High order Ostrowski type inequalities

George A. Anastassiou*
Department of Mathematical Sciences, The University of Memphis, Memphis, TN 38152, USA
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Abstract
We generalize Ostrowski inequality for higher order derivatives, by using a generalized Euler type identity. Some of the inequalities produced are sharp, namely attained by basic functions. The rest of the estimates are tight. We give applications to trapezoidal and mid-point rules. Estimates are given with respect to $L_\infty$ norm.
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1. Introduction
We mention as inspiration to our work the great Ostrowski inequality [8]:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[ \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a)\|f'\|_{\infty},$$ (1)

where $f \in C'([a, b]), x \in [a, b]$, which is a sharp inequality; see [3]. Other motivations come from [1–5].

We use here the sequence $\{B_k(t), k \geq 0\}$ of Bernoulli polynomials which is uniquely determined by the following identities:

$$B'_k(t) = kB_{k-1}(t), \quad k \geq 1, \quad B_0(t) = 1$$

and

$$B_k(t + 1) - B_k(t) = kt^{k-1}, \quad k \geq 0.$$ 

The values $B_k = B_k(0), k \geq 0$, are the known Bernoulli numbers. We need to mention

$$B_0(t) = 1, \quad B_1(t) = t - \frac{1}{2}, \quad B_2(t) = t^2 - t + \frac{1}{6},$$

$$B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t, \quad B_4(t) = t^4 - 2t^3 + t^2 - \frac{1}{30}.$$ 

* Tel.: +1 901 678 314; fax: +1 901 678 2480. 
E-mail address: ganastss@memphis.edu.

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\[ B_5(t) = t^5 - \frac{5}{2}t^4 + \frac{5}{3}t^3 - \frac{1}{6}t, \quad \text{and} \quad B_6(t) = t^6 - 3t^5 + \frac{5}{2}t^4 - \frac{1}{2}t + \frac{1}{42}. \]

Let \( \{B_k^*(t), k \geq 0 \} \) be a sequence of periodic functions of period 1, related to Bernoulli polynomials by

\[ B_k^*(t) = B_k(t), \quad 0 \leq t < 1, \quad B_k^*(t+1) = B_k^*(t), \quad t \in \mathbb{R}. \]

We have that \( B_k^*(t) = 1, B_k^* \) is a discontinuous function with a jump of \(-1\) at each integer, while \( B_k^*, k \geq 2 \), are continuous functions. Notice that \( B_k(0) = B_k(1) = B_k, k \geq 2 \).

We use the following result.

**Theorem 1.** Let \( f: [a, b] \to \mathbb{R} \) be such that \( f^{(n-1)}, n \geq 1 \), is a continuous function and \( f^{(n)}(x) \) exists and is finite for all but a countable set of \( x \) in \( (a, b) \) and that \( f^{(n)} \in L_1([a, b]) \). Then for every \( x \in [a, b] \) we have

\[ f(x) = \frac{1}{b-a} \int_a^b f(t) \, dt + \sum_{k=1}^{n-1} \frac{(b-a)^{k-1}}{k!} B_k \left( \frac{x-a}{b-a} \right) \left[ f^{(k-1)}(b) - f^{(k-1)}(a) \right] + \frac{(b-a)^{n-1}}{n!} \int_{[a,b]} \left[ B_n \left( \frac{x-a}{b-a} \right) - B_n^* \left( \frac{x-t}{b-a} \right) \right] f^{(n)}(t) \, dt. \]

The sum in (2) when \( n = 1 \) is zero.

