathrm{im}(h_*) = \mathrm{im}(i_*)$. So we obtain the desired exact sequence. This completes the proof. □

Corollary 3.16. *Consider the following conditions for a left coherent ring R and an integer $n \geqslant 2$:*

(1) *gl left $\mathcal{FI}$-$\dim_R \mathcal{M} \leqslant n - 2$.*
(2) *$l.FP$-$\dim(R) \leqslant n$.*
(3) *left $\mathcal{FI}$-$\dim N \leqslant n - 2$ for all pure-injective left R-modules N.*
(4) *$\mathrm{Ext}_{n+k}(M, N) = 0$ for all left R-modules M, all pure-injective left R-modules N and all $k \geqslant -1$.*
(5) *$\mathrm{Ext}_{n-1}(M, N) = 0$ for all left R-modules M and all pure-injective left R-modules N.*

Then (1) ⇒ (2) ⇒ (3) ⇔ (4) ⇔ (5).

Proof. It follows from Corollary 3.11 and Theorem 3.15. □

Lemma 3.17. *Let R be a left coherent ring. If M is a pure-injective left R-module, then $id(M) \leqslant n$ $(n \geqslant 0)$ if and only if for the minimal left $\mathcal{FI}$-resolution $\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to N \to 0$ of any pure-injective left R-module N,* $\mathrm{Hom}(M, F_n) \to \mathrm{Hom}(M, K_n)$ *is an epimorphism.*

Proof. The proof is modeled on that of [11, Lemma 8.4.34].

We will proceed by induction on n. Let $n = 0$. If M is injective, it is clear that $\mathrm{Hom}(M, F_0) \to \mathrm{Hom}(M, K_0)$ is an epimorphism. Conversely, put $N = M$. Then $\mathrm{Hom}(M, F_0) \to \mathrm{Hom}(M, M)$ is an epimorphism, and so M is injective.

Let $n \geqslant 1$. There is an exact sequence $0 \to M \to E \to L \to 0$ with E injective. Then we have the following exact commutative diagrams:

$$\begin{array}{ccccc}
\mathrm{Hom}(E, F_n) & \longrightarrow & \mathrm{Hom}(E, K_n) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \\
\mathrm{Hom}(M, F_n) & \longrightarrow & \mathrm{Hom}(M, K_n) & & \\
\downarrow & & & & \\
0 & & & &
\end{array}$$

and

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(L, K_n) & \longrightarrow & \mathrm{Hom}(L, F_{n-1}) & \longrightarrow & \mathrm{Hom}(L, K_{n-1}) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(E, K_n) & \longrightarrow & \mathrm{Hom}(E, F_{n-1}) & \longrightarrow & \mathrm{Hom}(E, K_{n-1}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(M, K_n) & \longrightarrow & \mathrm{Hom}(M, F_{n-1}) & \longrightarrow & \mathrm{Hom}(M, K_{n-1}). & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

Note that L is pure-injective by [13, Lemma 3.2.10]. Thus $id(M) \leqslant n$ if and only if $id(L) \leqslant n-1$ if and only if $\mathrm{Hom}(L, F_{n-1}) \to \mathrm{Hom}(L, K_{n-1})$ is an epimorphism by induction if and only if $\mathrm{Hom}(E, K_n) \to \mathrm{Hom}(M, K_n)$ is an epimorphism by the second diagram if and only if $\mathrm{Hom}(M, F_n) \to \mathrm{Hom}(M, K_n)$ is an epimorphism by the first diagram. □

Theorem 3.18. *Let R be a left coherent ring. Then the following are equivalent for an integer $n \geqslant 2$:*

(1) $l.FP\text{-}\dim(R) \leqslant n$.

(2) *left $\mathcal{FI}$-dim $N \leqslant n-2$ for all pure-injective left R-modules N.*
(3) $\mathrm{Ext}_{n+k}(M, N) = 0$ *for all left R-modules M, all pure-injective left R-modules N and all $k \geqslant -1$.*
(4) $\mathrm{Ext}_{n-1}(M, N) = 0$ *for all left R-modules M and all pure-injective left R-modules N.*
(5) $\mathrm{Ext}_{n+k}(M, N) = 0$ *for all pure-injective left R-modules M and N, and all $k \geqslant -1$.*
(6) $\mathrm{Ext}_{n-1}(M, N) = 0$ *for all pure-injective left R-modules M and N.*
(7) *For the minimal left $\mathcal{FI}$-resolution $\cdots \to F_n \to F_{n-1} \to F_{n-2} \to \cdots \to F_0 \to N \to 0$ of any pure-injective left R-module N,* $\mathrm{Hom}(M, F_n) \to \mathrm{Hom}(M, K_n)$ *is an epimorphism for any pure-injective left R-module M.*

Proof. (1) ⇒ (2) ⇒ (3) hold by Corollary 3.16.