**Proof.** By using Theorem 2 of [4], Exercise 18.41(d), p. 299 in [6], and Problem 14(c), p. 264 in [9], and that \( f^{(n-1)} \) as implied is absolutely continuous, it is also of bounded variation. \( \blacksquare \)

If \( f^{(n-1)} \) is just absolutely continuous then (2) is valid again. Formula (2) is a generalized Euler type identity; see also [7]. We define

\[ \Delta_a(x) := f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt - \sum_{k=1}^{n-1} \frac{(b-a)^{k-1}}{k!} B_k \left( \frac{x-a}{b-a} \right) \left[ f^{(k-1)}(b) - f^{(k-1)}(a) \right], \quad x \in [a, b]. \]

We have by (2) that

\[ \Delta_a(x) = \frac{(b-a)^{n-1}}{n!} \int_{[a,b]} \left[ B_n \left( \frac{x-a}{b-a} \right) - B_n^* \left( \frac{x-t}{b-a} \right) \right] f^{(n)}(t) \, dt. \]

In this work we find sharp, namely attained, upper bounds for \( |\Delta_4(x)| \) and tight upper bounds for \( |\Delta_n(x)|, n \geq 5, x \in [a, b] \), with respect to \( L_\infty \) norm. That generalizes (1) for higher order derivatives. High computational difficulties in this direction prevent us from establishing sharpness for \( n \geq 5 \) cases.

### 2. Main results

We give

**Theorem 2.** Let \( f: [a, b] \to \mathbb{R} \) be such that \( f^{(n-1)}, n \geq 1 \), is a continuous function and \( f^{(n)}(x) \) exists and is finite for all but a countable set of \( x \) in \( (a, b) \) and that \( f^{(n)} \in L_\infty([a, b]) \). Then for every \( x \in [a, b] \) we have

\[ |\Delta_n(x)| \leq \frac{(b-a)^{n-1}}{n!} \left( \int_a^b \left| B_n \left( \frac{x-a}{b-a} \right) - B_n^* \left( \frac{x-t}{b-a} \right) \right| \, dt \right) \| f^{(n)} \|_\infty. \]

**Proof.** By Theorem 1. \( \blacksquare \)

Performing the change of variable method on the integral of (5) we obtain
Corollary 1. All assumptions as in Theorem 2. Then for every \( x \in [a, b] \) we have

\[
|\Delta_n(x)| \leq \frac{(b-a)^n}{n!} \left( \int_0^1 \left| B_n(t) - B_n \left( \frac{x-a}{b-a} \right) \right| dt \right) \| f^{(n)} \|_\infty, \quad n \geq 1.
\] (6)

Note. Inequality (6) appeared first as Theorem 7, p. 350, in [4], wrongly under the sole assumption of \( f^{(n)} \in L_\infty([a, b]) \). Also in the rest of [4] the complete assumptions of our Theorem 2 are missing, whenever it applies.

Using the Cauchy–Schwartz inequality we get that

\[
\int_0^1 \left| B_n(t) - B_n \left( \frac{x-a}{b-a} \right) \right| dt \leq \left( \int_0^1 \left( B_n(t) - B_n \left( \frac{x-a}{b-a} \right) \right)^2 dt \right)^{1/2} = \frac{(n!)^2}{(2n)!} |B_{2n}| + B_n^2 \left( \frac{x-a}{b-a} \right), \quad n \geq 1,
\] (7)

where the last part comes by [4], p. 352.

We give

Corollary 2. All assumptions as in Theorem 2. Then for every \( x \in [a, b] \) we have

\[
|\Delta_n(x)| \leq \frac{(b-a)^n}{n!} \left( \frac{(n!)^2}{(2n)!} |B_{2n}| + B_n^2 \left( \frac{x-a}{b-a} \right) \right) \| f^{(n)} \|_\infty, \quad n \geq 1.
\] (8)

Here we elaborate on (6) and (8). We introduce the parameter

\[
\lambda := \frac{x-a}{b-a}, \quad a \leq x \leq b.
\] (9)

We have

\[
\lambda = 0 \quad \text{iff} \quad x = a,
\]

\[
\lambda = 1 \quad \text{iff} \quad x = b,
\]

and

\[
\lambda = \frac{1}{2} \quad \text{iff} \quad x = \frac{a+b}{2}.
\]

Consider

\[
p_4(t) := B_4(t) - B_4(\lambda) = t^4 - 2t^3 + t^2 - \lambda^4 + 2\lambda^3 - \lambda^2.
\] (10)

We need to compute

\[
I_4(\lambda) := \int_0^1 |p_4(t)| dt, \quad 0 \leq \lambda \leq 1.
\] (11)

Lemma 1. We find

\[
I_4(\lambda) = \begin{cases} 
\frac{16\lambda^5}{5} - 7\lambda^4 + \frac{14}{3}\lambda^3 - \lambda^2 + \frac{1}{30}, & 0 \leq \lambda \leq 1/2, \\
-\frac{16\lambda^5}{5} + 9\lambda^4 - \frac{26}{3}\lambda^3 + 3\lambda^2 - \frac{1}{10}, & 1/2 \leq \lambda \leq 1,
\end{cases}
\] (12)
which is continuous in $\lambda \in [0, 1]$. We get
\[ I_4(0) = I_4(1) = \frac{1}{30}, \]
\[ I_4 \left( \frac{1}{2} \right) = \frac{7}{240}. \]  

**Proof.** Here all calculations are done using Mathematica 4. The equation $p_4(t) = 0$ has four real roots,
\[ r_1 = 1 - \lambda, \quad r_2 = \lambda, \quad r_3 = \frac{1}{2} (1 - \sqrt{1 + 4\lambda - 4\lambda^2}) \]
and
\[ r_4 = \frac{1}{2} (1 + \sqrt{1 + 4\lambda - 4\lambda^2}). \]

We find the following orders:

(i) if $0 \leq \lambda \leq 1/2$, then
\[ r_3 \leq 0 \leq r_2 \leq r_1 \leq 1 \leq r_4, \]
and
(ii) if $1/2 \leq \lambda \leq 1$, then
\[ r_3 \leq 0 \leq r_1 \leq r_2 \leq 1 \leq r_4. \]

So we have $p_4(t) = (t - r_1)(t - r_2)(t - r_3)(t - r_4), t \in [0, 1]$. We easily find that when $0 \leq \lambda \leq 1/2$, $p_4(t)$ is greater than or equal to zero over $[\lambda, 1 - \lambda]$ and smaller than or equal to zero over $[0, \lambda]$ and $[1 - \lambda, 1]$.

Similarly when $1/2 \leq \lambda \leq 1$ we get that $p_4(t) \geq 0$ over $[1 - \lambda, \lambda]$ and $p_4(t) \leq 0$ over $[0, 1 - \lambda]$ and $[\lambda, 1]$. Therefore when $0 \leq \lambda \leq 1/2$ we get
\[ I_4(\lambda) = \int_0^\lambda p_4(t) \, dt + \int_{\lambda}^{1-\lambda} p_4(t) \, dt + \int_{1-\lambda}^1 p_4(t) \, dt, \]
while when $1/2 \leq \lambda \leq 1$ we have
\[ I_4(\lambda) = \int_{1-\lambda}^0 p_4(t) \, dt + \int_{0}^{\lambda} p_4(t) \, dt + \int_{\lambda}^1 p_4(t) \, dt, \]
proving (12) after the computations are done. ■

Using basic calculus we find

**Lemma 2.**
\[ \min_{\lambda \in [0, 1]} I_4(\lambda) = I_4 \left( \frac{1}{4} \right) = I_4 \left( \frac{3}{4} \right) = \frac{5}{256} = 0.01953125, \]  
and
\[ \max_{\lambda \in [0, 1]} I_4(\lambda) = I_4(0) = I_4(1) = \frac{1}{30} = 0.033333. \]  