(3) ⇒ (4) ⇒ (6), and (3) ⇒ (5) ⇒ (6) are trivial.

(6) ⇒ (7). Let M and N be pure-injective left R-modules and $\cdots \to F_n \xrightarrow{f} F_{n-1} \xrightarrow{g} F_{n-2} \to \cdots \to F_0 \to N \to 0$ be the minimal left $\mathcal{FI}$-resolution of N. Then the sequence

$$\mathrm{Hom}(M, F_n) \xrightarrow{f_*} \mathrm{Hom}(M, F_{n-1}) \xrightarrow{g_*} \mathrm{Hom}(M, F_{n-2})$$

is exact since $\mathrm{Ext}_{n-1}(M, N) = 0$. Note that the sequence

$$0 \longrightarrow \mathrm{Hom}(M, K_n) \longrightarrow \mathrm{Hom}(M, F_{n-1}) \xrightarrow{g_*} \mathrm{Hom}(M, F_{n-2})$$

is exact. It is easy to see that the sequence $\mathrm{Hom}(M, F_n) \to \mathrm{Hom}(M, K_n) \to 0$ is exact.

(7) ⇒ (1) follows from [20, Theorem 3.3.2], Lemmas 3.17 and 3.4. □

Recall that a homomorphism $\phi: M \to C$ with $C \in \mathcal{C}$ is said to a *$\mathcal{C}$-envelope with the unique mapping property* [5] if for any homomorphism $f: M \to C'$ with $C' \in \mathcal{C}$, there is a unique homomorphism $g: C \to C'$ such that $g\phi = f$. Dually we have the definition of a *$\mathcal{C}$-cover with the unique mapping property*.

It has been proven that R is a left coherent ring and $l.FP$-dim$(R) \leqslant 2$ if and only if every right R-module has a flat envelope with the unique mapping property (see [2]). Now we have

Theorem 3.19. *The following are equivalent for a ring R:*

(1) *R is left coherent and $l.FP$-dim$(R) \leqslant 2$.*
(2) *Every left R-module has an FP-injective cover with the unique mapping property.*
(3) *R is left coherent and* $\mathrm{Ext}_1(M, N) = 0$ *for all left R-modules M and N.*
(4) *R is left coherent and* $\mathrm{Ext}_k(M, N) = 0$ *for all left R-modules M, N and all $k \geqslant 1$.*
(5) *R is left coherent and every finitely presented FI-flat right R-module has an epic flat (pre)envelope.*

Proof. (1) ⇔ (3) ⇔ (4) follow from Corollary 3.11.

(1) ⇒ (5). Let M be a finitely presented *FI*-flat right R-module. By the proof of (1) ⇒ (3) in Proposition 3.5, we can construct a right $\mathcal{F}lat$-resolution of a right R-module K:

$$0 \longrightarrow K \xrightarrow{f} F^0 \xrightarrow{g} F^1 \longrightarrow \cdots$$

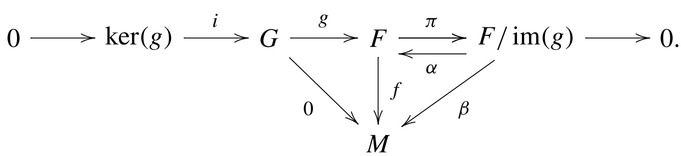
such that $\operatorname{coker}(f) = M$. Note that the complex is exact at F^i for $i \geqslant 1$ by (1) and [11, Theorem 8.4.31]. So we get an exact sequence

$$0 \longrightarrow \operatorname{im}(g) \longrightarrow F^1 \longrightarrow F^2 \longrightarrow L^3 \longrightarrow 0.$$

Thus $\operatorname{im}(g)$ is flat since $fd(L^3) \leqslant 2$. It follows that $M \to \operatorname{im}(g)$ is an epic flat (pre)envelope.