Consequently by Lemmas 1 and 2 we obtain

**Theorem 3.** Assumptions as in Theorem 2, case of $n = 4$. For every $x \in [a, b]$ it holds that
\[ |\Delta_4(x)| \leq \frac{(b - a)^4}{24} I_4(\lambda) \| f^{(4)} \|_{\infty}. \]
where $I_4(\lambda)$ is given by (12) with $\lambda = \frac{x-a}{b-a}$. Furthermore we have that

$$|\Delta_4(x)| \leq \frac{(b-a)^4}{720} \|f^{(4)}\|_{\infty}, \quad \forall x \in [a, b].$$

(17)

Optimality is achieved in

**Theorem 4.** Assumptions as in Theorem 2, case of $n = 4$. Inequality (17) is sharp, namely it is attained when $x = a, b$ by the functions $(t-a)^4$ and $(t-b)^4$.

**Proof.** We have

$$\Delta_4(a) = \Delta_4(b) = \left( \frac{f(a) + f(b)}{2} \right) - \frac{(b-a)}{12} (f'(b) - f'(a)) - \frac{1}{b-a} \int_a^b f(t) dt.$$ 

(18)

So by (17) we have

$$|\Delta_4(a)| = |\Delta_4(b)| \leq \frac{(b-a)^4}{720} \|f^{(4)}\|_{\infty}.$$ 

(19)

Let $f(t) = (t-a)^4$ or $f(t) = (t-b)^4$; then

$$|\Delta_4(a)| = |\Delta_4(b)| = \frac{(b-a)^4}{30} = \text{R.H.S. (19)},$$

proving that (19) is attained, that is proving (17) sharp. □

The trapezoid and mid-point inequalities follow.

**Corollary 3.** Assumptions as in Theorem 2, case of $n = 4$. It holds that

$$\left| \left( \frac{f(a) + f(b)}{2} \right) - \frac{(b-a)}{12} (f'(b) - f'(a)) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{(b-a)^4}{720} \|f^{(4)}\|_{\infty},$$

(20)

and the last inequality is attained by $(t-a)^4$ and $(t-b)^4$, that is, sharp.

Furthermore we have

$$\left| f \left( \frac{a + b}{2} \right) + \frac{(b-a)}{24} (f'(b) - f'(a)) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{7}{5760} (b-a)^4 \|f^{(4)}\|_{\infty}.$$ 

(21)

**Remark 1.** We do find the trapezoidal formula

$$\Delta_5(a) = \Delta_5(b) = \Delta_6(a) = \Delta_6(b) = \frac{(f(a) + f(b))}{2} - \frac{(b-a)}{12} (f'(b) - f'(a)) + \frac{(b-a)^3}{720} (f''(b) - f''(a)) - \frac{1}{b-a} \int_a^b f(t) dt.$$ 

(22)

We also find the mid-point formula

$$\Delta_5 \left( \frac{a + b}{2} \right) = \Delta_6 \left( \frac{a + b}{2} \right) = f \left( \frac{a + b}{2} \right) + \frac{(b-a)}{24} (f'(b) - f'(a)) - \frac{7(b-a)^3}{5760} (f'''(b) - f'''(a)) - \frac{1}{b-a} \int_a^b f(t) dt.$$ 

(23)

Using (8) and Mathematica 4 we get
Theorem 5. Assumptions as in Theorem 2, cases of $n = 5, 6$. It holds that

$$
\begin{cases}
|\Delta_5(a)|, \\
\left|\Delta_5 \left(\frac{a + b}{2}\right)\right| 
\end{cases}
\leq \frac{(b - a)^5}{144\sqrt{2310}} \|f^{(5)}\|_{\infty},
$$

(24)

and

$$
|\Delta_6(a)| \leq \frac{1}{1440} \sqrt{\frac{101}{30030}} (b - a)^6 \|f^{(6)}\|_{\infty},
$$

(25)

with

$$
\left|\Delta_6 \left(\frac{a + b}{2}\right)\right| \leq \frac{1}{46080} \sqrt{\frac{7081}{2145}} (b - a)^6 \|f^{(6)}\|_{\infty}.
$$

(26)

References