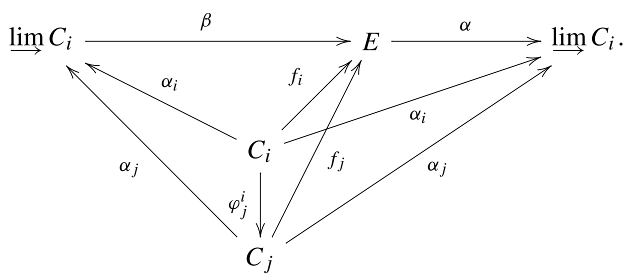
(5) $\Rightarrow$ (1). For any finitely presented right R-module M, there is a right $\mathcal{P}roj_{fg}$-resolution $0 \to M \to P^0 \to P^1 \to 0$ with P^0 and P^1 finitely generated and projective by (5) and Proposition 2.7(1). Thus gl right $\mathcal{P}roj_{fg}$-dim $\mathcal{M}_{R_{fp}} \leqslant 1$, and so $l.FP$-dim$(R) \leqslant 3$ by [11, Corollary 8.4.28]. Hence $l.FP$-dim$(R) = FP$-$id({}_R R)$ by [19, Proposition 3.5]. On the other hand, let $0 \to M \to F^0 \to F^1 \to \cdots$ be any right $\mathcal{P}roj_{fg}$-resolution of a finitely presented right R-module M. Since L^1 has an epic flat preenvelope $L^1 \to G$ by (5) and $L^1 \to F^1$ is a flat preenvelope with $L^2 = \operatorname{coker}(L^1 \to F^1)$, we have $G \oplus L^2 \cong F^1$ by Lemma 3.6(2). Hence L^2 is finitely generated and projective. It follows that the above complex is exact at F^i for $i \geqslant 1$, and so FP-$id({}_R R) \leqslant 2$ by [11, Theorem 8.4.31]. Therefore, $l.FP$-dim$(R) \leqslant 2$.

(1) $\Rightarrow$ (2). Let M be any left R-module. Then M has an FP-injective cover $f : F \to M$. It is enough to show that, for any FP-injective left R-module G and any homomorphism $g : G \to F$ such that $fg = 0$, we have $g = 0$. In fact, there exists $\beta : F/\operatorname{im}(g) \to M$ such that $\beta\pi = f$ since $\operatorname{im}(g) \subseteq \ker(f)$, where $\pi : F \to F/\operatorname{im}(g)$ is the natural map. Since $l.FP$-dim$(R) \leqslant 2$, $F/\operatorname{im}(g)$ is FP-injective. Thus there exists $\alpha : F/\operatorname{im}(g) \to F$ such that $\beta = f\alpha$, and so we get the commutative diagram with an exact row:



Thus $f\alpha\pi = f$, and hence $\alpha\pi$ is an isomorphism. Therefore, π is monic, and so $g = 0$.

(2) $\Rightarrow$ (1). We first prove that R is a left coherent ring. Let $\{C_i, \varphi^i_j\}$ be a direct system with each C_i FP-injective. By hypothesis, $\varinjlim C_i$ has an FP-injective cover $\alpha : E \to \varinjlim C_i$ with the unique mapping property. Let $\alpha_i : C_i \to \varinjlim C_i$ satisfy $\alpha_i = \alpha_j \varphi^i_j$ whenever $i \leqslant j$. Then there exists $f_i : C_i \to E$ such that $\alpha_i = \alpha f_i$ for any i. It follows that $\alpha f_i = \alpha f_j \varphi^i_j$, and so $f_i = f_j \varphi^i_j$ whenever $i \leqslant j$. Therefore, by the definition of direct limits, there exists $\beta : \varinjlim C_i \to E$ such that the following diagram is commutative:



Thus $f_i = \beta\alpha_i$, and so $(\alpha\beta)\alpha_i = \alpha(\beta\alpha_i) = \alpha f_i = \alpha_i$ for any i. Therefore $\alpha\beta = 1_{\varinjlim C_i}$ by the definition of direct limits, and hence $\varinjlim C_i$ is a direct summand of E. So $\varinjlim C_i$ is FP-injective. Thus R is a left coherent ring by [19, Theorem 3.2].

Next we prove that $l.FP\text{-dim}(R) \leqslant 2$. Let M be any left R-module. Then M has an FP-injective cover $f : F \to M$ with the unique mapping property. So $0 \to F \to M \to 0$ is a left $\mathcal{FI}$-resolution. Thus gl left $\mathcal{FI}\text{-dim}_R\, \mathcal{M} = 0$, and hence $l.FP\text{-dim}(R) \leqslant 2$ by Corollary 3.16. □

We conclude the paper with the following

Remark 3.20. It would be interesting to compare the results of Corollary 3.2, Proposition 3.5, Theorems 3.7 and 3.19. Let R be a left coherent ring. Then ${}_RR$ is FP-injective (respectively $FP\text{-}id({}_RR) \leqslant 1$) if and only if every finitely presented (respectively finitely presented FI-flat) right R-module has a flat preenvelope which is a monomorphism by Corollary 3.2 and Proposition 3.5; R is left semihereditary (respectively $l.FP\text{-dim}(R) \leqslant 2$) if and only if every finitely presented (respectively finitely presented FI-flat) right R-module has a flat preenvelope which is an epimorphism by Theorems 3.7 and 3.19. On the other hand, in view of Theorem 3.7(6), R is von Neumann regular (respectively left semihereditary) if and only if every finitely presented (respectively finitely presented FI-flat) right R-module has a flat preenvelope which is an isomorphism. This observation may be viewed as an illustration of the usefulness of FI-flat modules.

Acknowledgments

This research was partially supported by SRFDP (No. 20050284015), NSFC (No. 10331030), China Postdoctoral Science Foundation (No. 20060390926), Collegial Natural Science Research Program of Education Department of Jiangsu Province (No. 06KJB110033), and Jiangsu Planned Projects for Postdoctoral Research Funds (No. 0601021B). The authors thank the referee for the very helpful comments and suggestions.

References

[1] J. Asensio Mayor, J. Martinez Hernandez, Monomorphic flat envelopes in commutative rings, Arch. Math. (Basel) 54 (1990) 430–435.
[2] J. Asensio Mayor, J. Martinez Hernandez, On flat and projective envelopes, J. Algebra 160 (1993) 434–440.
[3] J.L. Chen, N.Q. Ding, A note on existence of envelopes and covers, Bull. Austral. Math. Soc. 54 (1996) 383–390.
[4] R.R. Colby, Rings which have flat injective modules, J. Algebra 35 (1975) 239–252.
[5] N.Q. Ding, On envelopes with the unique mapping property, Comm. Algebra 24 (4) (1996) 1459–1470.
[6] N.Q. Ding, J.L. Chen, Relative coherence and preenvelopes, Manuscripta Math. 81 (1993) 243–262.
[7] E.E. Enochs, Injective and flat covers, envelopes and resolvents, Israel J. Math. 39 (1981) 189–209.
[8] E.E. Enochs, O.M.G. Jenda, Resolvents and dimensions of modules and rings, Arch. Math. (Basel) 56 (1991) 528–532.
[9] E.E. Enochs, O.M.G. Jenda, Copure injective modules, Quaest. Math. 14 (1991) 401–409.
[10] E.E. Enochs, O.M.G. Jenda, Copure injective resolutions, flat resolutions and dimensions, Comment. Math. Univ. Carolin. 34 (1993) 203–211.
[11] E.E. Enochs, O.M.G. Jenda, Relative Homological Algebra, de Gruyter Exp. Math., vol. 30, de Gruyter, Berlin, 2000.
[12] D.J. Fieldhouse, Character modules, dimension and purity, Glasg. Math. J. 13 (1972) 144–146.
[13] R. Göbel, J. Trlifaj, Approximations and Endomorphism Algebras of Modules, de Gruyter Exp. Math., vol. 41, de Gruyter, Berlin, 2006.
[14] S. Jain, Flat and FP-injectivity, Proc. Amer. Math. Soc. 41 (1973) 437–442.

[15] C. Megibben, Absolutely pure modules, Proc. Amer. Math. Soc. 26 (1970) 561–566.
[16] K.R. Pinzon, Absolutely pure modules, Thesis, University of Kentucky, 2005.
[17] J. Rada, M. Saorin, Rings characterized by (pre)envelopes and (pre)covers of their modules, Comm. Algebra 26 (3) (1998) 899–912.
[18] J.J. Rotman, An Introduction to Homological Algebra, Academic Press, New York, 1979.
[19] B. Stenström, Coherent rings and *FP*-injective modules, J. London Math. Soc. 2 (1970) 323–329.
[20] J. Xu, Flat Covers of Modules, Lecture Notes in Math., vol. 1634, Springer-Verlag, Berlin, 1996.